Indian J. pure appl. Math., **35**(2) : 225-236, February 2004

GLOBAL RESULTS FOR EIGENVECTORS OF SUPERLINEAR OPERATORS AND APPLICATIONS*

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(*Received* 13 *December* 2002; *accepted* 25 *July* 2003)

In this paper, we investigate some global characteristics of the eigenvectors of a class of superlinear operators by means of the theory of fixed point index. Applications are given to singular differential equations.

Key Words : Continuum; Eigenvector; Fixed Point Index; Singular Equation

1. INTRODUCTION

Let E be a real Banach space with a cone P, and $A : P \to P$ be a completely continuous operator If there exist $\lambda > 0$ and $x \in P \setminus \{\theta\}$, such that $\lambda Ax = x$, then x is said to a positive eigenvector of A corresponding to λ. Let $L = \overline{\{(\lambda, x) \mid \lambda \in R^+, x \in P, x \neq \theta, x = \lambda Ax\}}$, where $R^+ = [0, +\infty)$, θ denotes the zero element of E. It is clear that L is closed and locally compact. Recall that a continuum is a maximal connected set of L.

In[1], the authors studied the global results for the eigenvectors of a class of superlinear operators in the whole Banach space and obtained the existence of the unbounded continuum of L. But it is imperative that A satisfies :

$$(H_0) : A\theta = \theta \text{ and } A_\theta \text{ exists.}$$

It is well known that many nonlinear operators do not satisfy conditions (H_0). Our purpose here is to investigate the global results for the eigenvectors of this kind of nonlinear operators. By applying topological methods on cones we show the existence and some properties of the continuum C of L emanating from $(0, \theta)$.

*The Project Supported by the National Natural Science Foundation of P.R. China (10071042).

As applications, we will discuss in detail a class of singular boundary value problems, a further structure theorem is obtained.

2. The Main Theorems

First we list the following known definition and lemmas for later use.

Definition 1[1] — Let E be a Banach space with a cone P, and $B : E \to E$ be a linear operator. If there exists $u_0 \in P \setminus \{\theta\}$, such that for any given $\varphi \in P \setminus \{\theta\}$, there exist $\alpha > 0, \beta > 0$ and a positive integer n, such that $\alpha u_0 \le B^n \varphi \le \beta u_0$. Then B is said to be u_0-bounded.

Lemma 1[1] — Let B be a completely and u_0-bounded linear operator and $B : E \to P$, then B has exactly one positive eigenvalue corresponding to positive eigenvectors.

Lemma 2[2] — Let K be a compact metric space and A and B disjoint closed subsets of K. Then either there exists a subcontinuum of K meeting both A and B or $K = K_A \cup K_B$, where K_A, K_B are disjoint compact subsets of K containing A and B, respectively.

Lemma 3[3] — Let Ω be a bounded open set of E. $\theta \in \Omega$ and $A : P \cap \overline{\Omega} \to P$ be a condensing operator. Suppose that $Ax \ne tx$, $\forall x \in P \cap \partial\Omega, t \ge 1$. Then $i(A, P \cap \Omega, P) = 1$.

Lemma 4[4] — Let $\lambda_1 \in R^1, \lambda_2 \in R^1, \lambda_1 < \lambda_2$, let U be an open subset of $[\lambda_1, \lambda_2] \times P$ and $U(\lambda) = U \cap (\{\lambda\} \times P)$. Suppose $A : \overline{U} \to P$ is a completely continuous operator such that $A(\lambda, x) \ne x$ for every $(\lambda, x) \in \partial U$. Then $i(A(\lambda, \cdot), U(\lambda), P)$, $\lambda \in [\lambda_1, \lambda_2]$ is well defined and independent of $\lambda \in [\lambda_1, \lambda_2]$.

Lemma 5[3] — Let Ω be a bounded open set of $E, A : P \cap \overline{\Omega} \to P$ be completely continuous. Suppose that

$$(i) \quad \underset{x \in P \cap \partial\Omega}{\infty} \quad \| Ax \| > 0,$$

$$(ii) \quad Ax \ne \mu x, \quad \forall x \in P \cap \partial\Omega, \ 0 < \mu \le 1.$$

Then $i(A, P \cap \Omega, P) = 0$.

Let $A : P \to P$. For convenience, let us list some conditions for later use.

$$(H_1) \qquad \lim_{x \in P, \ \|x\| \to 0^+} \frac{\| Ax \|}{\| x \|} = +\infty;$$

$$(H_2) \qquad \lim_{x \in P, \ \|x\| \to +\infty} \frac{\| Ax \|}{\| x \|} = +\infty;$$

(H_3) there exists a completely continuous and u_0-bounded linear operator $B: P \to P$, such that $Ax \geq Bx$, $\forall x \in P$.

***Theorem* 1** — *Let E be a Banach space with a cone P, and $A: P \to P$ be a completely continuous operator, $A\theta = \theta$. Assume that (H_1) - (H_3) are satisfied.*

Then the continuum C of *L* emanating from $(0, \theta)$ has the following properties:

(i) $C \subset L \subset ([0, \lambda_0] \times P)$, *where $\lambda_0 > 0$ is the unique positive eigenvalue corresponding to positive eigenvectors of B (see Lemma 1);*

(ii) C is unbounded in $[0, \lambda_0] \times P$;

(iii) $\lambda = 0$ *is the unique asymptotic bifurcation point of A in* $[0, +\infty)$;

(iv) there exists $\lambda^* \in (0, \lambda_0]$ such that for any $\lambda \in (0, \lambda^*)$, there exists $x_{\lambda, 1}, x_{\lambda, 2} \in P \setminus \{\theta\}$ with $\| x_{\lambda, 1} \| > \| x_{\lambda, 2} \|$ and $(\lambda, x_{\lambda, 1}), (\lambda, x_{\lambda, 2}) \in C$;

(v) $\lim_{\lambda \to 0^+, (\lambda, x_{\lambda, 1}) \in C} \| x_{\lambda, 1} \| = +\infty$, $\lim_{\lambda \to 0^+, (\lambda, x_{\lambda, 2}) \in C} \| x_{\lambda, 2} \| = 0$;

(vi) $\lambda = 0$ *is the unique bifurcation point of A in* $[0, +\infty)$.

PROOF : (1) Suppose that assertion *(i)* is not true, then there exist $\lambda_1 > \lambda_0$ and $\varphi_1 \in P \setminus \{\theta\}$ such that $\varphi_1 = \lambda_1 A\varphi_1$, noting (H_3), we have

$$\varphi_1 = \lambda_1 A\varphi_1 \geq \lambda_1 B\varphi_1. \qquad \text{... (2.1)}$$

By Lemma 1 and (H_3), λ_0 is the unique eigenvalue of B corresponding to positive eigenvectors. Let φ_0 be a positive eigenvector of B corresponding to λ_0. Since B is u_0-bounded, therefore, there exist $\alpha_1 > 0, \alpha_2 > 0, \beta_1 > 0, \beta_2 > 0$ and $n_1, n_2 \in N$, such that

$$\alpha_1 u_0 \leq B^{n_1} \varphi_0 \leq \beta_1 u_0; \qquad \alpha_2 u_0 \leq B^{n_2} \varphi_0 \leq \beta_2 u_0. \qquad \text{... (2.2)}$$

Since $\varphi_0 = \lambda_0 B\varphi_0$ and B is a linear operator, we have $\varphi_0 = \lambda_0^{n_1} B^{n_1} \varphi_0$. From (2.2), we have

$$\lambda_0^{n_1} \alpha_1 u_0 \leq \varphi_0 \leq \lambda_0^{n_1} \beta_1 u_0.$$

So,

$$B^{n_2} \varphi_1 \geq \alpha_2 u_0 \geq \alpha_2 \lambda_0^{-n_1} \beta_1^{-1} \varphi_0.$$

From (2.1), we get

$$\varphi_1 \geq \lambda_1 B\varphi_1 \geq \lambda_1^{n_2} B^{n_2} \varphi_1 \geq \lambda_1^{n_2} \alpha_2 \lambda_0^{-n_1} \beta_1^{-1} \varphi_0. \qquad \text{... (2.3)}$$

Let

$$\beta^* = \sup\{\beta > 0 \mid \varphi_1 \geq \beta\varphi_0\}, \qquad \text{... (2.4)}$$

then it follows from (2.3) that $\beta^* > 0$, and

$$\varphi_1 \geq \beta^* \varphi_0. \qquad \text{... (2.5)}$$

From (H_3) and (2.1) (2.5), we have

$$\varphi_1 \geq \lambda_1 B\varphi_1 \geq \lambda_1 B(\beta^* \varphi_0) = \lambda_1 \beta^* B\varphi_0 = \lambda_1 \beta^* \lambda_0^{-1} \varphi_0.$$

From (2.4), we get

$$\lambda_1 \beta^* \lambda_0^{-1} \leq \beta^*$$

i.e. $\lambda_1 \leq \lambda_0$ and we get a contradiction. So we have

$$C \subset L \subset ([0, \lambda_0] \times P).$$

(2) For any μ_1, μ_2 with $0 < \mu_1 < \mu_2 < +\infty$, we now prove that $L \cap ([\mu_1, \mu_2] \times P)$ is bounded. In fact, by the condition (H_2), there exists $\overline{R} > 0$, such that $\| \mu_1 A\varphi \| > \| \varphi \|$ for $\varphi \subset P$ and $\| \varphi \| \geq \overline{R}$. Suppose that $L \cap ([\mu_1, \mu_2] \times P)$ is unbounded, then there exist $\mu \in [\mu_1, \mu_2]$ and $x_\mu \in P$ such that $(\mu, x_\mu) \in L$ and $\| x_\mu \| > \overline{R}$, so $\| \mu A x_\mu \| \geq \| \mu_1 A x_\mu \| > \| x_\mu \|$, and this contradicts to $(\mu, x_\mu) \in L$.

(3) For any μ_3, μ_4 with $0 < \mu_3 < \mu_4 < +\infty$, we now prove that there exists $\tau > 0$ such that $([\mu_3 < \mu_4] \times T_r) \cap = \phi$, where $T_r = \{x \in P \mid \| x \| < \tau\}$. If it is not truem then there exists a sequence $\mu_n \in [\mu_3, \mu_4], x_n \neq \theta, x_n \to \theta$ such that $(\mu_n, x_n) \in L$. Assume without loss of generality that $\mu_n \to \mu' \in [\mu_3, \mu_4]$. Then furthermore, we get $\lim\limits_{\| x_n \| \to 0} \frac{\| Ax_n \|}{\| x_n \|} = \frac{1}{\mu'}$ in contradiction with the condition (H_1).

(4) Since $(0, \theta)$ is a solution of the equation $\lambda Ax = x$, let C be the continuum of L emanating from $(0, \theta)$. Suppose that C is bounded. Then there exist a δ-neighbourhood U_1 of C in $R^+ \times P$ and $R_1 > 0$ such that $U_1 \subset [0, R_1] \times P$. If $\partial U_1 \cap L \neq \theta$, then it is easy to know that $X = L \cap \overline{U}_1$ is a compact metric space, C and $D = \partial U_1 \cap L$ are nonempty disjoint closed subsets of the compact metric space X. Since C is a continuum, by Lemma 2, there exist disjoint compact subsets K_1, K_2 of X such that $X = K_1 \cup K_2$, $C \subset K_1$, $D \subset K_2$. Let $\delta_1 = \frac{1}{3} d(K_1, K_2)$ and U be δ_1-neighbourhood of

K_1 in $R^+ \times P$. Then we have

$$C \subset U,\ \partial U \cap L = \phi,\ U \cap (\{R\} \times P) = \phi. \qquad \text{... (2.6)}$$

where $R = R_1 + \delta_1$. If $\partial U_1 \cap L = \phi$, let $U = U_1$, then (2.6) is also true.

Thaking sufficiently small $\varepsilon > 0$, such that $\{(\lambda, \theta) \in U \mid 0 \le \lambda \le \varepsilon\}$, then by Lemma 4,

$$i(\theta, U_0, P) = i(\varepsilon A, U_\varepsilon, P), \qquad \text{... (2.7)}$$

where $U_\lambda = U \cap (\{\lambda\} \times P)$. By the proof of the part (3), we know that $([\varepsilon, R] \times \{\theta\}) \cap (U \cap L) = \phi$, so,

$$d_1 = d([\varepsilon, R] \times \{\theta\},\quad U \cap L) > 0.$$

Clearly,

$$d_2 = d([0, \varepsilon] \times \{\theta\},\ \partial U) > 0.$$

From (H_1), for the above $\varepsilon > 0$, there exists $d_3 > 0$ such that

$$\| \varepsilon A x \| > \| x \|,\quad \forall\ \| x \| \le d_3. \qquad \text{... (2.8)}$$

Let $r = \frac{1}{2} \min\{d_1, d_2, d_3\}$, $T_r = \{x \in P \mid \| x \| < r\}$, $U^* = U \setminus ([0, R] \times \overline{T}_r)$, then U^* is a bounded open set of $[0, R] \times P$. Evidently,

$$\partial U^* \cap ([\varepsilon, R] \times P) \cap L = \phi,$$

$$\partial U^* \cap ([\varepsilon, R] \times P) \cap ([\varepsilon, R] \times \{\theta\}) = \phi,$$

where ∂U^* represents the boundary of U^* in $[0, R] \times P$. Hence, by Lemma 4,

$$i(\varepsilon A, U_\varepsilon^*, P) = i(RA, U_R^*, P), \qquad \text{... (2.9)}$$

where $U_\varepsilon^* = U^* \cap (\{\varepsilon\} \times P)$. Since $\{\varepsilon\} \times T_r \subset U_\varepsilon$, by the additivity of the fixed point index,

$$i(\varepsilon A, U_\varepsilon, P) = i(\varepsilon A, U_\varepsilon^*, P) + i(\varepsilon A, T_r, P). \qquad \text{... (2.10)}$$

By (2.6), $U_R = \phi$, so $U_R^* = \phi$, we get

$$i(RA, U_R^*, P) = 0. \qquad \text{... (2.11)}$$

By (2.8), we have

$\| \varepsilon A X \| > \| x \|,\quad \forall x \in \partial T_r$, then, by Lemma 5,

$$i(\varepsilon A, T_r, P) = 0. \qquad \text{... (2.12)}$$

From (2.7), (2.9)-(2.12), we get

$$i(\theta, U_0, P) = 0. \qquad \text{... (2.13)}$$

By Lemma 3, it is easy to know that

$$i(\theta, U_0, P) = 1, \qquad \text{... (2.14)}$$

this contradicts to (2.13). So assertion *(ii)* is true.

From the proof of (1)(2), we know that $L \subset [0, \lambda_0] \times P$ and $L \cap [\mu_1, \mu_2] \times P$ is bounded for any μ_1, μ_2 with $0 < \mu_1 < \mu_2 < +\infty$. Since $C \subset L$ is unbounded, we get $\lambda = 0$ is the unique asymptotic bifurcation point of A, i.e., assertion *(iii)* is true. Since C is the continuum of L emanating from $(0, \theta)$ and assertions *(i)* *(ii)* and *(iii)*, we easily know that *(iv)* and *(v)* are true. In view of the proof of (3) and $A\theta = \theta$, we know that $\lambda = 0$ is the unique bifurcation point of A. The proof is completed.

In Theorem 1, we suppose $A\theta = \theta$. If $A\theta \neq \theta$, we have

Theorem 2 — *Let E be a Banach space with a cone P, and $A : P \to P$ be a completely continuous operator, $A\theta \neq \theta$. Assume that $(H_2, (H_3)$ are satisfied. Then the continuum C of L emanating from $(0, \theta)$ has the properties (i)-(v) in Theorem 1.*

PROOF : Similar to the proof of (1)(2) of Theorem 1, we easily get that

(a) $C \subset L \subset [0, \lambda_0] \times P$,

(b) For any μ_1, μ_2 with $0 < \mu_2 < \mu_2 < +\infty$, $L \cap [\mu_1, \mu_2] \times P$ is bounded.

Since $(0, \theta)$ is a solution of the equation $\lambda Ax = x$, let C be the continuum of L emanating from $(0, \theta)$. Suppose that C is bounded. Then there exist a δ-neighbourhood U_1 of C and $R > 0$ such that $U_1 \subset [0, R] \times P$. If $\partial U_1 \cap L \neq \phi$, then it is easy to know that $X = L \cap \overline{U}_1$ is a compact metric space, C and $D = \partial U_1 \cap L$ are nonempty disjoint closed subsets of the compact metric space $X = L \cap U_1$. Since C is a continuum, by Lemma 2, there exist disjoint compact subsets K_1, K_2 of X such that $X = K_1 \cup K_2$, $C \subset K_1$, $D \subset K_2$. Let $\delta_1 = \frac{1}{3} d(K_1, K_2)$ and U_2 be δ_1-neighbourhood of K_1, $U = U_1 \cap U_2$. Then we have

$$C \subset U, \ \partial U \cap L = \theta, \ U \cap (\{R\} \times P) = \theta. \qquad \text{... (2.15)}$$

If $\partial U_1 \cap L = \theta$, (2.15) is also true.

Since $A\theta \neq \theta$. Thus U is a bounded open subset of $[0, R] \times P$ such that $x \neq \lambda Ax$ for $(\lambda, x) \in \partial U$ and all $\lambda \in [0, R]$. Hence by Lemma 4, we get

$$i(\theta, U_0, P) = i(RA, U_R, P). \quad \text{... (2.16)}$$

From (2.15), we have

$$i(RA, U_R, P) = i(RA, \phi, P) = 0. \quad \text{... (2.17)}$$

By Lemma 3, it is easy to know that $i(\theta, U_0, P) = 1$, then by (2.16), $i(RA, U_R, P) = 1$, this contradicts to (2.17). So assertion (*ii*) is true.

From (*a*), (*b*) and assertion (*ii*), we easily get $\lambda = 0$ is the unique asymptotic bifurcation point of A, i.e., assertion (*iii*) is true. Since C is the continuum of L^+ emanating from $(0, \theta)$ and assertions (*i*)(*ii*) and (*iii*), we easily know that (*iv*) and (*v*) are true. The proof is completed.

3. APPLICATIONS

In this section, we are concerned with the structure of positive solutions for singular boundary value problems of the form:

$$\left.\begin{aligned} &u''(t) + \lambda p(t) f(u(t)) = 0, \quad t \in (0, 1), \\ &u(0) = u(1) = 0, \end{aligned}\right\} \quad \text{... (3.1)}$$

where $\lambda \in R^+$ is a parameter, $R^+ = [0, +\infty)$, p is singular at 0 and 1.

Wong[5], Robert[6] and Ha and Lee[7] studied the BVP (3.1) and proved that there exists a poisitive real number λ_0 such that (3.1) has no solution for $\lambda > \lambda_0$ and at least one solution for $0 < \lambda < \lambda_0$ under some assumptions. It is imperative that f is nondecreasing in these papers.

Our purpose here is to use Theorem 2 to give the existence of multiple positive solutions to the BVP(3.1) by means of investigating the structure of the solution set when f satisfies weaker assumptions.

It's well known that $C[0, 1]$ is a Banach space with norm $\| u \| = \max_{t \in [0,1]} | u(t) |$. A function $u(t) \in C[0, 1] \cap C^2(0, 1)$ is called a spoitive solution of (1.1) if $u(t)$ satisfies (3.1) and $u(t) > 0$ nontrivial concave function, of course, cannot equal zero at any point $t \in (0, 1)$. Let

$$L = \overline{\{(\lambda, u) \in (R^+ \times P) \mid u \neq \theta, (\lambda, u) \text{ satisfies } BVP\ (3.1)\}}.$$

It is clear that $(0, \theta) \in L$, where θ denotes the zero element of $C[0, 1]$. Let C be the continuum of L emanating from $(0, \theta)$.

It is well known that the problem (3.1) is equivalent to the integral equation

$$u(t) = \lambda \int_0^1 G(t, s)\, p(s) f(u(s))\, ds$$

where

$$G(t, s) = \begin{cases} s(1-t), & \text{for } 0 \le s \le t \le 1, \\ t(1-s), & \text{for } 0 \le t \le s \le 1. \end{cases} \quad \text{... (3.2)}$$

It is easy to verify that

$$\min_{t \in [\sigma, 1-\sigma]} G(t, s) \ge \sigma G(s, s), \quad \forall s \in [0, 1]. \quad \text{... (3.3)}$$

Consequently, the positive solutions for singular boundary value problems (3.1) are equivalent to the fixed points of the operator equation

$$u = \lambda T(u), \quad \forall u \in P,$$

where

$$T(u(t)) = \int_0^1 G(t, s)\, p(s) f(u(s))\, ds.$$

It is easy to verify that operator T maps P into P and T is completely continuous (see[6]). Thus, L is closed and locally compact.

***Theorem* 3** — *Let p and f satisfy the following assumptions* :

(A_1) $p \in C((0, 1), (0, +\infty)), p(0) = p(1) = +\infty$ and $\int_0^1 s(1-s)\, p(s)\, ds < +\infty$.

(A_2) $f \in C([0, +\infty), (0, +\infty))$,

and

$$\lim_{u \to +\infty} \frac{f(u)}{u} = +\infty.$$

Then the continuum C of L emanating from $(0, \theta)$, has the following properties

(*i*) there exist $\lambda_0 > 0$ such that $C \subset [0, \lambda_0] \times P$;

(*ii*) C is unbounded in $[0, \lambda_0] \times P$;

(*iii*) $\lambda = 0$ is the unique asymptotic bifurcation point of T. Where T is the integral operator of (3.1);

(*iv*) there exists $\lambda^* \in (0, \lambda_0]$. Such that (3.1) has at least two positive solutions $u_\lambda^*, u_\lambda^{**}$ with $\| u_\lambda^* \| > \| u_\lambda^{**} \|$ for $\lambda \in (0, \lambda^*)$ and $(\lambda, u_\lambda^*), (\lambda, u_\lambda^{**}) \in C$;

(*v*) $\lim_{\lambda \to 0, (\lambda, u_\lambda^*) \in C} = +\infty, \qquad \lim_{\lambda \to 0, (\lambda, u_\lambda^{**}) \in C} \| u_\lambda^{**} \| = 0.$

PROOF : By the condition (A_2), for given $M_1 > 0$, there exists $u^* > 0$ such that

$$f(u) \geq M_1 u, \quad \forall u \geq u^*. \qquad \text{... (3.4)}$$

Taking the constant $b > 0$ such that $\lim\limits_{0 \leq u \leq u^*} [f(u) - M_1 u] \geq -b$. Thus, we have

$$f(u) \geq M_1 u - b, \quad \forall u \geq 0. \qquad \text{... (3.5)}$$

Hence, from (3.5)(3.3), we know that for any $u \in P$,

$$T(u(t)) = \int_0^1 G(t,s)\,p(s)\,f(u(s))\,ds$$

$$\geq M_1 \int_\sigma^{1-\sigma} G(t,s)\,p(s)\,u(s)\,ds + b \int_\sigma^{1-\sigma} G(t,s)\,p(s)\,ds$$

$$\geq M_1\, \sigma^2 \,\|u\| \int_\sigma^{1-\sigma} G(s,s)\,p(s)\,ds + b \int_\sigma^{1-\sigma} G(t,s)\,p(s)\,ds.$$

Since M_1 is arbitrary, therefore, $\lim\limits_{u \in P, \|u\| \to +\infty} \dfrac{\|Tu\|}{\|u\|} = +\infty$, i.e. the condition (H_2) is satisfied for the operator T.

By (3.4) and the condition (A_2), let $m = \min\limits_{u \in [0, u^*]} f(u)$, $r = \min\left\{M_1, \dfrac{m}{u^*}\right\}$, then $r > 0$ and $f(u) \geq ru$ for $u \geq 0$. By the condition (A_1), $p(t)$ is an infinitely large quantity for $t = 0$ or $t = 1$, it is easy to know that $d = \inf\limits_{t \in [0,1]} p(t) > 0$. Consequently,

$$T(u(t)) = \int_0^1 G(t,s)\,p(s)\,f(u(s))\,ds \geq rd \int_0^1 G(t,s)\,u(s)\,ds.$$

Let $Ku(t) = rd \displaystyle\int_0^1 G(t,s)\,u(s)\,ds$, then it is easy to prove that $K : P \to P$ is a completely continuous linear operator.

By (3.2), we have

$$\int_0^1 G(t,s)ds = \int_0^1 s(1-t)ds + \int_0^1 t(1-s)\,ds = \frac{1}{2}t(1-t) \qquad \text{... (3.6)}$$

For any given $0<\zeta<\eta<1$, if $\eta \le t \le 1$, then

$$\int_{\zeta}^{\eta} G(t,s)ds = \int_{\zeta}^{\eta} s(1-t)ds = \frac{1}{2}(\eta+\zeta)(\eta-\zeta)(1-t)$$

$$\ge \zeta(\eta-\zeta)(1-t);$$

if $\zeta \le t \le \eta$, then

$$\int_{\zeta}^{\eta} G(t,s)ds \ge \int_{\zeta}^{\eta} \zeta(1-\eta)\,ds = \zeta(1-\eta)(\eta-\zeta);$$

if $0 \le t \le \zeta$, then

$$\int_{\zeta}^{\eta} G(t,s)ds \ge \int_{\zeta}^{\eta} t(1-\eta)\,ds = \frac{1}{2}(2-\eta)-\zeta)(\eta-\zeta)\,t$$

$$\ge (1-\eta)(\eta-\zeta)\,t.$$

So,

$$\int_{\zeta}^{\eta} G(t,s)ds \ge 2\zeta(1-\eta)(\eta-\zeta)\int_{0}^{1} G(t,s)\,ds, \quad \forall t \in [0,1].$$

We denote $u_0(t) = \int_0^1 G(t,s)\,ds$, for any given $u(t) \in P \setminus \theta$, we have

$$Ku(t) = rd\int_0^1 G(t,s)\,u(s)\,ds \ge \int_{\sigma}^{1-\sigma} G(t,s)\,u(s)\,ds$$

$$\ge rd\sigma \| u \| \int_{\sigma}^{1-\sigma} G(t,s)\,ds$$

$$\ge 2rd\sigma^3(1-2\sigma)\| u \| \int_0^1 G(t,s)ds$$

$$= 2rd\sigma^3(1-2\sigma)\| u \| u_0(t), \quad \forall t \in [0,1], \qquad \text{... (3.7)}$$

on the other hand, we have

$$Ku(t) = rd\int_0^1 G(t,s)\,u(s)\,ds \le rd\| u \| \int_0 G(t,s)\,ds$$

$$= rd\| u \| u_0(t), \quad \forall t \in [0,1]. \qquad \text{... (3.8)}$$

At the same time, we have

$$\left\| rd \| \|u\| u_0(t) - Ku(t) \right\| = \left\| rd \int_0^1 G(t, s) (\| u \| - u(s)) \, ds \right\|$$

$$\leq rd \int_0^1 s(1-s) (\| u \| - u(s)) \, ds.$$

Using (3.3) and the above inequality we obtain

$$\min_{t \in [\sigma, 1-\sigma]} [rd \| u \| u_0(t) - Ku(t)] \geq \sigma \| rd \| u \| u_0(t) - Ku(t) \|. \quad \text{... (3.9)}$$

Similarly, we easily obtain

$$\min_{t \in [\sigma, 1-\sigma]} [Ku(t) - 2rd\sigma^3 (1-2\sigma) \| u \| u_0(t)]$$

$$\geq \sigma \left\| Ku(t) - 2rd\sigma^3 (1-2\sigma) \| u \| u_0(t) \right\|. \quad \text{... (3.10)}$$

By (3.7)-(3.10), we easily see that K is u_0-bounded in the partial ordering defined by

$$P = \left\{ u \in C[0, 1] \mid u(t) \geq 0, \min_{t \in [\sigma, 1-\sigma]} u(t) \geq \sigma \| u \| \right\}.$$

Thus, the condition (H_3) is satisfied for the operator T. Since $f(0) \neq 0$, we have $T\theta \neq \theta$. So, the operator T satisfies the all conditions of Theorem 2. By Theorem 2, we know that the conclusions of this theorem are true. The proof is completed.

Remark 1 : If p is singular at $t = 0$ $(t = 1)$, then Theorem 3 is valid replacing (H_1) by

$$p \in C[0, 1] \; (p \in C[0, 1)), p > 0 \text{ on } (0, 1)$$

and

$$\int_0^1 sp(s) \, ds < +\infty \left(\int_0^1 (1-s) p(s) \, ds < +\infty \right).$$

Remark 2 : It is imperative that f is nondecreasing in papers [5-7], but we obtain the better results without the assumption that f is nondecreasing. So the results in this section improve the generalize the main results in papers[5-7].

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