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RICCI CURVATURE OF SUBMANIFOLDS IN LOCALLY CONFORMAL ALMOST COSYMPLECTIC MANIFOLDS

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We obtain certain inequalities involving several intrinsic invariants namely scalar curvature, Ricci curvature and k-Ricci curvature, and main extrinsic invariant namely squared mean curvature for submanifolds in a locally conformal almost cosymplectic manifold with pointwise constant φ-sectional curvature. Applying these inequalities we obtain several inequalities for slant, invariant, anti-invariant and CR-submanifolds. The equality cases are also discussed.

Key Words: Locally Conformal Almost Cosymplectic Manifold; Invariant Submanifold; Anti-invariant Submanifold; Slant Submanifold, *CR*-Submanifold; Ricci Curvature; Squared Mean Curvature; Relative Null Space; Totally Umbilical Submanifold; Minimal Submanifold; Totally Geodesic Submanifold

1. INTRODUCTION

"To establish simple relationship between the main intrinsic invariants and the main extrinsic invariants of a submanifold" is one of the most fundamental problems in submanifold theory as recalled by Chen[4]. Scalar curvature and Ricci curvature are among the main intrinsic invariants, while the squared mean curvature is the main extrinsic invariant of a submanifold. For more details we refer to the book by Chen[5].

On the other hand, there is an interesting class of almost contact metric manifolds which are locally conformal to almost cosymplectic manifolds[6]. These manifolds are called locally conformal almost cosymplectic manifolds[9].

In this paper, we continue the study[11] of submanifolds tangent to the structure vector field ξ in locally conformal almost cosymplectic manifolds of pointwise constant φ-sectional curvature and obtain certain relations of intrinsic invariants, namely scalar, Ricci and k-Ricci curvatures, with the main extrinsic invariant, namely squared mean curvature for these submanifolds. The paper is organized as follows. Section 2 contains a brief introduction to locally conformal almost cosymplectic manifolds, while in section 3 some necessary details about different kind of submanifolds are presented. In section 4, we obtain inequalities with left hand side containing scalar curvature and right hand side containing squared mean curvature for slant, invariant, anti-invariant and CR-submanifolds in locally conformal almost cosymplectic manifolds if pointwise constant φ-sectional curvature. The equality cases hold if and only if these submanifolds are totally geodesic. In section 5, we obtain an inequality involving Ricci curvature and squared mean curvature along with discussion of equality cases. We then apply these inequalities to find several inequalities for slant, invariant, anti-invariant and CR-submanifolds. In the last section, we find a relationship between the

k-Ricci curvature and the squared mean curvature for slant, invariant, anti-invariant and CR-submanifolds.

2. Locally Conformal Almost Cosymplectic Manifolds

Let $\tilde{M}$ be a $(2m+1)$-dimensional almost contact manifold[2] endowed with an almost contact structure (φ, ξ, η) consisting of a (1, 1) tensor field φ, a vector field ξ, and a 1-form η satisfying $\varphi^2 = -I + \eta \otimes \xi$ and (one of) $\eta(\xi) = 1$, $\varphi\xi = 0$, $\eta \circ \varphi = 0$. The almost contact structure induces a natural almost complex structure J on the product manifold $\tilde{M} \times \mathbb{R}$ defined by $J(X, \lambda d/dt) = (\varphi X - \lambda\xi, \eta(X)d/dt)$, where X is tangent to $\tilde{M}$, t the coordinate of $\mathbb{R}$ and λ a smooth function on $\tilde{M} \times \mathbb{R}$. The almost contact structure is said to be normal[10] if the almost complex structure J is integrable. Let $\langle , \rangle$ be a compatible Riemannian metric with (φ, ξ, η), i.e., $\langle X, Y\rangle = \langle \varphi X, \varphi Y\rangle + \eta(X)\eta(Y)$ or equivalently, $\Phi(X, Y) \equiv \langle X, \varphi Y\rangle = -\langle \varphi X, Y\rangle$ along with $\langle X, \xi\rangle = \eta(X)$ for all $X, Y \in T\tilde{M}$. Then, $(\varphi, \xi, \eta\langle , \rangle)$ is an almost contact metric structure on $\tilde{M}$, and $\tilde{M}$ is an almost contact metric manifold.

If the fundamental 2-form Φ and the 1-form η are closed, then $\tilde{M}$ is said to be almost cosymplectic manifold[6]. A normal almost cosymplectic manifold is cosymplectic[2]. An almost contact metric structure is cosymplectic if and only if $\tilde{\nabla}\varphi = 0$, where $\tilde{\nabla}$ is the Levi-Civita connection of the Riemannian metric $\langle , \rangle$. An example of a manifold which has an almost cosymplectic structure which is not cosymplectic, can be found[8]. A conformal change of an almost contact metric structure is defined by $\varphi^* = \varphi$, $\xi^* = e^{-\rho}\xi$, $\eta^* = e^{\rho}\eta$, $\langle , \rangle^* = e^{2\rho}\langle , \rangle$, where ρ is a differentiable function. $\tilde{M}$ is said to be a locally conformal almost cosymplectic manifold if every point of $\tilde{M}$ has a neighbourhood $\mathcal{U}$ such that $(\mathcal{U}, \varphi^*, \xi^*, \eta^*, \langle , \rangle^*)$ is almost cosymplectic for some function ρ on $\mathcal{U}$. Equivalently, $\tilde{M}$ is locally conformal almost cosymplectic manifold if there exists a 1-form ω such that $d\Phi = 2\omega \wedge \Phi$, $d\eta = \omega \wedge \eta$ and $d\omega = 0$[9].

A plane section π in $T_p\tilde{M}$ of an almost contact metric manifold $\tilde{M}$ is called a φ-section if $\pi \perp \xi$ and $\varphi(\pi) = \pi$. $\tilde{M}$ is of pointwise constant φ-sectional curvature if at each point $p \in \tilde{M}$, the sectional curvature $\tilde{K}(\rho)$ does not depend on the choice of the φ-section π of $T_p\tilde{M}$, and in this case for $p \in \tilde{M}$, and for any φ-section ρ of $p \in \tilde{M}$, the function c defined by $c(p) = \tilde{K}(\pi)$ is called the φ-sectional curvature of $\tilde{M}$. A locally conformal almost cosymplectic manifold $\tilde{M}$ of dimension ≥ 5 is of pointwise constant φ-sectional curvature if and only if its curvature tensor $\tilde{R}$ is of the form[9]

$$\tilde{R}(X, Y)Z = \frac{c-3f^2}{4}\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\}$$

$$+ \frac{c+f^2}{4}\{2\langle X, \varphi Y\rangle \varphi Z + \langle X, \varphi Z\rangle \varphi Y - \langle Y, \varphi Z\rangle \varphi X\}$$

$$+\left(\frac{c+f^2}{4}+f'\right)\{\eta(X)\,\eta(Z)\,Y-\eta(Y)\,\eta(Z)X$$

$$+\langle X,Z\rangle\,\eta(Y)\,\xi-\langle Y,Z\rangle\,\eta(X)\,\xi\} \quad \ldots (1)$$

for all $X, Y, X \in T\tilde{M}$, where f is the function such that $\omega = f\eta, f' = \xi f$; and c is the pointwise φ-sectional curvature of $\tilde{M}$.

3. SUBMANIFOLDS

Let M be an n-dimensional Riemannian manifold. The scalar curvature τ at p is given by $\tau = \sum_{i<j} K_{ij}$, where K_{ij} is the sectional curvature of the plane section spanned by e_i and e_j at $p \in M$ for any orthonormal basis $\{e_1, \ldots, e_n\}$ for T_pM. Now, if M is immersed in an m-dimensional Riemannian manifold $(\tilde{M}, \langle\,,\rangle)$, then Gauss and Weingarten formulas are given respectively by $\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$ and $\tilde{\nabla}_X N = -A_N + \overset{\perp}{\nabla}_X N$ for all $X, Y \in TM$ and $N \in T^{\perp} M$, where $\tilde{\nabla}$, ∇ and $\nabla^{\perp}$ are Riemannian, induced Riemannian and induced normal connections in $\tilde{M}, M$ and the normal bundle $T^{\perp} M$ of M respectively, and σ is the second fundamental form related to the shape operator A_N in the direction of M by $\langle \sigma(X, Y), N\rangle = \langle A_N X, Y\rangle$. Then, the Gauss equation is

$$\tilde{R}\,(X, Y, Z, W) = R(X, Y, Z, W) - \langle\,\sigma(X, W), \sigma(Y, Z)\,\rangle$$

$$+\langle\,\sigma(X, Z), \sigma(Y, W)\,\rangle \quad \ldots (2)$$

for all $X, Y, Z, W \in TM$, where $\tilde{R}$ and R are the curvature tensors of $\tilde{M}$ and M respectively. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM. The mean curvature vector H at $p \in M$ is expressed by

$$nH = \text{trace}\,(\sigma) = \sum_{i=1}^{n} \sigma(e_i, e_i). \quad \ldots (3)$$

The submanifold M is totally geodesic in $\tilde{M}$ if $\sigma = 0$, and minimal if $H = 0$. If $\sigma(X, Y) = \langle X, Y\rangle\, H$ for all $X, Y \in TM$, then M is totally umbilical. We put

$$\sigma_{ij}^{r} = g(\sigma(e_i, e_j), e_r) \text{ and } \|\sigma\|^2 = \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), \sigma(e_i, e_j)),$$

where e_r belongs to an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of the normal space $\overset{\perp}{T_p} M$.

Let $\mathcal{L}$ be a k-plane section of T_pM and X a unit vector in $\mathcal{L}$. We choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of $\mathcal{L}$ such that $e_1 = X$. The Ricci curvature $\mathrm{Ric}_{\mathcal{L}}$ of $\mathcal{L}$ at X is given by

$$\mathrm{Ric}_{\mathcal{L}}(X) = K_{12} + K_{13} + \ldots + K_{1k}, \quad \ldots (4)$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i, e_j. $\mathrm{Ric}_{\mathcal{L}}(X)$ is called a k-Ricci curvature. The scalar curvature τ of the k-plane section $\mathcal{L}$ is given by

$$\tau(\mathcal{L}) = \sum_{1 \le i < j \le k} K_{ij}. \qquad \dots (5)$$

For each integer k, $2 \le k \le n$, the Riemannian invariant θ_k on an n-dimensional Riemannian manifold M is defined by

$$\theta_k(p) = \left(\frac{1}{k-1}\right) \inf_{\mathcal{L}, X} \mathrm{Ric}_{\mathcal{L}}(X), \quad p \in M, \qquad \dots (6)$$

where $\mathcal{L}$ runs over all k-plane sections in T_pM and X runs over all unit vectors in $\mathcal{L}$.

Now, let M be an n-dimensional submanifold in an almost contact metric manifold. For a vector field X in M, we put

$$\varphi X = PX + FX, \qquad PX \in TM,\ FX \in T^{\perp} M.$$

Thus, P is an endomorphism of the tangent bundle of M and satisfies $\langle X, PY \rangle = -\langle PX, Y \rangle$ for all $X, Y \in TM$. We can define the squared norm of P by

$$\| P \|^2 = \sum_{i,j=1}^{n} \langle e_i, Pe_j \rangle^2$$

for any local orthonormal basis $\{e_1, e_2, \dots, e_n\}$ for T_pM. If the structure vector field ξ is tangential to M, then we write the orthogonal direct decomposition $TM = \varepsilon \oplus \varepsilon^{\perp}$, where ε is the distribution spanned by ξ.

A submanifold M of an almost contact metric manifold with $\xi \in TM$ is called a semi-invariant submanifold[1] or a contact CR-submanifold[12] if there exists two differentiable distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ on M such that (*i*) $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \varepsilon$, (*ii*) the distribution $\mathcal{D}$ is invariant by φ, i.e., $\varphi(\mathcal{D}) = \mathcal{D}$, and (*iii*) the distribution $\mathcal{D}^{\perp}$ is anti-invariant by φ, i.e., $\varphi(\mathcal{D}^{\perp}) \subseteq T^{\perp} M$.

The submanifold M tangent to ξ is said to be invariant or anti-invariant[12] according as $F = 0$ or $P = 0$. Thus, a CR-submanifold is invariant or anti-invariant according as $\mathcal{D} = \{0\}$. A proper CR-submanifold is neither invariant nor anti-invariant.

For each non zero vector $X \in T_pM$, such that X is not proportional to ξ_p, we denote the angle between φX and T_pM by $\theta(X)$. Then M is said to be slant[3,7] if the angle $\theta(X)$ is constant, i.e., it is independent of the choice of $p \in M$ and $X \in T_pM - \{\xi\}$. The angle θ of a slant immersion is called the slant angle of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \pi/2$ respectively. A proper slant immersion is neither invariant nor anti-invariant.

4. Scalar Curvature

Let M be an n-dimensional submanifold in a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c such that the structure vector field ξ is tangential to M. In view of (1) and (2), it follows that

$$\begin{aligned}\tilde{R}(X, Y, Z, W) &= \frac{c-3f^2}{4}\{\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle\}\\
&+ \frac{c+f^2}{4}\{\langle X, PW\rangle\langle Y, PZ\rangle - \langle X, PZ\rangle\langle Y, PW\rangle\\
&-2\langle X, PY\rangle\langle Z, PW\rangle\}\\
&+\left(\frac{c+f^2}{4}+f'\right)\{\eta(X)\,\eta(Z)\langle Y, W\rangle - \eta(Y)\,\eta(Z)\langle X, W\rangle\\
&+\langle X, Z\rangle\,\eta(Y)\,\eta(W) - \langle Y, Z\rangle\,\eta(X)\,\eta(W)\}\\
&+\langle\sigma(X, W), \sigma(Y, Z)\rangle - \langle\sigma(X, Z), \sigma(Y, W)\rangle \qquad \text{... (7)}\end{aligned}$$

for all $X, Y, Z, W \in TM$. Thus, the scalar curvature and the mean curvature of M at p satisfy[11]

$$\begin{aligned}n^2\|H\|^2 &= 2\tau + \|\sigma\|^2 - \frac{1}{4}n(n-1)(c-3f^2)\\
&-\frac{3}{4}\|P\|^2(c+f^2) + 2(n-1)\left(\frac{c+f^2}{4}+f'\right). \qquad \text{... (8)}\end{aligned}$$

Thus, we are able to state the following:

Theorem 4.1 — *For an n-dimensional submanifold M in a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c such that $\xi \in TM$, the following statements are true.*

1. *We have*

$$\begin{aligned}\tau &\le \frac{1}{2}n^2\|H\|^2 + \frac{1}{8}n(n-1)(c-3f^2)\\
&+\frac{1}{8}\{3\|P\|^2 - 2(n-1)\}(c+f^2) - (n-1)f'. \qquad \text{... (9)}\end{aligned}$$

2. *If M is a θ-slant submanifold, then*

$$\tau \le \frac{1}{2}n^2\|H\|^2 + \frac{n-1}{8}\{n(c-3f^2) + (3\cos^2\theta - 2)(c+f^2) - 8f'\}. \qquad \text{... (10)}$$

3. *If M is an invariant submanifold, then*

$$\tau \leq \frac{1}{2} n^2 \| H \|^2 + \frac{n-1}{8} \{(n+1)\, c - (3n-1) f^2 - 8f'\}. \quad \dots (11)$$

4. *If M is an anti-invariant submanifold, then*

$$\tau \leq \frac{1}{2} n^2 \| H \|^2 + \frac{n-1}{8} \{(n-2)\, c - (3n+2) f^2 - 8f'\}. \quad \dots (12)$$

5. *If M is a CR-submanifold, then*

$$\tau \leq \frac{1}{2} n^2 \| H \|^2 + \frac{1}{8} n(n-1)\,(c - 3f^2)$$

$$+ \frac{1}{8} \{6h - 2\,(n-1)\}\ (c + f^2) - (n-1) f', \quad \dots (13)$$

where $2h = \dim(\mathcal{D})$.

6. *The equality cases of (9), (10), (11), (12) and (13) hold if and only if M is totally geodesic.*

PROOF : It is easy to verify that an n-dimensional θ-slant submanifold M of an almost contact metric manifold satisfies

$$\langle PX, PY \rangle = \cos^2\theta \langle \varphi X, \varphi Y \rangle,\ \ \langle FX, FY \rangle = \sin^2\theta \langle \varphi X, \varphi Y \rangle \quad \dots (14)$$

for all $X, Y \in TM$. Consequently,

$$\| P \|^2 = (n-1) \cos^2\theta. \quad \dots (15)$$

If M is a CR-submanifold, then

$$\| P \|^2\, 2h = \dim(\mathcal{D}). \quad \dots (16)$$

Inequality (9) follows from (8). Using (15) in (9), we get (10). In (10), putting $\theta = 0$ and $\theta = \pi/2$ we get (11) and (12) respectively. Using (16) in (9), we get (13). The sixth statement is obvious in view of (8). □

5. RICCI CURVATURE

For an n-dimensional submanifold in an m-dimensional Riemannian manifold, let $\{e_1, \dots, e_n\}$ be a local orthonormal basis for T_pM and $\{e_{n+1}, \dots, e_m\}$ an orthonormal basis for the normal space T_pM. Then it is easy to verify that at each point $p \in M$ the squared second fundamental form and the squared mean curvature satisfy

$$\| \sigma \|^2 = \frac{1}{2} n^2 \| H \|^2 + \frac{1}{2} \sum_{r=n+1}^{m} \left(\sigma^r_{11} - \sigma^r_{22} - \dots - \sigma^r_{nn} \right)^2$$

$$+ 2 \sum_{r=n+1}^{n} \sum_{j=2}^{n} (\sigma_{1j}^{r})^2 - 2 \sum_{r=n+1}^{m} \sum_{2 \le i < j \le n} \left(\sigma_{ii}^{r} \sigma_{jj}^{r} - (\sigma_{ij}^{r})^2 \right). \quad \text{... (17)}$$

The relative null space of M at a point $p \in M$ is defined by

$$N_p = \left\{ X \in T_p M \mid \sigma(X, Y) = 0 \text{ for all } Y \in T_p M \right\}.$$

Chen[4] established a relationship between Ricci curvature and the squared mean curvature for a submanifold in a real space form as follows:

Theorem 5.1 — *Let M be an n-dimensional submanifold in a real space form $R^m(c)$. Then*

1. *For each unit vector $X \in T_pM$, we have*

$$\| H \|^2 \ge \frac{4}{n^2} \{\mathrm{Ric}(X) - (n-1)c\}. \quad \text{... (18)}$$

2. *If $H(p) = 0$, then a unit vector $X \in T_pM$ satisfies the equality case of (18) if and only if X lies in the relative null space N_p at p.*

3. *The equality case of (18) holds for all unit vectors $X \in T_pM$, if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

In this section, we find similar results for several kind of submanifolds in a locally conformal almost cosymplectic manifold.

Theorem 5.2 — *Let M be an n-dimensional submanifold in a $(2m + 1)$-dimensional locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c tangential to the structure vector field ξ. Then, the following statements are true*

(i) For each unit vector $X \in T_pM$, we have

$$\mathrm{Ric}(X) \le \frac{1}{4}\Big\{ n^2 \| H \|^2 + (n-1)(c - 3f^2)$$

$$+ (3 \| PX \|^2 - (n-2)\eta(X)^2 - 1)(c + f^2)\Big\}$$

$$- (1 + (n-2)\eta(X)^2) f'. \quad \text{... (19)}$$

(ii) For $H(p) = 0$, a unit vector $X \in T_pM$ satisfies the equality case of (19) if and only if X belongs to the relative null space N_p at p.

(iii) The equality case of (19) holds identically for all unit vectors $X \in T_pM$, if and only if either $n = 2$ and p is totally umbilical point or. p is a totally geodesic point.

PROOF : We choose an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{2m+1}\}$ such that $e_1, ..., e_n$ are tangential to M at p. From (7), we get

$$K_{ij} = \sum_{r=n+1}^{2m+1} \left(\sigma_{ii}^{r} \sigma_{jj}^{r} - (\sigma_{ij}^{r})^2 \right) + \frac{c-3f^2}{4} + \frac{3(c+f^2)}{4} \langle e_i, Pe_j \rangle^2$$

$$- \left(\frac{c+f^2}{4} + f' \right) (\eta(e_i)^2 + \eta(e_j)^2),$$

which gives

$$\sum_{2 \le i < j \le n} K_{ij} = \sum_{r=n+1}^{2m+1} \sum_{2 \le i < j \le n} \left(\sigma_{ii}^{r} \sigma_{jj}^{r} - (\sigma_{ij}^{r})^2 \right) + \frac{1}{8}(n-1)(n-2)(c-3f^2)$$

$$+ \frac{1}{8}(3 \| P \|^2 - 6 \| Pe_1 \|^2)(c+f^2)$$

$$- (n-2) \left(\frac{c+f^2}{4} + f' \right) (1 - \eta(e_1)^2). \qquad \ldots (20)$$

From (8) and (17), we get

$$\frac{1}{4} n^2 \| H \|^2 = \tau - \frac{1}{8} n(n-1)(c-3f^2) - \frac{1}{8}(3 \| P \|^2 - 2(n-1))(c+f^2)$$

$$+ (n-1) f' + \frac{1}{4} \sum_{r=n+1}^{2m+1} \left(\sigma_{11}^{r} - \sigma_{22}^{r} - \ldots - \sigma_{nn}^{r} \right)^2$$

$$+ \sum_{r=n+1}^{2m+1} \sum_{j=2}^{n} (\sigma_{1j}^{r})^2 - \sum_{r=n+1}^{2m+1} \sum_{2 \le j < \le n} \left(\sigma_{ii}^{r} \sigma_{jj}^{r} - (\sigma_{ij}^{r})^2 \right), \qquad \ldots (21)$$

which in view of (20) provides

$$\text{Ric}(e_1) = \frac{1}{4} \{ n^2 \| H \|^2 + (n-1)(c-3f^2)$$

$$+ (3 \| Pe_1 \|^2 - (n-2) \eta(e_1)^2 - 1)(c+f^2) \}$$

$$- (1 + (n-2) \eta(e_1)^2 f'$$

$$- \frac{1}{4} \sum_{r=n+1}^{2m+1} \left(\sigma_{11}^{r} - \sigma_{22}^{r} - \ldots - \sigma_{nn}^{r} \right)^2 - \sum_{r=n+1}^{2m+1} \sum_{j=2}^{n} (\sigma_{1j}^{r})^2. \qquad \ldots (22)$$

Since $e_1 = X$ can be chosen to be any arbitrary unit vector in T_pM, therefore the above equation implies (19).

In view of (22), the equality case of (19) is valid if and only if

$$\sigma_{12}^{r} = \ldots \sigma_{1n}^{r} = 0 \text{ and } \sigma_{11}^{r} = \sigma_{22}^{r} + \ldots + \sigma_{nn}^{r}, \qquad r \in \{n+1 \ldots, 2m+1\}. \qquad \ldots (23)$$

If $H(p) = 0$, (23) implies that $e_1 = X$ belongs to the relative null space N_p at p. Conversely, if $e_1 = X$ lies in the relative null space, then (23) holds because $H(p) = 0$ is assumed. This proves statement (*ii*).

Now, we prove (*iii*). Assume that the equality case of (19) for all unit tangent vectors to M at $p \in M$ is true. Then, in view of (22), for each $r \in \{n+1 ..., 2m+1\}$, we have

$$\sigma_{ij}^{r} = 0, \quad i \neq j, \qquad \text{... (24)}$$

$$2\sigma_{ii}^{r} = \sigma_{12}^{r} + \ldots + \sigma_{nn}^{r} \qquad i \in \{1, ..., n\}. \qquad \text{... (25)}$$

Thus, we have two cases, namely either $n = 2$ or $n \neq 2$. In the first case p is a totally umbilical point, while in the second case p is a totally geodesic point. The converse is obvious.

□

The above theorem implies the following results for slant, invariant and anti-invariant submanifolds.

Theorem 5.3 — *Let M be an n-dimensional submanifold in a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c such that $\xi \in TM$. Then, the following statements are true.*

1. *If M is θ-slant, then for each unit vector $X \in T_pM$, we have*

$$\begin{aligned}\text{Ric}(X) \leq \frac{1}{4}\{ & n^2 \| H \|^2 + (n-1)(c - 3f^2) \\ & + (3\cos^2\theta - (n + 3\cos^2\theta - 2)\,\eta(X)^2 - 1)(c + f^2)\} \\ & - (1 + (n-2)\,\eta(X)^2 f'. \end{aligned} \qquad \text{... (26)}$$

2. *If M is invariant, then for each unit vector $X \in T_pM$, we have*

$$\begin{aligned}\text{Ric}(X) \leq \frac{1}{4}\{ & n^2 \| H \|^2 + (n-1)(c - 3f^2) \\ & + (2 - (n+1)\,\eta(X)^2)(c + f^2)\} \\ & - (1 + (n-2)\,\eta(X)^2 f'. \end{aligned} \qquad \text{... (27)}$$

3. *If M is anti-invariant, then for each unit vector $X \in T_pM$, we have*

$$\begin{aligned}\text{Ric}(X) \leq \frac{1}{4}\{ & n^2 \| H \|^2 + (n-1)(c - 3f^2) \\ & - (1 + (n-2)\,\eta(X)^2)(c + f^2 + 4f')\}. \end{aligned} \qquad \text{... (28)}$$

4. *If* $H(p) = 0$, *a unit vector* $X \in T_pM$ *satisfies the equality case of (26), (27) and (28) if and only if* $X \in N_p$.

5. *The equality case of (26), (27) and (28) holds identically for all unit vectors* $X \in T_pM$, *if and only if either* $n = 2$ *and* p *is a totally umbilical point or* p *is a totally geodesic point.*

PROOF : In a θ-slant submanifold, for a unit vector $X \in T_pM$, in view of (14), for a unit vector $X \in T_pM$, we get

$$\| PX \|^2 = \langle PX, PX \rangle = \cos^2 (1 - \eta (X)^2).$$

Using this in (19), we get (26). Putting $\theta = 0$ and $\theta = \pi/2$ in (26), we have (27) and (28) respectively. Fourth and fifth statements are obvious. □

We also have the following:

Corollary 5.4 — Let M be an n-dimensional CR-submanifold in a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c. Then, the following statements are true.

1. For each unit vector $X \in \mathcal{D}$, we have

$$4 \,\mathrm{Ric}\,(X) \le n^2 \| H \|^2 + (n+1)\, c - (3n-5) f^2 - 4f'. \qquad \text{... (29)}$$

2. For each unit vector $X \in \mathcal{D}^{\perp}$, we have

$$4 \,\mathrm{Ric}\,(X) \le n^2 \| H \|^2 + (n-2)\, c - (3n-2) f^2 - 4f'. \qquad \text{... (30)}$$

3. If $H(p) = 0$, a unit vector $X \in \mathcal{D}$ (resp. $\mathcal{D}^{\perp}$) satisfies the equality case of (29) (resp. (30)) if and only if $X \in N_p$.

6. k-RICCI CURVATURE

In this section, we prove a relationship between the k-Ricci curvature and the squared mean curvature for submanifolds in a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c such that $\xi \in TM$.

Theorem 6.1 — *Let* M *be an* n*-dimensional submanifold in a* $(2m + 1)$*-dimensional localy conformal almost cosymplectic manifold* $\tilde{M}(c)$ *of pointwise constant* φ*-sectional curvature* c *such that* $\xi \in TM$.. *Then we have*

$$\| H \|^2 \ge \frac{2\tau}{n(n-1)} - \frac{c - 3f^2}{4} - \frac{3 \| P \|^2 (c + f^2)}{4n(n-1)} + \frac{2}{n}\left(\frac{c + f^2}{4} + f' \right). \qquad \text{... (31)}$$

PROOF : Choose an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{2m+1}\}$ at p such that e_{n+1} is parallel to the mean curvature vector $H(p)$ and $e_1, ..., e_n$ diagonalize the shape operator A_{n+1}. Then the shape operators take the forms

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & a_n \end{pmatrix}, \quad \dots (32)$$

$$A_r = \left(h_{ij}^r \right), i, j = 1, \dots, n; \ r = n+2, \dots, 2m+1,$$

$$\text{trace } A_r = \sum_{i=1}^{n} h_{ii}^r = 0. \quad \dots (33)$$

From (8), we get

$$n^2 \| H \|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} \left(h_{ij}^r \right)^2 - \frac{1}{4} n(n-1)(c - 3f^2)$$

$$- \frac{3}{4} \| P \|^2 (c + f^2) + 2(n-1) \left(\frac{c+f^2}{4} + f' \right). \quad \dots (34)$$

Since

$$0 \le \sum_{i<j} (a_i - a_j)^2 = (n-1) \sum_{i} a_i^2 - 2 \sum_{i<j} a_i a_j,$$

therefore, we get

$$n^2 \| H \|^2 = \left(\sum_{i=1}^{n} a_i \right)^2 = \sum_{i=1}^{n} a_i^2 + 2 \sum_{i<j} a_i a_j \le n \sum_{i=1}^{n} a_i^2 ,$$

$$\sum_{i=1}^{n} a_i^2 \ge \| H \|^2$$

which implies

$$n^2 \| H \|^2 \ge 2\tau + n \| H \|^2 - \frac{1}{4} n(n-1)(c - 3f^2)$$

$$- \frac{3}{4} \| P \|^2 (c + f^2) + 2(n-1) \left(\frac{c+f^2}{4} + f' \right), \quad \dots (36)$$

which gives (31). □

Next, we prove the following:

***Theorem* 6.2** — *Let M be an n-dimensional submanifold in a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c such that $\xi \in TM$.. Then,* *for each integer* $k, 2 \le k \le n$, *and every point* $p \in M$, *we have*

$$\| H \|^2 \ge \theta_k(p) - \frac{c-3f^2}{4} - \frac{3\| P \|^2 (c+f^2)}{4n(n-1)} + \frac{2}{n}\left(\frac{c+f^2}{4} + f'\right). \quad \text{... (37)}$$

PROOF : Let $\{e_1, ..., e_n\}$ be an orthonormal basis of T_pM. We denote by $\mathcal{L}_{i_1 \dots i_k}$ the k-plane section spanned by $e_{i_1}, ..., e_{i_k}$. From (4) and (5), it follows that

$$\tau\left(\mathcal{L}_{i_1 \dots i_k}\right) = \frac{1}{2} \sum_{i \in \{i_1, ..., i_k\}} \mathrm{Ric}_{\mathcal{L}_{i_1 \dots i_k}} (e_i), \quad \text{... (38)}$$

$$\tau(p) = \frac{1}{C_{k-2}^{n-2}} \sum_{1 \le i_1 < \dots < i_k \le n} \tau\left(\mathcal{L}_{i_1 \dots i_k}\right). \quad \text{... (39)}$$

Combining (6), (38) and (39), we obtain

$$\tau(p) \ge \frac{n(n-1)}{2} \theta_k(p). \quad \text{... (40)}$$

From (31) and (40), we get (37). □

As an application, we have the following Corollary.

Corollary 6.3 — Let M be an n-dimensional submanifold in a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c such that $\xi \in TM$, and let k be any integer such that $2 \le k \le n$. Then at each point $p \in M$, we have the following statements.

1. If M is θ-slant, then

$$\| H \|^2 \ge \theta_k - \frac{c-3f^2}{4} - \frac{3(c+f^2)\cos^2\theta}{4n} + \frac{2}{n}\left(\frac{c+f^2}{4} + f'\right). \quad \text{... (41)}$$

2. If M is invariant, then

$$\| H \|^2 \ge \theta_k - \frac{c-3f^2}{4} - \frac{3(c+f^2)}{4n} + \frac{2}{n}\left(\frac{c+f^2}{4} + f'\right). \quad \text{... (42)}$$

3. If M is anti-invariant, then

$$\| H \|^2 \ge \theta_k - \frac{c-3f^2}{4} + \frac{2}{n}\left(\frac{c+f^2}{4} + f'\right). \quad \text{... (43)}$$

4. If M is a CR-submanifold, then

$$\| H \|^2 \ge \theta_k - \frac{c-3f^2}{4} - \frac{6h(c+f^2)}{4n(n-1)} + \frac{2}{n}\left(\frac{c+f^2}{4} + f'\right), \quad \text{... (44)}$$

where $2h = \dim(\mathcal{D})$.

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