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AN EXISTENCE THEOREM FOR SOLVING THE SET-VALUED VARIATIONAL INCLUSIONS

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We prove an existence theorem of solution for the set-valued Variational inclusions. No continuousness assumption will be imposed on the multivalued m-accretive mapping.

Key Words: Variational Inclusions; m-Accretive Mapping; Resolvent Operator

We assume that E is a real Banach space with dual $E^*, \langle \cdot\cdot \rangle$ is the dual pair between E and E^*. The mapping $j: E \to 2^{E^*}$ defined by

$$J(x = \{f \in E^* \mid \langle f, x \rangle = \|f\| \cdot \|x\|, \|f\| = \|x\|\}$$

is called the normalized duality map. Let $T: D(T) \subset E \to 2^E$ be a set-valued mapping and $\varphi: [0, \infty) \to [0, \infty)$ be a strictly increasing function with $\varphi(0) = 0$. The mapping T is said to be accretive if, for any $x, y \in D(T)$, there exists $j(x\ y) \in J(x - y)$ such that

$$\langle u - v, j(x-y) \rangle \geq \varphi(\,||x-y||\,)\,||x-y||\,.$$

The mapping T is said to be m-accretive if T is accretive and $(I + \lambda T)\, D(T) = E$ for all $\lambda > 0$, where I is the identity mapping.

For an m-accretive mapping T, the "resolvents" $J_\lambda : E \to E$ of T are defined by $J_\lambda = (I + \lambda T)^{-1}$ for all $\lambda \in (0, \infty)$. The "yosida approximations" $T_\lambda : E \to E$ of T are defined by $T_\lambda = (1/\lambda)(I - J_\lambda)$. We defined $|Tx|$ by $|Tx| = \inf\{\|y\| : y \in Tx\}$. Some of the main properties of J_λ and T_λ are given below:

(1) $\|J_\lambda x - J_\lambda y\| \leq \|x - y\|$ for all $x, y \in E$;

(2) $\|T_\lambda x\| \leq |Tx|$ for all $x \in D(T)$;

(3) T_λ is m-accretive on E and $\|T_\lambda x - T_\lambda y\| \leq (2/\lambda)\|x - y\|$ for all $\lambda > 0$, $x, y \in E$;

(4) $T_\lambda x \in TJ_\lambda$ for all $x \in D(T)$.

Let X and Y be two topological spaces. A set-valued mapping $T: X \to 2^Y$ is said to lower semicontinuous (l.s.c.) at $x_0 \in D(T)$ if, for any open subset $U \subset Y$, such that $U \cap Tx_0 \neq \phi$, there exists a nbeighbourhood $V(x_0)$ such that for any $x \in V(x_0)$, $U \cap Tx \neq \phi$ Chang[1], [p. 451]. T is said to be upper semicontinuous (u.s.c.) at $x_0 \in D(T)$, if, for every open set U in Y with $Tx \subset U$, there exists a neighbourhood $V(x_0)$ of x_0 such that $T(V(x_0)) \subset U$. T is said to be continuous at $x_0 \in D(T)$ if it is both u.s.c. and l.s.c. at x_0.

Recently, Chang[1] introduced and studied the following class of set-valued variational inclusion problem.

Let, $T, F: X \to 2^E$ be two set-valued mappings, $A: D(T) \to 2^E$ an m-accretive mapping, $g: E \to D(A)$ a single-valued mapping and $N(.,.): E \times E \to E$ a nonlinear mapping. For any given $f \in E$ and $\rho > 0$, we consider the following problem: Find $q \in E, w \in T(q)$, $q \in T(q)$, such that

$$f \in N(w, v) + \rho A(g(q)). \qquad \text{... (1)}$$

Some special cases:

(1) If $E = H$ is a Hilbert space and $A: D(A) = H \to H$ is maximal monotone mapping, the problem (1) is equivalent to finding $q \in H, w \in T(q), v \in T(q)$ such that

$$f \in N(w, v) + \rho A(g(q)) \qquad \text{... (2)}$$

This problem was introduced and studied in Noor[5] and Noor *et al.*[6] under some additional conditions.

(2) If $E = H$ is a Hilbert space, $\rho = 1$ and $A = \partial \varphi$ the subdifferential of a proper convex lower semi-continuous functional $\varphi: H \to R \cup \{+\infty\}$, then the problem (1) is equivalent to finding $g \in H, w \in T(q), v \in F(q)$ such that

$$\langle N(w, v) - f, x - g(q) \rangle \geq \varphi(g(g)) - \varphi(x) \text{ for all } x \in H. \qquad \text{... (3)}$$

This problem is called the generalized set-valued mixed variational inequality, which was introduced and studied by Noor *et al.*[6].

(3) If H is a Hilbert space, $T, F, M: H \to 2^H$ are three set-valued mappings, and $M, S, g: H \to H$ are three single-valued mappings, $K(z) = M(z) + K$ where K is closed subset of $H, N(x, y) = Sx + Gy$ and

$$\varphi(x) = I_{K(z)}(x) = \begin{cases} 0 & \text{if } x \in K(z) \\ \infty & \text{if } x \notin K(z), \end{cases}$$

then the problem (3) is equivalent to finding $q \in H, w \in T(q), v \in F(q), z \in M(q)$ such that

$$g(q) \in K(q), \langle Sw + Gv - f, x - g(q) \geq 0, x \in K(z). \qquad \text{... (4)}$$

This problem is called generalized strongly nonlinear implicit quasi-variational inequality, which was studied in Huang[2].

In this paper we will use the concept of the resolvent operator technique to prove an existence theorem for solving the set-valued variational inclusions. No continuousness assumption will be imposed on the multivalued m-accretive operator. Our result improve and modify the Theorem 3.1 of Chang[1].

In the sequel shall denote the single-valued generalized quality mapping by j; $\rightarrow$ and $\xrightarrow{w}$ will denote strong and weak convergence, respectively.

Theorem 1 — *Let E be a real Banach space with uniformly convex, dual $E^*, T, F: \rightarrow 2^E$ and $A: D(A) \rightarrow 2^E$ three set-valued mappings, $g: E \rightarrow D(A)$ a single-valued mapping and $N(.,.): E \times E \rightarrow E$ a single-valued mapping satisfying the following conditions*:

(*i*) $A \circ g\, E \rightarrow 2^E$ is m-accretive;

(*ii*) The mapping $x \rightarrow N(x, y)$ is φ-strongly accretive, the mapping $y \rightarrow N(x, y)$ is accretive, where function φ is strictly increasing and continuous with $\varphi(0) = 0$;

(*iii*) $Sx = N(Tx, Nx)$ is m-accretive.

Fix $f \in E$ and $\rho > 0$. Then for any $\lambda > 0$, there exists a unique $x_\lambda \in E$ such that

$$f \in S_\lambda x_\lambda + \rho (A \circ g)(x_\lambda) \qquad \dots (5)$$

If $\{S_\lambda x_\lambda\}$ is bounded when $\lambda \rightarrow 0$, then there exist $x \in E, w \in T(x), v \in F(x)$ which is a solution of the set-valued variational inclusion (1). Hence, $S_\lambda = (1/\lambda)(I - J_\lambda)$, $J_\lambda = (I + \lambda S)^{-1}$.

PROOF : *Step* 1 — For any $\lambda > 0$, eq. (5) has unique solution x_λ. It follows from condition (*ii*) and Chang[1] [Lemma 2.2] that the mapping S is φ-strongly accretive. Since S is m-accretive, for any $u, v \in E$, there exist $w_1 \in Sv, w_2 \in Sv$ and $x, y \in E$ such that $u = J_\lambda x, v = J_\lambda y$ and $x = u + \lambda w_1 \in (I + \lambda S)u$, $y = v + \lambda w_2 \in (I + \lambda S)v$. By φ-strong accretiveness of S, we have

$$\langle x - y, j(u - v) \rangle = \langle u - v + \lambda (w_1 - w_2), j(u - v) \rangle$$

$$= \| u - v \|^2 + \lambda \langle w_1 - w_2, j(u - v) \rangle \geq \| u - v \|^2 + \lambda\, \varphi(\| u - v \|)\, \| u - v \|.$$

Therefore

$$\| u - v \| + \lambda\, \varphi(\| u - v \|) \leq \| x - y \|.$$

Let $\varphi_o(t) = t + \lambda\, \varphi(t)$. Then $\varphi_o(t)$ is a strictly increasing function with $\varphi_o(0) = 0$. Thus, $\| u - v \| \leq \varphi_o^{-1}(\| x - y \|)$, i.e.

$$\left|\left| J_\lambda x - J_\lambda y \right|\right| \leq \varphi_o^{-1}(\| x - y \|) \qquad \dots (6)$$

For any $x, y \in E$, by (6),

$$\langle S_\lambda x - S_\lambda y, j(x-y)\rangle \geq (1/\lambda)\langle x - j_\lambda x - (y - j_\lambda y), j(x-y)\rangle$$

$$= (1/\lambda)(\|x-y\|^2 - \langle j_\lambda x - j_\lambda y, j(x-y)\rangle)$$

$$\geq (1/\lambda)(\|x-y\|^2 - \|j_\lambda x - j_\lambda y\| \cdot \|x-y\|)$$

$$\geq (1/\lambda)(\|x-y\| - \varphi_\circ^{-1}\|x-y\|)\|x-y\|$$

$$= \varphi_1(\|x-y\|)\|x-y\|$$

where $\varphi_1(t) = (1/\lambda)(t - \varphi_\circ^{-1}(t))$. Hence, the S_λ is φ_1-strongly accretive. We may show that φ_1 is strictly increasing with $\varphi_1(0) = 0$ (see, Remark 1 (1)). By Kobayashi[3], $S_\lambda + \varphi(A \circ g))$ is m-accretive and φ_1-strongly accretive. Therefore, it is also m-accretive and φ_1-expansive mapping. By Kobayashi[3] (see, 1. Lemma 2.6]) $S_\lambda + \varphi(A \circ g) : E \to 2^E$ is surjective. Thus, by strong accretiveness of S_λ, for any $\lambda > 0$, there exists a unique $x_\lambda \in E$ such that (5) holds.

Step 2 : There exists $x \in D(A)$ such that $x_\lambda \to x \in E$ (as $\lambda \to 0$).

Since $f \in S_\lambda x_\lambda + \rho(A \circ g)x_\lambda$, $f \in S_\mu x_\mu + \rho(A \circ g)x_\mu$ for any $\lambda, \mu > 0$ and $\rho(A \circ g)$ is accretive, we have

$$\langle -S_\lambda x_\lambda + S_\mu x_\mu, j(x_\lambda - x_\mu)\rangle \geq 0.$$

Therefore, by $S_\lambda x_\lambda \in SJ_\lambda x_\lambda$ and ρ-strong accretiveness of S,

$$0 \leq -\langle S_\lambda x_\lambda + S_\mu x_\mu, j(x_\lambda - x_\mu)\rangle$$

$$= -\langle S_\lambda x_\lambda + S_\mu x_\mu, j(x_\lambda - x_\mu)\rangle + \langle S_\lambda x_\lambda + S_\mu x_\mu, j(J_\lambda x_\lambda - J_\mu x_\mu)\rangle$$

$$-\langle S_\lambda x_\lambda + S_\mu x_\mu, j(J_\lambda x_\lambda - J_\mu x_\mu)\rangle$$

$$\leq \langle S_\lambda x_\lambda + S_\mu x_\mu, j(J_\lambda x_\lambda - J_\mu x_\mu) - j(x_\lambda - x_\mu)\rangle$$

$$> -\varphi(\|J_\lambda x_\lambda - J_\mu x_\mu\|)\|J_\lambda x_\lambda - J_\mu x_\mu\|. \quad \ldots (7)$$

Since $J_\lambda x - x = \lambda S_\lambda x$, $\{S_\lambda x_\lambda\}$ is bounded when $\lambda \to 0$, and the j is uniformly continuous in bounded set (7) reduces to

$$\varphi(\|J_\lambda x_\lambda - J_\mu x_\mu\|) \leq O(\|x_\lambda x_\mu - J_\lambda x_\lambda + J_\mu x_\mu\|) = O(\lambda + \mu)$$

So, $J_\lambda x_\lambda \to x$ (as $\lambda \to 0$) for some $x \in E$, since φ is strictly increasing and continuous with $\varphi(0) = 0$. By $x_\lambda - J_\lambda x_\lambda = \lambda S_\lambda x_\lambda$ and assumption of Theorem 1, $x_\lambda \to x$ (as $\lambda \to 0$).

Step 3 : The x above is the unique solution of (1).

Since E is reflective and $\{S_\lambda x_\lambda\}$ is bounded, there exist $\lambda_n > 0$ (n = 1, 2, ...) such that $\lambda_n \to 0$ and $x_{\lambda_n} x_{\lambda_n} \xrightarrow{w} z$ for some $z \in E$. Let $w_{\lambda_n} = f - S_{\lambda_n} x_{\lambda_n} \in (A^\circ g)(x_{\lambda_n})$. Then $w_{\lambda_n} \xrightarrow{w} w$ for some $w \in E$. Since $(A \circ g)$ is demiclosed, $x \in D(A), x \in E = D(S)$, $z \in Sx$ and $w' \in \rho(A \circ g)x$. So, by $z \in Sx = N(Tx, Fx)$, there exists $w \in Tx, v \in Fx$ such that $z = N(w, v)$.

Consequently,

$$f \in N(W, v) + \rho w' \in N(Tx, Fx) + \rho (A \circ g)\ (x).$$

This completes the proof.

Corollary 1 — Suppose that $E, T, F, A, g, N, S, \rho, \varphi$ and f are the same as in Theorem 1. If, for any $r > 0$, there exist $L < 1$ ($L > 0$) and $M > 0$ such that

$$| Sx | \leq L | (A \circ g) x | + M \text{ for all } x \in D(A), \| x \| \leq r. \qquad ... (8)$$

Then the set-valued variational inclusion (1) has a solution.

PROOF : It suffices to show that the $\{S_\lambda x_\lambda\}$ $(\lambda \to 0)$ in Theorem 1 is bounded. Since $x_\lambda \to x$ when $\lambda \to 0$, $\{x_\lambda\}$ is bounded. Therefore, by (8), $| S_\lambda x_\lambda | \leq L | (A \circ g) x_\lambda | + M$. Using $| (A \circ g) x_\lambda | \leq \| f - S_\lambda x_\lambda \| \leq \| f \| + \| S_\lambda x_\lambda \|$. Therefore, $\| S_\lambda x_\lambda \| \leq L \| f \| + L \| S_\lambda x_\lambda \| + M$. So, $\| S_\lambda x_\lambda \| \leq \dfrac{L \| f \| + M}{1 - L}$. This proves that $\{S_\lambda x_\lambda\}$ $(\lambda \to 0)$ is bounded. This completes the proof.

Remark 1 : (1) The $\varphi_1(t)$ in the proof of Theorem 1 is strictly increasing with $\varphi_1(0) = 0$. Indeed, clearly $\varphi_0^{-1}(t)$ is strictly increasing and $\varphi_0^{-1}(0) = 0$. Let $r = \varphi_1(t) = (1/\lambda)\ (t - \varphi_0^{-1}(t))$. Then $t = \varphi_0(t - \lambda_r) = t - \lambda_r + \varphi(t - \lambda_r)$, and hence $r = (1/\lambda)\, \varphi(t - \lambda_r) = (1/\lambda)\, \varphi(t - \lambda(1/\lambda)(t - \varphi_0^{-1}(t))) = (1/\lambda(\varphi_0^{-1}(t))$. By the properties of φ and φ_0^{-1}, $r = \varphi_1(t)$ is strictly increasing and $\varphi_1(0) = 0$.

(2) It is well known that the accretive mapping which is lower semicontinuous cannot be multivalued (see, 4, 5]). Hence, the following proposition show that our result improve and modify the Theorem 3.1 of [11].

Proposition 1 — Let E be a real Banach space and $T : D(T) \subset E \to 2^E$ be a accretive mapping. If T is lower semicontinuous and int $D(T) \neq \phi$, then T cannot be multivalued in int $D(T)$.

PROOF : We show that the mapping T is single-valued in int $D(T)$, Suppose is multivalued. Then exists $x_0 \in$ int $D(T)$, and $u, v \in Tx_0$ such that $u \neq v$. Since T is l.s.c. at x_0, for $V = Tx_0 + \{x : \| x \| < \varepsilon\}$, where $\varepsilon = \dfrac{\| u - v \|}{2}$, can take $\delta \in \left(0, \dfrac{\varepsilon}{2}\right)$, so that for any $x \in B(x_0, \delta) \subset$ int $D(T)$, we have $Tx_0 \cap V \neq \phi$. So, there exists $z_x \in Tx$ such that

$$\| z_x - u \| < \varepsilon. \quad \ldots (9)$$

Since T is accretive, then there exists $j_1', j_2' \in J(x - x_0)$ such that

$$\langle z_x - u, j_1' (x - x_0) \rangle \geq 0, \quad \ldots (10)$$

$$\langle z_x - u, j_2' (x - x_0) \rangle \geq 0. \quad \ldots (11)$$

We take $\delta_1 \in (0, \delta)$ and $x = \delta_1 \dfrac{v-u}{\| u - v \|} + x_0 \in B(x_0, \delta)$. Then we can choose $z_x \in Tx$ as in inequality (9) (10) and (11). We note that

$$j_1 = \frac{\| v - u \|}{\delta} j_1' \in J(v-u), j_2 = \frac{\| v - u \|}{\delta} j_2' \in J(v-u).$$

Therefore

$$\langle z_x - u, j_1 \rangle \geq 0, \quad \ldots (12)$$

$$\langle z_x - u, j_2 \rangle \geq 0, \quad \ldots (13)$$

By (13), we have

$$0 \leq \langle z_x - v, j_1 \rangle = \langle z_x - v, j_2 \rangle + \langle v - u, j_2 \rangle - \langle v - u, j_2 \rangle$$

$$= \langle z_x - u, j_2 \rangle - \langle v - u, j_2 \rangle,$$

$$\langle z_x - u, j_2 \rangle \geq \langle v - u, j_2 \rangle. \quad \ldots (14)$$

(12) added to (14), we get

$$\langle z_x - u, j_1 + j_2 \rangle \geq \langle v - u, j_2 \rangle = \| v - u \|^2.$$

And hence $\| z_x - u \| \geq \varepsilon$. Thus we obtain a contradiction to (9). This completes the Proof.

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