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SPECTRAL PROPERTIES OF A FINITE SYSTEM OF STURM-LIOUVILLE DIFFERENTIAL OPERATORS

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In this article, we investigate the eigenvalues and the spectral singularities of the operator $\mathcal{L}$ generated by the finite system of Sturm-Liouville differential expressions in the space $L^2(\mathbb{R}_+, \mathbb{C}^N)$.

Key Words : Spectral Theory; Eigenvalues; Spectral Singularities

1. INTRODUCTION

The spectral analysis of non-selfadjoint abstract operators with purely discrete spectrum has been considered by Keldysh[12,13]. He studied the spectrum and principal functions of operators involving a polynomial dependent on the spectral parameter, and also showed the completeness of the principal functions of these operators.

Study of the spectral analysis of the non-selfadjoint Sturm-Liouville operators (SLO) with continuous and discrete spectrum was begun by Naimark[15]. He proved that the spectrum of the non-selfadjoint SLO consists of the continuous spectrum, the eigenvalues, and the spectral singularities. The spectral singularities are poles of the kernel of the resolvent and are also imbedded in the continuous spectrum, but they are not eigenvalues. In[14] the effect of the spectral singularities in the spectral expansion of SLO in terms of the principal vectors was considered. The spectral analysis of the quadratic pencil of Schödinger and Klein-Gordon operators with spectral singularities was studied in [3]-[5], [8], [10]. The spectral singularities of the discrete Schrödinger, Dirac and infinite Jacobi matrices have been investigated in detail in [1], [6], [7], [9].

Let $L^2(\mathbb{R}_+, \mathbb{C}^N)$ denote the Hilbert space of all complex vector functions

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_N \end{pmatrix},$$

with the norm

$$\|f\|^2 := \int_0^\infty \sum_{n=1}^{N} \left| f_n(x) \right|^2 dx.$$

We will consider the finite system of Sturm-Liouville differential expressions

$$l_n(y_n) := -y_n'' + q_n(x)\, y_n,\ n = 1, 2, \ldots, N,\ x \in \mathbb{R}_+ := [0, \infty),$$

where q_n are complex valued functions. Let $\mathcal{L}$ denote the operator generated in $L^2(\mathbb{R}_+, \mathbb{C}^N)$ by

$$l(y) := \begin{pmatrix} l_1(y_1) \\ l_2(y_2) \\ \\ \cdot \\ \cdot \\ l_N(y_N) \end{pmatrix},$$

and the boundary condition

$$y(0) = 0,$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_N \end{pmatrix}.$$

Since q_n, $n = 1, 2, \ldots, N$, are complex valued functions, so the operator $\mathcal{L}$ is non-selfadjoint.

In this paper, we investigate the eigenvalues and the spectral singularities of $\mathcal{L}$. In particular, we prove that $\mathcal{L}$ has a finite number of eigenvalues and spectral singularities with finite multiplicities, under the conditions

$$\sup_{x \in \mathbb{R}_+} \left\{ \exp(\varepsilon \sqrt{x})\, |q_n(x)| \right\} < \infty,\ n = 1, 2, \ldots, N,\ \varepsilon > 0.$$

2. Special Solutions of $l(y) = \rho^2 y$

Related with the operator $\mathcal{L}$ we will consider the equation

$$-y'' + Q(x)y = \rho^2 y,\ x \in \mathbb{R}_+, \qquad \ldots (2.1)$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_N \end{pmatrix}, \qquad Q(x) = \begin{bmatrix} q_1(x), & 0, & \ldots & 0 \\ 0, & q_2(x), & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \ldots & q_N(x) \end{bmatrix},$$

and ρ is a spectral parameter. Suppose that the functions q_n, $n = 1, 2, \ldots, N$, satisfy

$$\int_0^\infty x\,|q_n(x)|\,dx < \infty, \; n = 1, 2, ..., N. \qquad ... (2.2)$$

We will denote by $E(x, \rho)$, the matrix solution of (2.1) satisfying the condition

$$\lim_{x \to \infty} E(x, \rho)\, e^{-i\rho x} = I,$$

for all $\rho \in \overline{\mathbb{C}}_+ := \{\rho, \rho \in \mathbb{C}, \operatorname{Im} \rho \geq 0\}$, where I is a identity matrix in $\mathbb{C}^{\mathbb{N}}$.

The following result has been obtained in[2]: Under the condition (2.2) the solution $E(x, \rho)$ has the representation

$$E(x, \rho) = \begin{bmatrix} e_1(x, \rho), & 0, & \dots & 0 \\ 0, & e_2(x, \rho), & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0, & 0, & \dots & e_N(x, \rho) \end{bmatrix}, \qquad ... (2.3)$$

where

$$e_n(x, \rho) = e^{i\rho x} + \int_x^\infty K_n(x, t)\, e^{i\rho t}\, dt, \; n = 1, 2, ..., N.$$

The functions $K_n(x, t), n = 1, 3, ..., N$, satisfy

$$\left| K_n(x, t) \right| \leq C \int_{\frac{x+t}{2}}^\infty |q_n(u)|\, du, \qquad n = 1, 2, ..., N,$$

where $C > 0$ is a constant. Therefore the functions $e_n(x, \rho), n = 1, 2, ..., N$ are analytic in $\mathbb{C}_+ := \{\rho, \rho \in \mathbb{C}, \operatorname{Im} \rho > 0\}$ with respect to ρ and continuous in $\overline{\mathbb{C}}_+$.

3. Eigenvalues and Spectral Singularities of $\mathcal{L}$

Let us define the sets

$$A_n^1 = \{\rho, \rho \in \mathbb{C}_+, e_n(\rho) = 0\}, \qquad ... (3.1)$$

$$A_n^2 = \{\rho, \rho \in \mathbb{R}, e_n(\rho) = 0\}, \qquad ... (3.2)$$

where $\mathbb{R} = (-\infty, \infty)$ and

$$e_n(\rho) := e_n(0, \rho) = 1 + \int_0^\infty K_n(0, t)\, e^{i\rho t}\, dt, \qquad ... (3.3)$$

Lemma 3.1 —

$$\sigma_d(\mathcal{L}) = \bigcup_{n=1}^{N} B_n^1, \qquad \text{... (3.4)}$$

where $\sigma_d(\mathcal{L})$ denote the set of all eigenvalues of $\mathcal{L}$ and

$$B_n^1 = \left\{\lambda : \lambda = \rho^2, \rho \in A_n^1\right\}.$$

PROOF : By the definition of the eigenvalues of $\mathcal{L}$ we obtain that

$$\sigma_d(\mathcal{L}) = \left\{\lambda, \lambda = \rho^2, \rho \in \mathbb{C}_+ , \det E(\rho) = 0\right\}, \qquad \text{... (3.5)}$$

where

$$E(\rho) = E(0, \rho).$$

It follows from (2.3) that

$$\det E(\rho) = \prod_{n=1}^{N} e_n(\rho). \qquad \text{... (3.6)}$$

(3.4) is the direct consequence of (3.1), (3.2), (3.5) and (3.6). ■

Let $S(x, \rho)$ denote the matrix solution of (2.1) satisfying the initial conditions

$$S(0, \rho) = 0, \; S'(0, \rho) = I.$$

It is obvious that,

$$S(x, \rho) = \begin{bmatrix} S_1(x, \rho), & 0, & \dots & 0 \\ 0, & S_2(x, \rho), & \dots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \dots & S_N(x, \rho) \end{bmatrix},$$

where $S_n(x, \rho)$ denote the solution of the equation

$$-y_n'' + q_n(x)\, y_n = \rho^2 y_n, \; x \in \mathbb{R}_+,$$

satisfying the initial conditions

$$S_n(0, \rho) = 0, \; S_n'(0, \rho) = 1.$$

For all $\rho \in \mathbb{C}_+$ and $\det E(\rho) \neq 0$, we define

$$G(x, t; \rho) = \begin{cases} S(t, \rho)\, E(x, \rho)\, E^{-1}(\rho), & 0 \le t < x, \\ S(t, \rho)\, E(x, \rho)\, E^{-1}(\rho), & x \le t < \infty. \end{cases} \quad \text{... (3.7)}$$

Note that

$$\left(R_\rho(\mathcal{L})f)(x) := \int_0^\infty G(x, t; \rho) f(t)\, dt,\ f \in L^2(\mathbb{R}_+, \mathbb{C}^{\mathbb{N}} \right), \quad \text{... (3.8)}$$

is the resolvent of $\mathcal{L}$.

Using (3.4), (3.7) and (3.8) we get the following:

Corollary 3.2 —

$$\sigma_{ss}(\mathcal{L}) = \bigcup_{n=1}^{N} B_n^2 \setminus \{0\}, \quad \text{... (3.9)}$$

where $\sigma_{ss}(\mathcal{L})$ denote the set of all spectral singularities of $\mathcal{L}$ and

$$B_n^2 = \left\{ \lambda : \lambda = \rho^2, \rho \in A_n^2 \right\}. \quad \text{... (3.10)}$$

Definition 3.3 — The multiplicity of a zero of $e_n (n = 1, 2, ..., N)$ in $\mathbb{C}_+$ is defined as the multiplicity of the corresponding eigenvalue or spectral singularity of $\mathcal{L}$.

It follows from (3.1), (3.2), (3.4) and (3.9) that, in order to investigate the quantitative properties of the eigenvalues and the spectral singularities of $\mathcal{L}$, we need to discuss the quantitative properties of the zeros of e_n, $n = 1, 2, ..., N$, in $\mathbb{C}_+$.

Lemma 3.4 — Under the condition (2.2),

(*i*) The set A_n^1 is bounded and has at most a countable number of elements and its limit points can lie only in a bounded subinterval of the real axis.

(*ii*) The set A_n^2 is compact and $\mu(A_n^2) = 0$, where $\mu(A_n^2)$ denote the Lebesgue measure of A_n^2.

PROOF : We obtain that

$$e_n(\rho) = 1 + o(1),\ \rho \in \mathbb{C}_+,\ |\rho| \to \infty,$$

by (2.4) and (3.3). Now (3.10) shows the boundedness of the sets A_n^1 and A_n^2. Since e_n is analytic in $\mathbb{C}_+$, we get that the set A_n^1 has at most a countable number of elements. By the uniqueness of analytic functions we find that the limit points of A_n^1 can lie only in a bounded subinterval of the real axis. The closedness and the property of having zero Lebesgue measure of the set A_n^2 can be obtained from the boundary uniqueness theorem of analytic functions[11]. ■

By (3.4), (3.9) and Lemma 3.4 we have the following:

Theorem 3.5 — *Under the condition (2.2),*

(i) The set of eigenvalues of $\mathcal{L}$ is bounded and countable, and its limit points can lie only in a bounded subinterval of $\mathbb{R}_+$

(ii) The set of spectral singularities of $\mathcal{L}$ is bounded and $\mu(\sigma_{ss}(L)) = 0$.

Now we will assume that

$$\sup_{x \in \mathbb{R}_+} \left\{ \exp(\varepsilon \sqrt{x}) \, | q_n(x) | \right\} < \infty, \varepsilon > 0, n = 1, 2, ..., N, \qquad \text{... (3.11)}$$

holds. It follows from (2.4) and (3.3) that, under the condition (3.11) the functions e_n, ($n = 1$, 2, ..., N) are analytic in $\mathbb{C}_+$, and all of its derivatives are continuous in $\overline{\mathbb{C}}_+$. So

$$\left| \frac{d^r}{d\rho^r} e_n(\rho) \right| \leq M_r, \lambda \in \overline{\mathbb{C}}_+, n = 1, 2, ..., N, r = 0, 1, ...,$$

where

$$M_r = 2^r C \int_0^\infty t^r \exp\left(-\frac{\varepsilon}{2} \sqrt{t} \right) dt, \quad r = 0, 1, ..., \qquad \text{... (3.12)}$$

and $C > 0$ is a constant.

Let us denote the set of all limit points of A_n^1 by A_n^3, and the set of all zeros of e_n with infinity multiplicity in $\overline{\mathbb{C}}_+$ by A_n^4.

By the uniqueness of analytic functions we get

$$A_n^3 \subset A_n^2, A_n^4 \subset A_n^2, \ \mu(A_n^4) = 0.$$

Using of the continuity of all derivatives of e_n on the real axis, we obtain

$$A_n^3 \subset A_n^4. \qquad \text{... (3.13)}$$

Lemma 3.6 — If (3.11) holds, then $A_n^4 = \phi$.

PROOF : Using the uniqueness theorem for the analytic functions given in[3] we have

$$\int_0^h lnT(s)\, d\mu(A_{n,s}^4) > -\infty, \qquad \ldots (3.14)$$

where $h > 0$ is a constant, $T(s) = \inf_r \frac{M_r s^r}{r!}$, the constant M_r is defined by (3.12) and $\mu(A_{n,s}^4)$ is the Lebesgue measure of s-neighbourhood of A_n^4.

It is easy to verify that

$$M_r \le Bb^r r^r r!, \qquad \ldots (3.15)$$

where B and b are constants depending of C and ε. From (3.15) we find that

$$T(s) = \inf_r \frac{M_r s^r}{r!} \le B \inf_r \{b^r s^r r^r\} \le B \exp\{-b^{-1} e^{-1} s^{-1}\},$$

or by (3.14)

$$\int_0^h \frac{1}{s}\, d\mu\,(A_{n,s}^4) < \infty. \qquad \ldots (3.16)$$

(3.16) holds for an arbitrary s, if and only if $\mu\,(A_{n,s}^4) = 0$ or $(A_n^4) = \phi$. ■

***Theorem* 3.7** — *Under the condition (3.11) the operator $\mathcal{L}$ has a finite number of eigenvalues and spectral singularities, and each them is of finite multiplicity.*

PROOF : To be able to prove the theorem we have to show that the functions $e_n, n = 1, 2, \ldots, N$ have a finite number of zeros with finite multiplicities in $\mathbb{C}_+$.

From Lemma 3.6 and (3.13) we get that $A_n^3 = \phi$. So the bounded sets A_n^1 and A_n^2 have no limit points (see Lemma 3.4), i.e., the functions $e_n, n = 1, 2, \ldots, N$ have only a finite number of zeros in $\mathbb{C}_+$. Since $A_n^4 = \phi$, these zeros are of finite multiplicity. ■

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