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ON THE FOURTH POWER MEAN OF THE GENERAL KLOOSTERMAN SUMS*

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The main purpose of this paper is to study the calculating problem of the fourth power mean of the general Kloosterman sums, and give an exact calculating formula.

Key Words: General Kloosterman sums; Fourth Power Mean; Asymptotic Formula

1. INTRODUCTION

Let $q \geq 3$ be a positive integer. For any integers m and n, the general Kloosterman sums $S(m, n, \chi; q)$ is defined as follows:

$$S(m, n, \chi; q) = \sum_{a=1}^{q}{}' \ \chi(a) e\left(\frac{(ma + n\bar{a})}{q}\right),$$

where $\sum'$ denotes the summation over all a such that $(a, q) = 1$, χ denotes a Dirichlet character, q, $a\bar{a} \equiv 1 \bmod q$ and $e(y) = e^{2\pi i y}$.

This summation is very important, because it is a generalization of the classical Kloosterman sums $S(m, n, \chi_0; q) = S(m, n; q)$, where χ_0 is the principal character mod q. The various properties of $S(m, n; q)$ were investigated by many authors. Perhaps the most famous property of $S(m, n; q)$ is the estimate (see [1] and [2]):

$$|S(m, n; q)| \leq d(q)^{\frac{1}{2}} (m, n, q)^{\frac{1}{2}},$$

where $d(q)$ is the divisor function, (m, n, q) denotes the greatest common divisor of m, n and q. If q be a prime p, then Chowla[3] and Malyshev[4] also studied the general Kloosterman sums, and proved a similar result for $S(m, n, \chi; q)$. On the other hand, Iwaniec[5] studied the fourth power mean of $S(m, n; q)$, and proved the identity

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$$\sum_{a=1}^{p-1} S^4(a, 1; p) = 2p^3 - 3p^2 - 3p - 1. \quad \text{... (1)}$$

Salié[6] and Davenport (independently) obtained the estimate

$$\sum_{a=1}^{p-1} S^6(a, 1; p) \ll p^4.$$

Higher moments of the Kloosterman sums resisted any attack until 1979 when Katz skillfully employed Deligne's profound theory of exponential sums over varieties over finite fields. The main purpose of this paper is to study the calculating problem of the fourth power mean of the general Kloosterman sums, and prove the following:

Theorem — *Let $p \geq 3$ be a prime. Then for any fixed integer n with $(n, p) = 1$, we have the identity*

$$\sum_{m=1}^{p} | S(m, n, \chi; p) |^4$$

$$= \begin{cases} p(2p^2 - 3p - 3), & \text{if } \chi \text{ be the principal character mod } p; \\ p^2(3p - 7), & \text{if } \chi \text{ be the Legendre symbol;} \\ 2p^2(p - 3), & \text{if } \chi \text{ be a complex character mod } p. \end{cases}$$

From this theorem, we may immediately deduce the following.

Corollary — Let p be an odd prime large enough, then there exists an integer m with $(p, m) = 1$ such that

$$\left| S(m, 1, \chi_2; p) \right| \geq \left(3^{\frac{1}{4}} - \varepsilon \right) \sqrt{p},$$

where χ_2 be the Legendre symbol, ε be any fixed positive number.

For general integer $k \geq 3$, whether there exists an exact formula for

$$\sum_{m=1}^{p} | S(m, 1, \chi; p) |^{2k}$$

is an open problem.

2. Some Lemmas

To complete the proof of the theorem, we need the following Lemmas.

Lemma 1 — Let p be an odd prime. Then for any integers m and n with $(mn, p) = 1$, we have the identity

$$| S(m, n, \chi; p) |^2 = p + \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} e\left(\frac{mb - n\bar{b}\,\bar{a}\,(a-1)^2}{p} \right).$$

PROOF : From the properties of characters modulo p and note that $\bar{a} - 1 \equiv \bar{a}\ (1 - a)$ mod p, and if a pass through a reduced residue system modulo p, then ab $(1 \le b \le p - 1)$ is also pass through a reduced residue system modulo p, we have

$$| S(m, n, \chi; p) |^2 = \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \chi(\bar{b})\, e\left(\frac{m(a-b) + n(\bar{a} - \bar{b})}{p} \right)$$

$$= \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} e\left(\frac{mb(a-1) + n\bar{b}\,(\bar{a} - 1)}{p} \right)$$

$$= p - 1 + \sum_{a=2}^{p-1} \chi(a) \sum_{b=1}^{p-1} e\left(\frac{mb(a-1) + n\bar{b}\,(\bar{a} - 1)}{p} \right)$$

$$= p - 1 + \sum_{a=2}^{p-1} \chi(a) \sum_{b=1}^{p-1} e\left(\frac{mb - n\bar{b}\,\bar{a}\,(a-1)^2}{p} \right)$$

$$= p + \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} e\left(\frac{mb - n\bar{b}\,\bar{a}\,(a-1)^2}{p} \right),$$

where we have used the trigonometric identity

$$\sum_{a=1}^{q} e\left(\frac{sa}{q} \right) = \begin{cases} q, & \text{if } q \mid s; \\ 0, & \text{if } q \nmid s. \end{cases}$$

This completes the proof of Lemma 1.

Lemma 2 — Let p be an odd prime. Then for any non-principal character χ mod p and any integer n with $(n, p) = 1$, we have

$$\sum_{m=1}^{p} \left[\sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} e\left(\frac{mb - n\bar{b}\,\bar{a}\,(a-1)^2}{p} \right) \right]^2$$

$$= \begin{cases} p^2\,(p-3), & \text{if } \chi \text{ be a complex character mod } p; \\ 2p^2\,(p-2), & \text{if } \chi \text{ be the Legendre symbol.} \end{cases}$$

PROOF : From (2) and note that $-\overline{b} = -\overline{b}$ we have

$$\sum_{m=1}^{p}\left[\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1} e\left(\frac{mb - n\overline{b}\,\overline{a}\,(a-1)^2}{p}\right)\right]^2$$

$$= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\chi(ac)\sum_{b=1}^{p} e\left(\frac{m(b+d) - n(\overline{b}\,\overline{a}\,(a-1)^2 + \overline{d}\,\overline{c}\,(c-1)^2}{p}\right)$$

$$= p\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(ac)e\left(\frac{-n(\overline{b}\,\overline{a}\,(a-1)^2 + n\overline{b}\,\overline{c}\,(c-1)^2}{p}\right)$$

$$= p\sum_{a=1}^{p-1}\sum_{c=1}^{p-1}\sum_{b=1}^{p-1}\chi(ac)e\left(\frac{-n(\overline{b}\,(\overline{a}\,(a-1)^2 - \overline{c}\,(c-1)^2}{p}\right)$$

$$= p\sum_{a=1}^{p-1}\sum_{c=1}^{p-1}\sum_{b=1}^{p-1}\chi(ac)e\left(\frac{-b(\overline{a} + a - \overline{c} - c)}{p}\right)$$

$$= p\sum_{a=1}^{p-1}\sum_{c=1}^{p-1}\sum_{b=1}^{p-1}\chi(ac)e\left(\frac{-b(\overline{a} + a - \overline{c} - c)}{p}\right)$$

$$= p^2 \sum_{\substack{a=1 \\ p \mid (a+\overline{a}-\overline{c}-c)}}^{p-1}\sum_{c=1}^{p-1}\chi(ac)e$$

$$= p^2 \sum_{\substack{a=1 \\ p \mid (a-c)(1-\overline{ac})}}^{p-1}\sum_{c=1}^{p-1}\chi(ac)$$

$$= p^2\left[p - 3 + \sum_{a=1}^{p-1}\chi(a^2)\right]$$

$$= \begin{cases} p^2(p-3), & \text{if } \chi \text{ be a complex character mod } p; \\ 2p^2(p-2), & \text{if } \chi \text{ be the Legendre symbol.} \end{cases} \qquad \ldots (2)$$

This proves the Lemma 2.

3. PROOF OF THE THEOREM

From the Lemma 1 and Lemma 2 on the above section, we can complete the proof of the theorem. In fact for any integer n with $(n, p) = 1$, if χ be a non-principal character mod p, then applying Lemma 1, Lemma 2 and (2) we have

$$\sum_{m=1}^{p} |S(m,n,\chi;p)|^4 = |S(0,n,\chi;p)|^4 + \sum_{m=1}^{p-1} |S(m,n,\chi;p)|^4$$

$$= |\tau(\chi)|^4 + \sum_{m=1}^{p-1}\left[p + \sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1} e\left(\frac{mb - n\bar{b}\,\bar{a}\,(a-1)^2}{p}\right)\right]^2$$

$$= |\tau(\chi)|^4 + \sum_{m=1}^{p-1}\left[p + \sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1} e\left(\frac{mb - n\bar{b}\,\bar{a}\,(a-1)^2}{p}\right)\right]^2$$

$$-\left[p + \sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1} e\left(\frac{-n\bar{b}\,\bar{a}\,(a-1)^2}{p}\right)\right]^2$$

$$= |\tau(\chi)|^4 + p^3 + 2p\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\sum_{m=1}^{p} e\left(\frac{mb - n\bar{b}\,\bar{a}\,(a-1)^2}{p}\right)$$

$$+\sum_{m=1}^{p}\left[\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1} e\left(\frac{mb - n\bar{b}\,\bar{a}\,(a-1)^2}{p}\right)\right]^2 - \left[2p - \sum_{a=1}^{p-1}\chi(a)\right]^2$$

$$= |\tau(\chi)|^4 + p^3 + \sum_{m=1}^{p}\left[\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1} e\left(\frac{mb - n\bar{b}\,\bar{a}\,(a-1)^2}{p}\right)\right]^2 - 4p^2$$

$$= \begin{cases} 2p^2(p-3), & \text{if } \chi \text{ be a complex character mod } p; \\ p^2(3p-7), & \text{if } \chi \text{ be the Legendre symbol} \end{cases} \qquad \ldots (3)$$

where we have used the properties of the Gauss sums. That is,

$$|S(0,n,\chi;p)| = |\tau(\chi)| = \left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{a}{p}\right)\right| = \sqrt{p}.$$

Combining (1) and (3) we may immediately deduce

$$\sum_{m=1}^{p} |S(m,n,\chi;p)|^4$$

$$= \begin{cases} p(2p^2 - 3p - 3), & \text{if } \chi \text{ be the principal character mod } p; \\ p^2(3p - 7), & \text{if } \chi \text{ be the Legendre symbol;} \\ 2p^2(p - 3), & \text{if } \chi \text{ be a complex character mod } p. \end{cases}$$

This completes the proof the Theorem.

REFERENCES

1. T. Estermann, *Mathematica*, **8** (1961), 83-86.
2. S. Chowla, *Norkse Vid. Selbsk. Fak. Frondheim*, **40** (1967), 70-72.
3. A. V. Malyshev, *Vestnik Leningrad Univ.*, **15** (1960), 59-75.
4. H. Iwaniec, *Graduate Studies in Mathematics*, **17** (1997), 61-63.
5. H Salié, *Math. Z.*, **34** (1931), 91-109.
6. J. H. H. Chalk and R. A. Smith, *Acta Arith.*, **18** (1971), 191-212.
7. T. M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, New York, 1976.
8. Wolfang M. Schmidt, *Equations over finite fields*, Lecture Notes in Mathematics 536, Springer-Verlag, Berlin, 1976.