Indian J. pure appl. Math., **35**(3) : 273-283, March 2004

GLOBAL BEHAVIOUR OF A DISCRETE HAEMATOPOIESIS MODEL

M. R. XU, B. SHI* AND D. C. ZHANG

Department of Basic Sciences, Naval Aeronautical Engineering Institute, Yantai, Shandong 264001, *Peoples Republic of China*
**e-mail: baoshi781@sohu.com*

(*Received* 24 *January* 2003; *accepted* 25 *July* 2003)

In this paper, we study the difference equation:

$$\Delta y_n = -\alpha y_n + \frac{\beta y_{n-k}}{1 + y_{n-k}^p} \qquad \dots (1^*)$$

and we obtain the global asymptotic stability and oscillation of eq. (1*), where $\alpha \in (0, 1)$, $\beta \in (\alpha, \infty)$, $p \in (1, \infty)$, k is a nonnegative integer.

Key Words: Difference Equations; Oscillation; Global Asymptotic Stability

1. INTRODUCTION

In the monograph of Kocić and Ladas[1], they give a Research project (see[1], p. 111). To this end, we consider the equation:

$$\Delta y_n = -\alpha y_n + \frac{\beta y_{n-k}}{1 + y_{n-k}^p} \text{ for } n = 0, 1, \dots, \qquad \dots (1)$$

where Δ is the forward difference operator,

$$\alpha \in (0, 1),\ \beta \in (\alpha, \infty),\ p \in (1, \infty), k \in \{1, 2, \dots\}. \qquad \dots (2)$$

If the initial values $y_{-k}, \dots, y_{-1} \in [0, \infty), y_0 \in (0, \infty)$, then corresponding solution of the eq. (1) is positive, and eq. (1) has a unique positive equilibrium point $\bar{y}$,

$$\bar{y} = \left(\frac{\beta}{\alpha} - 1\right)^{\frac{1}{p}}. \qquad \dots (3)$$

The aim of this paper is to investigate the oscillation and the global asymptotic stability of the positive equilibrium of eq. (1).

2. SOME LEMMAS

Lemma 2.1 — (see[1], p 6). Assume that $p_i \in (0, \infty)$ and $k_i \in \{0, 1, \dots\}$ with $\sum_{i=1}^{m} (p_i + k_i) \neq 1$, for $i = 1, 2, \dots, m$. Let $\{p_i(n)\}$ be sequences of positive numbers such that

$$\liminf_{n \to \infty} p_i(n) \geq p_i \text{ for } i = 1, 2, ..., m.$$

Suppose that the linear difference inequality

$$z_{n+1} - z_n + \sum_{i=1}^{m} p_i(n)\, z_{n-k_i} \leq 0 \text{ for } n = 0, 1, ...$$

has an eventually positive solution. Then the difference equation:

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i x_{n-k_i} = 0$$

has a positive solution.

Lemma 2.2 — (see[1], p. 6). Assume that $p \in \mathbb{R}$ and k is a nonnegative integer. Then every solution of the delay difference equation:

$$y_{n+1} - y_n + p y_{n-k} = 0 \text{ for } n = 0, 1, ...$$

oscillates if and only if

$$p > \begin{cases} 1, & \text{if } k = 0 \\ \dfrac{k^k}{(k+1)^{k+1}}, & \text{if } k \geq 1. \end{cases}$$

Lemma 2.3 — (see[1], p. 12) Kocić and Ladas. Assume that $a, b \in \mathbb{R}$ and k is a nonnegative integer. Then $|a| + |b| < 1$ is a sufficient condition for the asymptotic stability of the difference equation:

$$x_{n+1} + a x_n + b x_{n-k} = 0 \quad \text{for } n = 0, 1, ...$$

Lemma 2.4 — Assume that (2) holds, and that $\frac{\beta}{\alpha} \geq 2^p - 1$. Then

$$\frac{1 + 2^p\left(\frac{\beta}{\alpha} - 1\right)}{\beta}\, \alpha \leq \frac{\beta}{\alpha}. \qquad \text{... (4)}$$

PROOF : To prove (4) holds, it suffices to show

$$1 + 2^p\left(\frac{\beta}{\alpha} - 1\right) \leq \left(\frac{\beta}{\alpha}\right)^2$$

Let $a = \frac{\beta}{\alpha}$ and $F(a) = a^2 - 2^p(a-1) - 1$. So it suffices to prove $F(a) \geq 0$ for $a > 1$.

Because $F'(a) = 2a - 2^p$, so if $a \geq 2^{p-1}$, then $F'(a) \geq 0$, which implies that $F(a)$ is increasing in $(2^{p-1}, \infty)$. Since $F(2^p - 1) = 0$ and $2^p - 1 \geq 2^{p-1}$, hence $F(a) \geq 0$ for $a \geq 2^p - 1$. This completes the proof.

Lemma 2.5 — Assume that (2) holds. Let $\{y_n\}$ be a solution of eq. (1) with positive initial conditions $y_{-k}, \ldots, y_0$. Let

$$R = \max\left\{ y_{-k} \ldots, y_0, \max_{0 \leq x \leq \bar{y}} \left\{ \frac{\frac{\beta}{\alpha} x}{1 + x^p} \right\} \right\}$$

and

$$S = \min\left\{ y_{-k} \ldots, y_0, \min_{\bar{y} \leq x \leq R} \left\{ \frac{\frac{\beta}{\alpha} x}{1 + x^p} \right\} \right\}.$$

Then

$$0 < S \leq y_n \leq R \text{ for } n = -k, -k+1, \ldots . \qquad \ldots (5)$$

PROOF : Clearly, (5) holds for $n = -k, -k + 1, \ldots, 0$. Suppose (5) holds for all integers less than or equal to n. Then if $0 \leq y_{n-k} \leq \bar{y}$, then

$$\frac{\frac{\beta}{\alpha} y_{n-k}}{1 + y_{n-k}^p} \leq \max_{0 \leq x \leq \bar{y}} \left\{ \frac{\frac{\beta}{\alpha} x}{1 + x^p} \right\} \leq R.$$

Else if $y_{n-k} > \bar{y}$, then

$$\frac{\frac{\beta}{\alpha} y_{n-k}}{1 + y_{n-k}^p} < \frac{\frac{\beta}{\alpha} y_{n-k}}{1 + \bar{y}^p} = y_{n-k} \leq R.$$

So, we have

$$y_{n+1} = (1 - \alpha) y_n + \frac{\beta y_{n-k}}{1 + y_{n-k}^p}$$

$$= (1 - \alpha) y_n + \alpha \frac{\frac{\beta}{\alpha} y_{n-k}}{1 + y_{n-k}^p} \leq (1 - \alpha) R + \alpha R = R.$$

In the following, we will prove that $y_{n+1} \geq S$.

If $\bar{y} \leq y_{n-k} \leq R$, then

$$\frac{\frac{\beta}{\alpha} y_{n-k}}{1+y_{n-k}^{p}} \geq \min_{\bar{y} \leq x \leq R} \left\{ \frac{\frac{\beta}{\alpha} x}{1+x^{p}} \right\} \geq S.$$

Else if $0 < y_{n-k} < \bar{y}$, then we can find

$$\frac{\frac{\beta}{\alpha} y_{n-k}}{1+y_{n-k}^{p}} \geq \frac{\frac{\beta}{\alpha} y_{n-k}}{1+\bar{y}^{p}} = y_{n-k} \geq S.$$

So we have

$$y_{n+1} = (1-\alpha) y_n + \frac{\beta y_{n-k}}{1+y_{n-k}^{p}}$$

$$= (1-\alpha) y_n + \alpha \frac{\frac{\beta}{\alpha} y_{n-k}}{1+y_{n-k}^{p}} \geq (1-\alpha) S + \alpha S = S.$$

Therefore, (5) holds for $n = -k, -k+1, \ldots$. This completes the proof.

3. Main Results

***Theorem* 3.1** — Assume that (2) holds, and that

$$\frac{(\beta-\alpha)p-\beta}{\alpha} \cdot \frac{(k+1)^{k+1}}{k^k} > (1-\alpha)^{k+1}. \quad \ldots (6)$$

Then every positive solution of eq. (1) oscillates about the positive equilibrium $\bar{y}$.

PROOF : Assume for the sake of contradiction that eq. (1) has a positive solution $\{y_n\}$ which does not oscillate about $\bar{y}$. We assume that $y_n > \bar{y}$ eventually. If $y_n < \bar{y}$ eventually, then the proof is similar and will be omitted. So there exists an $n_0 \geq 0$ such that $y_n > \bar{y}$ for $n \geq n_0$. This follows that $y_{n-k} > \bar{y}$ for $n \geq n_1$, where $n_1 = n_0 + k$.

First we claim that $\{y_n\}$ is a bounded sequence. Otherwise there exists a subsequence $\{y_{n_i}\}$ such that

$$\lim_{i \to \infty} y_{n_i+1} = \infty \text{ and } y_{n_i+1} \geq y_{n_i} \text{ for } n_i \geq n_1 \quad i = 1, 2, \ldots . \quad \ldots (7)$$

Eq. (1) can be rewritten in the form

$$y_{n+1} = (1-\alpha) y_n + \frac{\beta y_{n-k}}{1+y_{n-k}^{p}}. \quad \ldots (8)$$

From (7) and (8), we have

$$y_{n_i+1} \leq (1-\alpha)\, y_{n_i+1} + \frac{\beta y_{n_i-k}}{1+y^p_{n_i-k}}.$$

That is,

$$\alpha y_{n_i+1} \leq \frac{\beta y_{n_i-k}}{1+y^p_{n_i-k}} \qquad \text{... (9)}$$

$$\leq \alpha y_{n_i-k}. \qquad \text{... (10)}$$

From (7) and (10), we see that $\lim_{i\to\infty} y_{n_i-k} = \infty$. But

$$\lim_{i\to\infty} \frac{\beta y_{n_i-k}}{1+y^p_{n_i-k}} < \infty,$$

which contradicts (9).

Next we claim that

$$\lim_{n\to\infty} y_n = \bar{y}. \qquad \text{... (11)}$$

Otherwise, let

$$\mu = \limsup_{n\to\infty} y_n . \qquad \text{... (12)}$$

Then $\mu > \bar{y}$ and there exists a subsequence $\{y_{n_i}\}$ such that

$$\lim_{i\to\infty} y_{n_i+1} = \mu \text{ and } y_{n_i+1} \geq y_{n_i} \text{ for } n_i \geq n_1,\ i = 1, 2, \ldots .$$

From (10), we have $\mu \leq \limsup_{n\to\infty} y_{n_i-k}$. So. because of (12), we have

$$\limsup_{n\to\infty} y_{n_i-k} = \mu.$$

Hence we can choose a subsequence $\{y_{ni_j}\} \subset \{y_{n_i}\}$ $i, j = 1, 2, \ldots$, such that

$$\lim_{j\to\infty} y_{n_i-k} = \mu.$$

But then (9) leads to

$$\alpha\mu \leq \frac{\beta\mu}{1+\mu^p} < \frac{\beta\mu}{1+\bar{y}^p} = \alpha\mu,$$

which is an contradiction. Hence (11) holds.

Set

$$y_n = \bar{y} + x_n \quad n = -k, -k+1, \ldots . \qquad \ldots (13)$$

Then $\{x_n\}$ is a eventually positive solution of the difference equation

$$x_{n+1} - x_n + \alpha x_n + \alpha\bar{y} - \frac{\beta(\bar{y} + x_{n-k})}{1 + (\bar{y} + x_{n-k})^p} = 0 \quad n = 0, 1, \ldots . \qquad \ldots (14)$$

Eq. (14) can be rewritten in the form

$$x_{n+1} - (1-\alpha)x_n + p(n)x_{n-k} = 0 \quad n = 0, 1, \ldots, \qquad \ldots (15)$$

where

$$p(n) = \alpha\bar{y} - \frac{\dfrac{\beta(\bar{y} + x_{n-k})}{1 + (\bar{y} + x_{n-k})^p}}{x_{n-k}}.$$

From (6) we see

$$\lim_{n \to \infty} p(n) = \frac{p(\beta - \alpha) - \alpha}{\frac{\beta}{\alpha}} > 0.$$

One can easily see that the hypothesis of Lemma 1 are satisfied and so the linear equation

$$x_{n+1} - (1-\alpha)x_n + \frac{p(\beta-\alpha) - \beta}{\frac{\beta}{\alpha}} x_{n-k} = 0 \qquad \ldots (16)$$

has an eventually positive solution.

Let $\{x_n\}$ be an eventually positive solution of eq. (16). Then $z_n = (1-\alpha)^{-n} x_n$ is an eventually positive solution of

$$z_{n+1} - z_n + \frac{p(\beta-\alpha) - \beta}{\frac{\beta}{\alpha}} (1-\alpha)^{-k-1} z_{n-k} = 0 \quad n = 0, 1, \ldots . \qquad \ldots (17)$$

According to Lemma 2, eq. (17) has no nonoscillatory solution. This is a contradiction.

The proof is completed.

***Theorem* 3.2** — *Assume that (2) holds, and that*

(1) $\left[(1-\alpha)^{-k-1} - 1 \right] \dfrac{\beta - \alpha}{\alpha} < 1;$

(2) $1 < p < \min\left\{ \log_2\left(\frac{\beta}{\alpha}+1\right), \frac{2\beta}{\beta-\alpha} \right\}$.

Then the positive equilibrium $\bar{y}$ of eq. (1) is globally asymptotically stable.

PROOF : First, we will prove that the positive equilibrium $\bar{y}$ is a global attractor of all positive solutions of eq. (1). For the case $\{y_n\}$ is nonoscillatory, we have shown eq. (11) holds in Theorem 3.1. Therefore it remains to show eq. (11) holds when $\{y_n\}$ is strictly oscillatory. That is $\lim_{n\to\infty} x_n = 0$ holds, when $\{x_n\}$ is a strictly oscillatory solution of eq. (14).

To this end, let $\{x_{p_i+1}, x_{p_i+2}, ..., x_{q_i}\}$ be the ith positive semicycle of $\{x_n\}$ followed by the jth negative semicycle $\{x_{q_i+1}, x_{q_i+2}, ..., x_s\}$. Let x_{M_i}, x_{m_i} be the extreme values in these two semicycles with the smallest possible indices M_i and m_i. Then we claim that

$$M_i - p_i \leq k+1 \quad \text{and} \quad m_i - q_i \leq k+1. \qquad ...(18)$$

We will prove (18) for positive semicycle. The proof for negative semi-cycle is similar and will be omitted. Assume for the sake of contradiction that the first inequality in (18) is not true. Then $M_i - p_i > k+1$ and the terms $x_{M_i-k-1}, x_{M_i-k}, ..., x_{M_i-1}$ are in a positive semicycle. Since $x_{M_i} > x_{M_i-1}$, from (14), we obtain

$$\alpha x_{M_i-1} + \alpha\bar{y} \leq \frac{\beta(\bar{y}+x_{M_i-k-1})}{1+(\bar{y}+x_{M_i-k-1})^p} \leq \frac{\beta(\bar{y}+x_{M_i-k-1})}{1+\bar{y}^p} = \alpha(\bar{y}+x_{M_i-k-1}).$$

Furthermore, from (14) and the above, we find

$$x_{M_i} + \bar{y} = (1-\alpha)(x_{M_i-1}+\bar{y}) + \frac{\beta(\bar{y}+x_{M_i-k-1})}{1+(\bar{y}+x_{M_i-k-1})^p}$$

$$\leq (1-\alpha(\bar{y}+x_{M_i-k-1}) + \frac{\beta(\bar{y}+x_{M_i-k-1})}{1+\bar{y}^p}$$

$$= (1-\alpha)(\bar{y}+x_{M_i-k-1}) + \alpha(\bar{y}+x_{M_i-k-1})$$

$$= \bar{y} + x_{M_i-k-1},$$

which contradicts that M_i is the smallest possible indices of the extreme value in positive semicycle. So (18) is true. Noting that $\{y_n\}$ is bounded from Lemma 2.5, we can let

$$\lambda = \liminf_{n\to\infty} x_n = \liminf_{i\to\infty} x_{m_i}$$

and

$$\mu = \liminf_{n\to\infty} x_n = \liminf_{i\to\infty} x_{M_i}. \quad \ldots (19)$$

To prove that $\lim_{n\to\infty} x_n = 0$ holds, it is sufficient to show that $\lambda = \mu = 0$. Clearly, $\lambda \ge -\bar{y}$. From (19), it follows that if $\varepsilon \in (0, \lambda)$ is given, then there exists an $n_0 \ge 0$ such that

$$x_n \le \mu + \varepsilon \quad \text{for} \quad n \ge n_0 + k. \quad \ldots (20)$$

Eq. (14) can be written in the form

$$x_{n+1} - (1-\alpha)\, x_n = -\alpha \bar{y} + \frac{\beta(\bar{y} + x_{n-k})}{1 + (\bar{y} + x_{n-k})^p}. \quad \ldots (21)$$

Multiplying (21) by $(1-\alpha)^{-n-1}$ and then summing up from $n = p_i$ to $n = M_i - 1$ for i sufficiently large, we get

$$(1-\alpha)^{-M_i} x_{M_i} - (1-\alpha)^{-pi} x_{p_i} = \sum_{n=p_i}^{M_i-1} (-\alpha\bar{y})(1-\alpha)^{-n-1}$$

$$+ \sum_{n=p_i}^{M_i-1} \frac{\beta(\bar{y}+x_{n-k})}{1+(\bar{y}+x_{n-k})^p} (1-\alpha)^{-n-1}.$$

As $x_{p_i} < 0, x_{n-k} > 0$ for $n = p_i$ to $n = M_i - 1$, we have from (20)

$$(1-\alpha)^{-M_i} x_{M_i} \le (1-\alpha\bar{y}) \sum_{n=p_i}^{M_i-1} (1-\alpha)^{-n-1}$$

$$+ \sum_{n=p_i}^{M_i-1} \frac{\beta(\bar{y}+x_{n-k})}{1+(\bar{y}+x_{n-k})^p} (1-\alpha)^{-n-1}$$

$$\le (-\alpha\bar{y}) \sum_{n=p_i}^{M_i-1} (1-\alpha)^{-n-1} + \sum_{n=p_i}^{M_i-1} (\beta\bar{y} + \alpha x_{n-k})(1-\alpha)^{-n-1}$$

$$= (\beta-\alpha)\bar{y} \sum_{n=p_i}^{M_i-1} (1-\alpha)^{-n-1} + \sum_{n=p_i}^{M_i-1} \alpha x_{n-k} (1-\alpha)^{-n-1}$$

$$\le (\beta-\alpha)\bar{y} \sum_{n=p_i}^{M_i-1} (1-\alpha)^{-n-1} + \sum_{n=p_i}^{M_i-1} \alpha(\mu+\varepsilon)(1-\alpha)^{-n-1}.$$

So,

$$x_{M_i} \leq \left[\frac{\beta - \alpha}{\alpha} \bar{y} + \mu + \varepsilon \right] [1 - (1 - \alpha)^{M_i - p_i}].$$

By using (19) and $M_i - p_i \leq k+1$, we get

$$\mu \leq \left[\frac{\beta - \alpha}{\alpha} \bar{y} + \mu + \varepsilon \right] [1 - (1 - \alpha)^{k+1}].$$

As $\varepsilon > 0$ is arbitrary, we have

$$\mu \leq \frac{\beta - \alpha}{\alpha} \bar{y} \; [1 - (1 - \alpha)^{-k-1} - 1].$$

Set

$$Q_1 = \frac{\beta - \alpha}{\alpha} \bar{y} \; [(1 - \alpha)^{-k-1} - 1]. \qquad \text{... (22)}$$

Then

$$\mu \leq Q_1 \quad \text{and} \quad 0 < Q_1 < \bar{y}. \qquad \text{... (23)}$$

Furthermore, from (19) it follows that for n sufficiently large, and $\varepsilon > 0$ arbitrary small

$$\mu + \varepsilon > x_{n-k} \geq \lambda - \varepsilon.$$

After multiplying (21) by $(1 - \alpha)^{-n-1}$ and then summing up from $n = q_i$ to $n = m_i - 1$ for i sufficiently large, we obtain

$$(1 - \alpha)^{-m_i} x_{m_i} - (1 - \alpha)^{-q_i} x_{q_i} = (-\alpha \bar{y}) \sum_{n=q_i}^{m_i - 1} (1 - \alpha)^{-n-1}$$

$$+ \sum_{n=q_i}^{m_i - 1} \frac{\beta(\bar{y} + x_{n-k})}{1 + (\bar{y} + x_{n-k})^p} (1 - \alpha)^{-n-1}.$$

Since $x_{q_i} > 0, x_{m_i} < 0, m_i - q_i \leq k+1$ and $\varepsilon > 0$, we have

$$-\lambda \leq \left[\alpha \bar{y} - \frac{\beta(\bar{y} + \lambda)}{1 + (\bar{y} + Q_1)^p} \right] \frac{1 - (1 - \alpha)^{k+1}}{\alpha}.$$

That is

$$-\lambda \left[1 - \frac{\beta}{1 + (\bar{y} + Q_1)^p} \, \frac{1 - (1 - \alpha)^{k+1}}{\alpha} \right]$$

$$\leq \left[\alpha \bar{y} - \frac{\beta \bar{y}}{1+(\bar{y}+Q_1)^p} \right] \frac{1-(1-\alpha)^{k+1}}{\alpha}$$

It is easy to see that:

$$1 - \frac{\beta}{1+(\bar{y}+Q_1)^p} \frac{1-(1-\alpha)^{k+1}}{\alpha}$$

$$\geq \frac{\beta}{1+(\bar{y}+Q_1)^p} \frac{(1-\alpha)^{k+1}}{\alpha}.$$

Therefore,

$$-\lambda \leq \left[\alpha \bar{y} - \frac{\beta \bar{y}}{1+(\bar{y}+Q_1)^p} \right] \frac{1-(1-\alpha)^{k+1}}{\alpha} \frac{1+(\bar{y}+Q_1)^p}{\beta} \frac{\alpha}{(1-\alpha)^{k+1}}.$$

$$= \left[\alpha \bar{y} - \frac{\beta \bar{y}}{1+(\bar{y}+Q_1)^p} \right] \frac{1+(\bar{y}+Q_1)^p}{\beta} \left[(1-\alpha)^{-k-1} - 1 \right]$$

$$= \left[\alpha \frac{1+(\bar{y}+Q_1)^p}{\beta} - 1 \right] \bar{y} \left[(1-\alpha)^{-k-1} - 1 \right]$$

$$\leq \left(\frac{1+2^p \bar{y}^p}{\beta} \alpha - 1 \right) \bar{y} \left[(1-\alpha)^{-k-1} - 1 \right] \quad \ldots (24)$$

$$= \left[\frac{1+2^p \left(\frac{\beta}{\alpha} - 1 \right)}{\beta} \alpha - 1 \right] \bar{y} \left[(1-\alpha)^{-k-1} - 1 \right]$$

$$\leq \frac{\beta - \alpha}{\alpha} \bar{y} \left[(1-\alpha)^{-k-1} - 1 \right] = Q_1.$$

Hence

$$-Q_1 \leq \lambda \leq \mu \leq Q_1 \text{ and } 0 < Q_1 < \bar{y}. \quad \ldots (25)$$

If we define the sequence $\{Q_n\}$ by

$$Q_{n+1} = \left[\frac{1+(\bar{y}+Q_n)^p}{\beta} \alpha - 1 \right] \bar{y} \left[(1-\alpha)^{-k-1} - 1 \right],$$

where Q_1 is given by (22). Then from (24) we find $Q_2 \leq Q_1$. It easy to prove by induction that

$$-Q_n \leq \lambda \leq \mu \leq Q_n, \; 0 < Q_n < \bar{y} \text{ and } Q_{n+1} \leq Q_n, \quad \text{for} \quad n = 1, 2, \ldots .$$

So, $\{Q_n\}$ is a convergent sequence. Set $Q = \lim_{n \to \infty} Q_n$. Then Q is a solution of the equation

$$Q = \left[\frac{1+(\bar{y}+Q)^p}{\beta} \alpha - 1 \right] \bar{y} \left[(1-\alpha)^{-k-1} - 1 \right].$$

We can define a function F by

$$F(u) = \left[\frac{1+(\bar{y}+U)^p}{\beta} \alpha - 1 \right] \bar{y} \left[(1-\alpha)^{-k-1} - 1 \right] - u.$$

Then

$$F'(u) < 0 \text{ and } F(0) = 0.$$

So, $F(u)$ has only one zero in $[0, \bar{y}]$. Hence, $Q = 0$. So, $\lambda = \mu = 0$, that is, $\lim_{n \to \infty} x_n = 0$, which implies that $\bar{y}$ is a global attractor of all positive solutions of eq. (1).

To complete the proof, it remains to show that $\bar{y}$ is locally asymptotically stable.

The linearized equation associated with eq. (1) is

$$y_{n+1} = (1-\alpha) y_n + \left[\alpha - \frac{\alpha p (\beta - \alpha)}{\beta} \right] y_{n-k}$$

If $1 < p \leq \frac{\beta}{\beta - \alpha}$, then

$$0 \leq \alpha - \frac{\alpha p (\beta - \alpha)}{\beta}) < \frac{\alpha}{\beta} \alpha < \alpha.$$

Hence

$$0 < (1-\alpha) + \left[\alpha - \frac{\alpha p (\beta - \alpha)}{\beta} \right] < 1.$$

If $\frac{\beta}{\beta - \alpha} < p < \frac{2\beta}{\beta - \alpha}$, then $-\alpha < \alpha - \frac{\alpha p (\beta - \alpha)}{\beta} < 0$.

So,

$$0 < (1-\alpha) - \left[\alpha - \frac{\alpha p (\beta - \alpha)}{\beta} \right] < 1.$$

Therefore, by Lemma 2.3, the positive equilibrium $\bar{y}$ is locally asymptotically stable. The proof is complete.

REFERENCE

1. V. L. Kocić and G. Ladas *Global Behaviour of Nonlinear Difference Equations of Higher Order and Applications*, Kluwer Academic Publishers, Dordrecht, 1993.