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MULTIPLE POSITIVE UNBOUNDED SOLUTIONS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY*

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This paper presents some existence results of unbounded solutions of boundary value problem (BVP) on half-line for a class of functional differential equations with infinite delay and nonhomogeneous boundary data. By constructing a new Banach space, a special cone and applying fixed point index theory, we obtain the existence of multiple positive unbounded solutions.

Key Words : Boundary Value Problem; Functional-Differential Equation; Fixed Point Index; Infinite Delay; Cone

1. INTRODUCTION

Consider the boundary value problems of second order functional diferential equations with infinite delay

$$\left.\begin{aligned} &(Lx)(t)+f(t,x_t)=0, \quad t>0;\\ &x(0)-\beta \lim_{t\to 0^+} p(t)\,x'(t)=y\ge 0;\\ &\lim_{t\to +\infty} p(t)\,x'(t)=z\ge 0;\\ &x(t)=\phi(t), \quad t\in(-\infty,0) \end{aligned}\right\} \quad \ldots (1.1)$$

where $(Lx)(t)=:\dfrac{1}{p(t)}(p(t)\,x'(t))'$, $p\in C[0,+\infty)\cap C^1(0,+\infty), p(t)>0$ for $t\in(0,+\infty)$, $\beta\ge 0$ and $f\in C[R^+\times M_h, R^+]$, $\phi\in M_h, R^+=[0,+\infty)$ and M_h will be defined in Section 2. For any fixed $t\ge 0, x_t$ is a function defined on $(-\infty,0)$ by $x_t(s)=: x(t+s)$ for $s\in(-\infty,0)$.

A function $x(t)$ defined on $[0,+\infty)$ is said to be solution of BVP (1.1) if $x(t)$ satisfied (1.1), where

$$x_t(s)=\left.\begin{cases} x(t+s), & s\in[-t,0);\\ \phi(t+s), & s\in(-\infty,-t). \end{cases}\right\} \quad \ldots (1.2)$$

Also, assume that

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$$\int_0^1 \frac{dt}{p(t)} < +\infty, \quad \int_1^{+\infty} \frac{dt}{p(t)} = +\infty. \qquad \text{... (1.3)}$$

Up to now, most known results in this area concern only finite delay or bounded solutions; see[1-6] and references therein, for example. Very recently, Yan[6] discussed BVP (1.1) with impulsive condition $\Delta x|_{t=t_k} = I_k(x_{t_k})$. It is remarkable that his conditions also include $\int_0^{+\infty} \frac{dt}{p(t)} < +\infty$, which is very different from (1.3) in this paper. By using Leray-Shauder theorem and fixed point index theory, he obtained some existence principles of bounded solutions. However, as far as we know, there is no paper to concern the existence of multiple unbounded solutions of BVP (1.1).

In this paper, it will be presented that some existence results of multiple positive unbounded solutions of BVP (1.1). Our main techniques are the theory of M_h space, which is similar to that of[5,6], the fixed point index theory, a new defined Banach space and a special cone. Since the infinite delay and the solution obtained in[6] are both bounded, so our conditions and conclusions are more general than that of[6]. In present paper, not only the solutions obtained here are unbounded, but also the infinite delay may be unbounded. At the same time, the quantity of solutions may be more than two.

The structure of this paper is organized as follows. Section 2 states some preliminaries. Section 3 is devoted to our main results. Finally, in Section 4, some examples are given to illustrate our main results.

2. Preliminaries and some Lemmas

We first establish the theory of space M

Let $R^- = (-\infty, 0), R^+ = [0, +\infty), R = (-\infty, +\infty)$. Assume that $h : R^- \to (0, +\infty)$ is a continuous function with $l = \int_{-\infty}^{0} h(t)dt < +\infty$. Define

$$M[R^-, R] =: \{\varphi : R^- \to R \mid_{\varphi} \text{ is measurable on } R^-\}$$

and

$$\|\varphi\|_{[t,0)} =: \sup_{[t,0)} |\varphi(s)|, \qquad \forall t \in R^-, \quad \varphi \in M[R^-, R].$$

For any $\varphi \in M[R^-, R]$, by [6, Lemma 2.1], $\|\varphi\|_{[t,0)}$ is a measurable function on R^-. Therefore, it can be defined that

$$M_h =: \left\{ \varphi \in M[R^-, R] : \int_{-\infty}^{0} h(t) \|\varphi\|_{[t,0)} dt < +\infty \right\}$$

with norm $\| \varphi \|_h =: \int_{-\infty}^{0} h(t) \| \varphi \|_{[t, 0)} \, dt.$

Lemma 2.1 — $(M_h, \| \cdot \|_h)$ is a Banach space.

PROOF : First, it is obvious that $(M_h, \| \cdot \|_h)$ is a normed linear space. Therefore, we need only to show its completeness. Suppose $\{\varphi_n\}$ is a cauchy sequence of M_h. Then for $\forall a > 0$, we have

$$\left(\min_{t \in [-2a, -a]} h(t) \right) \cdot \| \varphi_n - \varphi_m \|_{[-a, 0)}$$

$$\leq \int_{-2a}^{-a} h(t) \| \varphi_n - \varphi_m \|_{[t, 0)} \, dt \leq \| \varphi_n - \varphi_m \|_h . \qquad \text{... (2.1)}$$

So, there exists a function $\varphi \in M\,[[-a, 0), R]$ such that

$$\lim_{n \to +\infty} \| \varphi_n - \varphi \|_{[-a, 0)} = 0. \qquad \text{... (2.2)}$$

Since a is arbitrary, there exists $\varphi \in M[R^-, R]$ such that (2.2) holds for every $a > 0$. Now it remains to show $\varphi \in M_h$ and $\lim_{n \to +\infty} \| \varphi_n - \varphi \|_h = 0$.

Because $\{\varphi_n\}$ is a cauchy sequence, for $\forall \varepsilon > 0$, there exists $N > 0$ satisfying $\int_{-\infty}^{0} h(t) \| \varphi_n - \varphi_m \|_{[t, 0)} \, dt \leq \varepsilon$ for $n, m \geq N$. Therefore, for any $T > 0$, we have

$$\int_{-T}^{0} h(t) \| \varphi_n - \varphi_m \|_{[t, 0)} \, dt \leq \varepsilon. \quad n, m \geq N. \qquad \text{... (2.3)}$$

Letting $m \to +\infty$ in (2.3), by Fatou Lemma, it follows that

$$\int_{-T}^{0} h(t) \| \varphi_n - \varphi \|_{[t, 0)} \, dt \leq \varepsilon, \quad \forall T > 0, \quad \forall n \geq N.$$

At the same time, since T is arbitrary, againt let $T \to +\infty$ to obtain

$$\int_{-T}^{0} h(t) \| \varphi_n - \varphi \|_{[t, 0)} \, dt \leq \varepsilon, \quad \forall n \geq N.$$

This implies that $\varphi_n - \varphi \in M_h$ for $n > N$. Consequently, $\varphi \in M_h$ and $\lim_{n \to +\infty} \| \varphi_n - \varphi \|_h = 0$.

To sum up, $(M_h, \| \cdot \|_h)$ is a Banach space. □

Now we are in position to consider BVP (1.1). Without loss of generality, suppose $\beta > 0$. Let

$$\tau(t) =: \int_0^1 \frac{1}{p(s)} ds, \quad \tau \in R^+. \qquad \text{... (2.4)}$$

$$G(t, s) =: p(s)\,(\beta + \tau(t \wedge s)), \ \ t, s \in R^+, \qquad \text{... (2.5)}$$

where $t \wedge s$ =: min $\{t, s\}$,

$$e(t) =: y + (\beta + \tau(t))\, z. \qquad \text{... (2.6)}$$

Then it is easy to see that the following conclusion holds.

Lemma 2.2 — $x \in C[R^+, R] \cap C^1 [(0, +\infty), R]$ is a solution of BVP (1.1) if and only if $x \in C[R^+, R]$ is a solution of the following integral equation

$$x(t) = e(t) + \int_0^{+\infty} G(t, s) f(s, x_s)\, ds, \qquad t \in [0, +\infty),$$

where $e(t)$, $G(t, s)$ and x_s are as defined as in (2.6), (2.5) and (1.2), respectively.

Let

$$FC[R^+, R] =: \left\{ x \in C[R^+, R] : \sup_{t \in R} \frac{|x(t)|}{\beta + \tau(t)} < +\infty \right\}. \qquad \text{... (2.7)}$$

Then $FC[R^+, R]$ is a Banach space with norm

$$\|x\|_F =: \sup_{t \in R^+} \frac{|x(t)|}{\beta + \tau(t)}.$$

In present paper, $FC[R^+, R]$ will be the basic space to study (1.1). Now we give the following result which combines $FC[R^+, R]$ with M_h.

Lemma 2.3 — Suppose $x, y \in FC[R^+, R]$, x_t, y_t is as defined as in (1.2). Then for $\forall y > 0$, we have $x_t \in M_h$; moreover,

$$l\,|x(t)| \le \|x_t\|_h \le \|\phi\|_h + 2l \cdot \sup_{s \in [0, t]} |x(s)| \le \|\phi\|_h$$

$$+ 2l\,(\beta + \tau(t))\, \|x\|_F, \quad t \in R^+, \qquad \text{... (2.8)}$$

$$l\,|x(t) - y(t)| \le \|x_t - y_t\|_h \le 2l \cdot \sup_{s \in [0, t]} |x(s) - y(s)|$$

$$\le 2l\,(\beta + \tau(t))\, \|x\|_F, \quad t \in R^+, \qquad \text{... (2.9)}$$

PROOF : First, for fixed $t > 0$, it is obvious that $x_t(s) = x(t+s)$ is measurable on R^-. Next, since

$$\begin{aligned}
\| x_t \|_h &= \int_{-\infty}^{0} h(s) \| x_t \|_{[s,0)} \, ds \\
&= \int_{-\infty}^{-t} h(s) \| x_t \|_{[s,0)} \, ds + \int_{-\infty}^{-t} h(s) \| x_t \|_{[s,0)} \, ds \\
&\le \int_{-\infty}^{-t} h(s) \| \phi \|_{[s,0)} + \| x_0 \|_{[0,t)} \, ds + \int_{-t}^{0} h(s) \| x \|_{[0,t)} \, ds \\
&\le \| \phi \|_h + 2l \cdot \sup_{s \in [0,t]} | x(s) | \\
&\le \| \phi \|_h + 2l (\beta + \tau(t)) \sup_{s \in [0,t]} \frac{| x(s) |}{\beta + \tau(s)} \\
&\le \| \phi \|_h + 2l (\beta + \tau(t)) \| x \|_F . \qquad \ldots (2.10)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\| x_t \|_h &= \int_{-\infty}^{0} h(s) \| x_t \|_{[s,0)} \, ds \\
&\ge | x(t) | \int_{-\infty}^{0} h(s) \, ds = l \, | x(t) | \, !
\end{aligned}$$

This together with (2.10) implies (2.8). The other inequality (2.9) follows similarly. □

Finally in this section we state the following result from the literature[7], [Theorem 2.3.4] which will be used in Section 3.

Lemma 2.4 — (Fixed point theorem of cone expansion and compression of norm type). Let Ω_1 and Ω_2 be two bounded open sets in Banach space E such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let operator $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be completely continuous, where θ denotes the zero element of E and P is a cone of E. Suppose that one of the two conditions

$$(i) \quad \| A_x \| \le \| x \|, \ \forall x \in P \cap \partial \Omega_1 \text{ and } \| A_x \| \ge \| x \|, \ \forall x \in P \cap \partial \Omega_2$$

and

$$(ii) \quad \| A_x \| \ge \| x \|, \ \forall x \in P \cap \partial \Omega_1 \text{ and } \| A_x \| \le \| x \|, \ \forall x \in P \cap \partial \Omega_2$$

is satisfied. Then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Results

For convenience, let us list the following condition.

(H_1) $f \in C[R^+ \times M_h, R^+]$ and there exist $g \in C[R^+, R^+]$ and a nondecreasing function $J \in C[R^+, R^+]$ such that

$$f(t, (\beta + \tau(t))\, \varphi) \leq g(t)\, J(\| \varphi \|_h), \quad \forall t \in R^+, \ \varphi \in M_h$$

and

$$\int_0^{+\infty} p(s)\, g(s)\, ds < +\infty.$$

For the sake of investigating positive solutions of BVP (1.1), we define a special cone P of $FC[R^+, R^+]$ by

$$P =: \{x \in FC[R^+, R^+] : x \text{ is nondecreasing on } R^+ \text{ and } x(t) \geq \frac{\beta x(s)}{\beta + \tau(s)}, \ \forall t, s \in R^+\}.$$

It is easy to see P is a nonempty closed convex subset of $FC[R^+, R^+]$; moreover, P is a cone of $FC[R^+, R^+]$ and for any $x \in P$, notice

$$x(t) \geq \beta \| x \|_F. \qquad \text{... (3.1)}$$

This means $x(t)$ is positive for $t \in (0, +\infty)$ if $x \in P$ and $x \neq \theta$, where θ denotes the zero element of $FC[R^+, R]$. Now, we define an operator A on P by

$$(Ax)(t) =: e(t) + \int_0^{+\infty} G(t, s) f(s, x_s)\, ds, \quad t \in R^+, \qquad \text{... (3.2)}$$

where, and in the following $e(t)$, $G(t, s)$ and x_s are as defined as in (2.6), (2.5) and (1.2), respectively.

Lemma 3.1 — Suppose (H_1) holds. Then $A : P \to P$ is bounded.

Proof : We first show $\int_0^{+\infty} G(t, s) f(s, x_s)\, ds$ is convergent for every $x \in FC[R^+, R^+]$ and $t \in R^+$.

In fact, for any given $x \in FC[R^+, R^+]$, notice $\| x \|_F =: \sup_{s \in R^+} \frac{| x(t) |}{\beta + \tau(t)} < +\infty$. Thus, condition (H_1) and Lemma 2.3 guarantee that

$$\left\| \frac{x_s}{\beta + \tau(s)} \right\|_h \leq \frac{1}{\beta} \| \phi \|_h + 2l \| x \|_f, \quad " s \in R^+,$$

and from the same reason, one can get

$$f(s, x_s) = f\left(s, (\beta + \tau(s)) \frac{x_s}{\beta + \tau(s)} \right) \leq g(s)\, J\left(\left\| \frac{x_s}{\beta + \tau(s)} \right\|_h \right)$$

$$\leq g(s)\, J\left(\frac{1}{\beta} \| \phi \|_h + 2l \| x \|_F \right), \quad s \in R^+.$$

Again by (H_1), it follows that $\int_0^{+\infty} G(t, s) f(s, x_s)\, ds$ is convergent. So Ax is well defined on $FC\,[R^+, R^+]$.

$$(Ax)'(t) = \frac{z}{p(t)} + \frac{1}{p(t)} \int_t^{+\infty} p(s) f(s, x_s)\, ds \geq 0, \quad t \in (0, +\infty),\ x \in P,$$

that is, $(Ax)(t)$ is nondecreasing on R^+.

On the other hand, since

$$\frac{G(t, s)}{G(\xi, s)} = \begin{cases} \dfrac{\beta + \tau(t)}{\beta + \tau(\xi)} \geq \dfrac{\beta}{\beta + \tau(\xi)}, & 0 < t, \xi < s; \\[2ex] \dfrac{\beta + \tau(t)}{\beta + (s)} \geq \dfrac{\beta}{\beta + \tau(\xi)}, & 0 < t < s < \xi; \\[2ex] \dfrac{\beta + \tau(s)}{\beta + \tau(\xi)} \geq \dfrac{\beta}{\beta + \tau(\xi)}, & 0 < \xi < s < t; \\[2ex] \dfrac{\beta + \tau(s)}{\beta + \tau(s)} = 1 \geq \dfrac{\beta}{\beta + \tau(\xi)}, & t, \xi > s > 0 \end{cases}$$

for $x \in FC\,[R^+, R^+]$, one can obtain

$$(Ax)(t) = e(t) \int_0^{+\infty} G(t, s) f(s, x_s)\, ds$$

$$\geq \frac{\beta}{\beta + \tau(\xi)} e(\xi) + \int_0^{+\infty} \frac{\beta}{\beta + \tau(\xi)} G(\xi, s) f(s, x_s)\, ds$$

$$= \frac{\beta}{\beta + \tau(\xi)} \left[e(\xi) + \int_0^{+\infty} G(\xi, s) f(s, x_s)\, ds \right]$$

$$= \frac{\beta}{\beta + \tau(\xi)} (Ax)(\xi), \quad \forall\, t, \xi \in R^+.$$

Therefore, $A(FC[R^+, R^+]) \subset P$. Immediately, $A: P \to P$. In the following we prove $A: P \to P$ is bounded. To see this suppose B is a bounded subset of P, that is, there exists $L > 0$ such that

$$\|x\|_F \leq L, \quad \forall x \in B. \qquad \text{... (3.3)}$$

This together with Lemma 2.3 implies that

$$\left\| \frac{x_s}{\beta + \tau(s)} \right\|_h \leq \frac{1}{\beta} \|\phi\|_h + 2l\|x\|_F \leq \frac{1}{\beta}\|\phi\|_h + 2lL, \quad \forall x \in B. \qquad \text{... (3.4)}$$

Combining condition (H_1) with (3.3), (3.4), there exists $M > 0$ satisfying

$$f(s, x_s) = f\left(s, (\beta + \tau(s)) \frac{x_s}{\beta + \tau(s)}\right) \leq g(s)\, J\left(\left\| \frac{x_s}{\beta + \tau(s)} \right\|_h \right)$$

$$\leq Mg(s), \qquad \forall x \in B, s \in R^+. \qquad \text{... (3.5)}$$

Consequently by (H_1) and (3.5) again

$$\left| \frac{(Ax)(t)}{\beta + \tau(t)} \right| = \frac{e(t)}{\beta + \tau(t)} + \int_0^{+\infty} \frac{G(t, s)}{\beta + \tau(t)} f(s, x_s)\, ds$$

$$\leq \frac{y}{\beta} + z + M \int_0^{+\infty} p(s)\, g(s)\, ds < +\infty, \quad \forall t \in R^+, \quad \forall x \in B, \qquad \text{... (3.6)}$$

which yields

$$\|Ax\|_F \leq \frac{y}{\beta} + z + M \int_0^{+\infty} p(s)\, g(s)\, ds.$$

This means that $A: P \to P$ is bounded. □

From Lemma 2.2 and Lemma 3.1, we need to prove only the existence of fixed point for operator A on P. For this sake, we still need the following two Lemmas.

Lemma 3.2 — Let (H_1) be satisfied, V is a bounded subset of $FC[R^+, R^+]$. Then $\frac{(AV)(t)}{\beta + \tau(t)}$ are equicontinuous on any finite subinterval of R^+ and also for any $\varepsilon > 0$, there exists $N > 0$ such that

$$\left| \frac{(Ax)(t_1)}{\beta + \tau(t_1)} - \frac{(Ax)(t_2)}{\beta + \tau(t_2)} \right| < \varepsilon$$

uniformly with respect to $x \in V$ as $t_1, t_2 \geq N$.

The proof is similar to that of[8], [Theorem 2.1]. So it is omitted.

Lemma 3.3 — Assume (H_1) holds. Then $A : P \to P$ is completely continuous.

PROOF : For any bounded subset $B \subset P$, using lemma 3.2 and a similar process as the proof of[8], [Theorem 2.1], one can get AB is relatively compact. So we need only to prove $A : P \to P$ is continuous.

To see this suppose $\{x_n\}$, $\{x\} \subset P$ and $\| x_n - x \|_F \to 0$ $(n \to +\infty)$. Hence, $\{x_n : n \geq 1\}$ is a bounded subset of $FC[R^+, R^+]$. Similar to (3.5), there exists $M > 0$ such that

$$f(s, x_{ns}) = f\left(s, (\beta + \tau(s)) \frac{x_{sn}}{\beta + \tau(s)} \right) \leq g(s)\, J\left(\left\| \frac{x_{sn}}{\beta + \tau(s)} \right\|_h \right)$$

$$\leq Mg(s), \quad s \in R^+, \ n \geq 1. \qquad \text{... (3.7)}$$

In addition, by (2.9), we know $\| x_{ns} - x_s \|_h \to 0$ as $n \to +\infty$. This together with (3.2)(3.7) and Lebesgue dominated convergence theorem gurantees that

$$(Ax_n)(t) \to (Ax)(t) \quad (n \to +\infty), \qquad t \in R^+. \qquad \text{... (3.8)}$$

Now we prove $\| Ax_n - Ax \|_F \to 0$ $(n \to +\infty)$.

In fact, if this is not true, then there exist $\varepsilon_0 > 0$ and $\{x_{n_i}\} \subset \{x_n\}$ such that $\| Ax_{n_i} - Ax \|_F \geq \varepsilon_0 (i = 1, 2, 3, ...)$. Since $\{Ax_n\}$ is relatively compact, there exists a subsequence of $\{Ax_{n_i}\}$ (without loss of generality, we relabel the subsequence still as $\{Ax_{n_i}\}$) and $u \in P$ with $Ax_{n_i} \to u (i \to +\infty)$, this is, $\| Ax_{n_i} - u \|_F \to 0$ $(i \to +\infty)$. Therefore,

$$\left| \frac{(Ax_{n_i})(t)}{\beta + \tau(t)} - \frac{u(t)}{\beta + \tau(t)} \right| \to 0 \quad (n \to +\infty), \qquad (n \to +\infty), \qquad \forall t \in R^+.$$

Combining with (3.8), one can get $u = Ax$. This is a contradiction.

Consequently, A is continuous. □

Now we are in position to state our main results.

Theorem 3.1 — *Let (H_1) be satisfied. In addition, suppose*

(H_2) $\overline{\lim_{u \to +\infty}} \frac{J(u)}{u} < \mu$, *where J is the same as in (H_1),* $\mu = \left(2l \int_0^{+\infty} p(s)\, g(s)\, ds \right)^{-1}$, $l = \int_{-\infty}^{0} h(s)\, ds.$

Then, BVP (1.1) has at least one nonnegative solution, which is positive unbounded if $z > 0$.

PROOF : If $y + z > 0$, choose $r \in \left(0, \frac{y}{\beta} + z\right)$. Then, by (3.2) one can get

$$\| Ax \|_F \geq r = \| x \|_F, \qquad \forall x \in \partial P_r, \qquad \text{... (3.9)}$$

where $\quad P_r = \{x \in P : \; \| x \|_F < r\}$.

On the other hand, by (H_2), choose $\mu' < \mu$ such that $\overline{\lim_{u \to +\infty}} \frac{J(u)}{u} < \mu'$. Then there exists $R' > 0$ such that $J(u) \leq \mu' u$ for $u \geq R'$. Let

$$R = \max \left\{ R', \frac{R'}{2l}, \left(1 - 2l\mu' \int_0^{+\infty} p(s)\, g(s)\, ds \right)^{-1} \left(\frac{y}{\beta} + z + \frac{u'}{\beta} \int_0^{+\infty} p(s)\, g(s) ds \cdot \| \phi \|_h \right) \right\}. \qquad \text{... (3.10)}$$

This implies that $\frac{1}{\beta} \| \phi \|_h + 2lR \geq R'$. In addition, (3.10) together with $(H_1)\,(H_2)$ and Lemma 2.3 guarantees that

$$\begin{aligned}
\left| \frac{(Ax)\,(t)}{\beta + \tau(t)} \right| &= \frac{e(t)}{\beta + \tau(t)} + \int_0^{+\infty} \frac{G(t,s)}{\beta + \tau(t)} f(s, x_s) ds \\
&\leq \frac{y}{\beta} + z + \int_0^{+\infty} p(s)\, g(s)\, J \left(\left\| \frac{x_s}{\beta + \tau(s)} \right\|_h \right) ds \\
&\leq \frac{y}{\beta} + z + \int_0^{+\infty} p(s)\, g(s)\, J \left(\frac{1}{\beta} \| \phi \|_h + 2l \| x \|_F \right) ds \\
&\leq \frac{y}{\beta} + z + u' \int_0^{+\infty} p(s)\, g(s) \left(\frac{1}{\beta} \| \phi \|_h + 2lR \right) ds \leq R, \quad \forall x \in \overline{P}_R, \qquad \text{... (3.11)}
\end{aligned}$$

that is,

$$\| Ax \|_F \leq R, \quad \forall x \in \overline{P}_R. \qquad \text{... (3.12)}$$

This together with (3.9) and Lemma 2.4 guarantees that BVP (1.1) has a positive solution $x \in FC\,[R^+, R^+]$.

If $y = z = 0$, the well known Leray-shauder theorem and (3.11) (3.12) guarantee that BVP (1.1) has at least one nonnegative solution $x \in FC\,[R^+, R^+]$. □

Corollary 3.1 — Suppose the conditions of Theorem 3.1 hold. Then the following initial value problem with infinite delay

$$\left.\begin{aligned} &(Lx)(t)+f(t,x_t)=0, \quad t>0; \\ &x(0)-\beta \lim_{t\to 0^+} p(t)\,x'(t)=y\geq 0; \\ &x(t)=\phi(t), \quad t\in(-\infty,0). \end{aligned}\right\} \qquad \text{... (3.13)}$$

has infinite positive unbounded solutions defined on R^+, where $(Lx)(t), p, \phi$ are the same as that of (1.1).

PROOF : By the proof of Theorem 2.1, for any $z > 0$, there exists at least one positive solution $x(t)$ of IVP (3.13) which satisfied $\lim_{t\to 0^+} p(t)\,x'(t)=z$. Thus, our conclusion follows. □

The following theorem is the uniqueness result for BVP (1.1).

Theorem 3.2 — *Suppose* $f\in C[R^+\times M_h, R^+]$ *and there exists* $v\in C[R^+\times R^+]$ *such that*

$$\left| f(t,(\beta+\tau(t))\,\varphi_1)-f(t,(\beta+\tau(t))\,\varphi_2)\right|,$$

$$\leq v(t)\left|\left|\varphi_1-\varphi_2\right|\right|_h, \quad \forall t\in R^+. \qquad \text{... (3.14)}$$

In addition,

$$2l\int_0^{+\infty} p(t)\,v(t)\,dt<1, \quad \int_0^{+\infty} f(t,0)\,dt<+\infty. \qquad \text{... (3.15)}$$

Then, BVP (1.1) has a unique nonnegative solution, which is positive unbounded if $z = 0$.

PROOF : Suppose the operator A is as defined as in (3.2). Similar to the proof of Lemma 3.1, under the conditions (3.14) (3.15), one can see $A:P\to P$. On the other hand, for any $x_1, x_2\in P$, from (3.14) and Lemma 2.3, it follows that

$$\| Ax_1-Ax_2\|_F = \sup_{t\in R^+}\frac{|(Ax_1)(t)-(Ax_2)(t)|}{\beta+\tau(t)}$$

$$\leq \int_0^{+\infty} p(s)\,|f(s,x_{1s})-f(s,x_{2s})|\,ds$$

$$= \int_0^{+\infty} p(s)\left| f\left(s,(\beta+\tau(s))\frac{x_{1s}}{\beta+\tau(s)}\right) - f\left(s,(\beta+\tau(s))\frac{x_{2s}}{\beta+\tau(s)}\right)\right| ds$$

$$\leq \int_0^{+\infty} p(s)\, v(s) \left\| \frac{x_{1s}}{\beta+\tau(s)} - \frac{x_{2s}}{\beta+\tau(s)} \right\|_h$$

$$\leq 2l \int_0^{+\infty} p(s)\, v(s)\, ds \cdot \| x_1 - x_2 \|_F.$$

This together with (3.15) and Banach fixed point theorem guarantees that A has a unique fixed point on P, that is, BVP (1.1) has a unique nonnegative solution. □

To give the existence of multiple positive solution, under the condition of (H_1), let

$$I(r) =: z + \sup_{t \in R^+} \int_a^b \frac{G(t, s)}{\beta+\tau(t)} \underline{f}(s, r)\, ds, \qquad \text{... (3.16)}$$

where $0 < a < b < +\infty$ and

$$\underline{f}(s, r) =: \inf\{f(s, x_s) : s \in [a, b], x \in P$$

and

$$l\beta r \leq \| x_s \|_h \leq \| \phi \|_h + 2rl\,(\beta + \tau(s))\}. \qquad \text{... (3.17)}$$

At the same time, let

$$F(R) =: \frac{y}{\beta} + z + J\left(\frac{1}{\beta} \| \phi \|_h + 2lR \right) \int_0^{+\infty} p(s)\, g(s) ds. \qquad \text{... (3.18)}$$

Now we are ready to state the result of multiple positive solutions for BVP (1.1).

Theorem 3.3 — *Suppose (H_1) holds.*

(i) If there exist $0 < r_1 < R_1 < r_2 < R_2 < \ldots < r_k < R_k < +\infty$ such that $I(r_1) > r_i$ and $F(R_i) < R_i$. Then BVP (1.1) has at least $2k - 1$ positive solutions belonging to $FC\,[R^+ \times R^+]$.

(ii) If there exist $0 < R_1 < r_1 \ldots < R_k < r_k < +\infty$ such that $I(r_i) > r_i$ and $F(R_i) < R_i$. Then BVP (1.1) has at least $2k - 1$ positive solutions belonging to $FC\,[R^+ \times R^+]$. In addition, BVP (1.1) has a nonnegative solution belonging to P_{R_1}.

PROOF : (*i*) First we prove for every $i = 1, 2, \ldots, k$,

$$\| Ax \|_F > \| x \|_F = r_i, \quad \forall x \in \partial P_{r_i}. \qquad \text{... (3.19)}$$

In fact, for $x \in \partial P_{r_i}$, (3.1) (3.2) and Lemma 2.3 guarantee that

$$l\beta r_i = l\beta \| x \|_F \leq l\, | x(s) | \leq \| x_s \|_h \leq \| \phi \|_h + 2lr_i\,(\beta + \tau(s)), \quad \forall s > 0.$$

So, by (3.16) (3.17) we have

$$f(s, x_s) \geq \underline{f}(s, r_i), \quad s \in [a, b],$$

and consequently

$$\|Ax\|_F = \sup_{t \in R^+} \left\{ \frac{e(t)}{\beta + \tau(t)} + \int_0^{+\infty} \frac{G(t, s)}{\beta + \tau(t)} f(s, x_s)\, ds \right\}$$

$$\geq z + \sup_{t \in R^+} \int_a^b \frac{G(t, s)}{\beta + \tau(t)} f(s, x_s)\, ds$$

$$\geq z + \sup_{t \in R^+} \int_a^b \frac{G(t, s)}{\beta + \tau(t)} \underline{f}(s, r_i)\, ds$$

$$= I(r_i) > r_i = \|x\|_F, \qquad \forall x \in \partial P_{r_i}.$$

Thus, (3.19) follows.

Next we prove for every $i = 1, 2, ..., k$,

$$\|Ax\|_F < \|x\|_F = R_i, \ \forall x \in \partial P_{r_i}. \qquad \text{... (3.20)}$$

Indeed, for $x \in \partial P_{R_i}$, similar to (3.11) we have

$$\|Ax\|_F \leq \frac{y}{\beta} + z + \int_o^{+\infty} p(s)\, g(s)\, J\left(\left\| \frac{x_z}{\beta + \tau(s)} \right\|_h \right) ds$$

$$\leq \frac{y}{\beta} + z + \int_o^{+\infty} p(s)\, g(s)\, J\left(\frac{1}{\beta} \|\phi\|_h + 2lR_i \right) ds = F(R_i) < R_i.$$

This means (3.20) holds.

Thus, (3.19) (3.20) together with Lemma 3.3. and Lemma 2.4 guarantee that there exists at least one positive solution of BVP (1.1) which lies in $\overline{P}_{R_i}\ P_{r_i}$ ($i = 1, 2, ..., k$) and $\overline{P}_{r_{i+1}} \setminus P_{R_i}$ ($i = 1, 2, ..., k - 1$), respectively. Also, (3.19) (3.20) implies that BVP (1.1) has no solution on ∂P_{R_i} and ∂P_{r_i} ($i = 1, 2, ..., k$). In conclusion, BVP (1.1) has at least $2k - 1$ positive solutions which lies in $P_{R_k} \setminus \overline{P}_{r_1}$.

The proof of (*ii*) is very similar to that of (*i*), so it is omitted. □

Corollary 3.2 — Suppose (H_2) holds and there exists $R > 0$ such that

(*i*) $F(R) < R$, where F is as defined as in (3.18).

(*ii*) There exists a subinterval $[a, b] \subset R^+$ such that

$$\lim_{\substack{\|x_t\|_h \to +\infty \\ x \in P}} \frac{f(t, x_t)}{\|x_t\|_h} = +\infty \text{ uniformly with respect to } t \in [a, b].$$

Then there exist at least two nonnegative solutions of BVP (1.1), which are both positive unbounded if $z > 0$.

PROOF : First we prove that there exists $R' > R$ such that

$$\|Ax\|_F \geq \|x\|_F = R', \quad \forall x \in \partial P_{R'}. \qquad \text{... (3.21)}$$

In fact, (3.1) and Lemma 2.3 implies that

$$\|x_s\|_h \geq l\,|x(s)| \geq l\beta \|x\|_F, \quad \forall x \in P. \qquad \text{... (3.22)}$$

Let $\mu = \left(l\beta \int_a^b \frac{\beta + \tau(a)}{\beta + \tau(b)}\, p(s)\, ds \right)^{-1}$. Then from condition (*ii*), there exists $M > 0$ such that

$$\frac{f(s, x_s)}{\|x_s\|_h} > \mu \quad \text{as} \quad s \in [a, b],\ x \in P \quad \text{and} \quad \|x_s\|_h > M. \qquad \text{... (3.23)}$$

Again let $R' =: \frac{R+M}{l\beta}$. Then (3.22) guarantees that

$$\|x_s\|_h \geq l\beta \|x\|_F = l\beta R' = R + M > M, \qquad \forall s \in [a, b], \quad x \in \partial P_{R'}.$$

This together with (3.23) yields

$$f(s, x_s) > \mu \|x_s\|_h \geq \mu\beta l R', \qquad \forall s \in [a, b], \quad x \in \partial P_{R'}.$$

Therefore, (3.2) gives

$$\|Ax\|_F = \sup_{t \in R^+} \left\{ \frac{e(t)}{\beta + \tau(t)} + \int_0^{+\infty} \frac{G(t, s)}{\beta + \tau(t)} f(s, x_s)\, ds \right\}$$

$$\geq z + \sup_{t \in R^+} \int_a^b \frac{G(t, s)}{\beta + \tau(t)} f(s, x_s)\, ds$$

$$\geq z + \int_a^b \frac{\beta + \tau(a)}{\beta + \tau(b)}\, p(s) f(s, x_s)\, ds$$

$$= \left(\int_a^b \frac{\beta + \tau(a)}{\beta + \tau(b)}\, p(s)\, ds \right) \mu\beta l R' = R' = \|x\|_F, \quad \forall x \in \partial P_{R'}.$$

Consequently, (3.21) follows.

At last, (3.21) together with the proof of Theorem 3.2 and Lemma 2.4 yields our results. □

Remark : If $\beta = 0$, instead of above, $FC[R^+ \times R^+]$ is defined by

$$FC[R^+ \times R^+] =: \left\{ x \in C[R^+ \times R^+] : \sup_{t \in R^+} \frac{x(t)}{1 + \int_0^t \frac{ds}{p(s)}} < +\infty \right\}$$

with norm $\|x\|_F = \sup_{t \in R^+} \dfrac{x(t)}{1 + \int_0^t \frac{ds}{p(s)}}$ and cone P is defined by

$$P =: \left\{ x \in C[R^+ \times R^+] : x(t) \geq \begin{cases} \dfrac{\tau(t)\, x(\xi)}{1 + \tau(\xi)}, & \tau(t) \in (0, 1); \\[2ex] \dfrac{x(\xi)}{1 + \tau(\xi)}, & \tau(t) \in [1, +\infty) \end{cases} \right\}.$$

Following the same steps as above, we also can give similar results.

4. Examples

Example 4.1 — Consider the following problem

$$\begin{cases} x''(t) + \dfrac{x'}{2t} + t^3 e^{-t} \ln\left(1 + \displaystyle\int_{-\infty}^{0} e^s |x_t(s)|\, ds\right) + \dfrac{\sqrt{|x(t)|}}{1 + t^2} = 0, \ t \in (0, +\infty); \\[2ex] x(0) - \lim\limits_{t \to 0^+} \sqrt{t}\, x'(t) = y \geq 0; \\[2ex] \lim\limits_{t \to +\infty} \sqrt{t}\, x'(t) = z \geq 0; \\[2ex] x(t) = \phi(t), \quad t \in (-\infty, 0), \end{cases} \qquad \text{... (4.1)}$$

where $\int_{-\infty}^{0} e^s \|\phi\|_{[s,0)}\, ds < +\infty$, $\|\phi\|_{[s,0)} = \sup_{t \in [s,0)} |\phi(t)|$.

Conclusion — BVP (4.1) has at least one nonnegative solution, which is positive unbounded as $z > 0$.

Proof : Let $h(t) = e^t, t \in (-\infty, 0)$. Then $l = \int_{-\infty}^{0} e^t\, dt = 1$. So we can define M_h-space as in Section 2. As a result, system (4.1) can be regard as an BVP of form (1.1) with $p(t) = \sqrt{t}, \beta = 1$,

$\tau(t) = \int_0^t \dfrac{ds}{p(s)} = 2\sqrt{t}$ and

$$f(t,\varphi)=t^3 e^{-t}\ln\left(1+\int_{-\infty}^{0} e^s|\varphi(s)|\,ds\right)+\frac{\sqrt{|\varphi(0)|}}{1+t^2}.$$

Obviously, $f\in C[R^+\times M_h, R^+]$ and

$$0\le f(t,(1+2\sqrt{t})\varphi)\le g(t)\,J(\|\varphi\|_h)$$

with $g(t)=t^3(1+2\sqrt{t})\,e^{-t}+\frac{1+2\sqrt[4]{t}}{1+t^2}$, $J(u)=\ln\left(1+\frac{3}{2}u\right)+\sqrt{u}$. It is easy to see the conditions $(H_1)\,(H_2)$ are satisfied. From Theorem 3.1 our conclusion follows. □

Example 4.2. — Consider the following problem

$$\left.\begin{aligned}
&x''(t)+\frac{1}{24e^t}\left[\frac{x^2(t)}{1+t}+\int_{-\infty}^{0} e^s x_t(s)\,ds\right]=0,\quad t\in(0,+\infty);\\
&x(0)-\lim_{t\to 0^+} x'(t)=y>0;\\
&\lim_{t\to+\infty} x'(t)=z\ge 0;\\
&x(t)\equiv\phi(t),\quad t\in(-\infty,0).
\end{aligned}\right\}\qquad \text{... (4.2)}$$

Conclusion : System (4.2) has at least two positive unbounded solutions if $y+z<\frac{1}{2}$.

PROOF : Let $h(t)=e^t, t\in(-\infty,0)$. Then $l=\int_{-\infty}^{0} e^t\,dt=1$ and we can define M_h-space. The system (4.2) can be regard as an BVP of form (1.1) with $p(t)\equiv 1, \beta=1, \tau(t)=t$ and $f(t,\varphi)=\frac{1}{24e^t}\left[\frac{\varphi^2(0)}{1+t}+\int_{-\infty}^{0} e^s\varphi(s)\,ds\right]$. This guarantees that

$$\begin{aligned}
f(t,(1+t)\varphi) &= \frac{1}{24e^t}\left[(1+t)\,\varphi^2(0)+(1+t)\int_{-\infty}^{0} e^s\varphi(s)\,ds\right]\\
&\le \frac{1+t}{24e^t}\,[\|\varphi\|_h^2+\|\varphi\|_h],
\end{aligned}$$

that is, as in (H_1),

$$g(t)=\frac{1+t}{24e^t},\quad J(u)=u^2+u,$$

so (H_1) is satisfied.

At the same time, from (3.16) and $y+z<\frac{1}{2}$, one can obtain $F(1)=y+z+\frac{1}{2}<1$. This means the condition (*i*) of Corollary 3.2 is satisfied.

On the other hand, notice that

$$\|x_t\|_h = \int_{-\infty}^{0} e^s \|x_t\|_{[s,0)} ds = \int_{-t}^{0} e^s |x(t)| ds = (1-e^{-t} x(t), \qquad t>0, \qquad x\in P.$$

Thus, $\|x_t\|_h \to \infty$ uniformly for $t\in [1, 0]$ is equivalent to that $x(t)$ tend to infinity uniformly for $t\in [1, 10]$, and so

$$\lim_{\substack{\|x_t\|_h \to +\infty \\ x\in P}} \frac{f(t,x_t)}{\|x_t\|_h} = \frac{1}{24e^t(1-e^{-t})x(t)} \left[(1+r)x^2(t) + (1+t)\int_{-\infty}^{0} e^s x_t(s) ds \right] = +\infty$$

uniformly with respect to $t\in [1, 10]$. This means the condition (*ii*) of Corollary 3.2 is satisfied. Therefore, by Corollary 3.2, our conclusion follows. □

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