Indian J. pure appl. Math., **35**(3) : 285-301, March 2004

ON FOURIER-HANKEL TRANSFORMATION OF ULTRA-DISTRIBUTIONS

B. N. BHOSALE

S. H. Kelkar College of Arts, Commerce and Science, Devgad-Sindhudurg (Maharashtra), India

AND

M S. CHAUDHARY

Department of Mathematics, Shivaji University, Kolhapur, (Maharashtra), India

(*Received* 18 *July* 2000; *after final revision* 15 *January* 2002; *accepted* 18 *January* 2003)

The aim of this paper is to study the Fourier-Hankel transform to spaces of Ultra-distributions. For this purpose, spaces $FH_{\mu, ak, A, ak', A'}$, $FH_{\mu,}^{bq, B, bq', B'}$, $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$ are constructed, on which Fourier-Hankel transform (fh_μ) is defined. It is proved that the so defined *F-H* transform fh_μ is a continuous linear mapping from the space $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$ into the space $FH_{\mu, bk, B, ak', bk' A1'}{}^{aq, A, aq'^2, B'}{}_1$. Further, generalized *F-H* transform is defined and its inversion formula is given. An operational transform formula is also established. In the end, a differential equation of the form $P(D_x, S_{\mu, y})\, u = g$ has been solved by using the so defined *F-H* transform.

Key Words: Fourier-Hankel; Ultra-distribution

1. INTRODUCTION

As referred to in Pathak[4], if the test function spaces are some classes of non-quasi-analytic functions with some natural topology, then the dual spaces have the properties analogous to those of distributions. The elements of these dual spaces are the ultra-distributions. Pathak[4] gave a comprehensive account of extensions of Fourier and Hankel transformations of ultra-distributions (of Roumieu type). With the similar motivation, following the idea of Roumieu[5] and Komatsu[2], we introduce the space of ultra-diferentiable functions on which the combined Fourier-Hankel transformation acts as a continuous linear mapping, so that the generalized *F-H* transformation on the corresponding dual spaces also acts as a continuous linear mapping.

2. NOTATIONS AND DEFINITIONS

The notations and the terminology of this work follow from those of Pathak[4] and Zemanian[6].

Fourier-Hankel transform : We define an integral transform for which the kernel is the product of the kernels of Fourier and the Hankel transformations as below:

Let $\phi(x, y)$ be a suitably restricted function on $-\infty < x < \infty, 0 < y < \infty$ then its Fourier-Hankel transform is given by:

$$fh_\mu \phi = \Phi(\xi, t) = \int_{-\infty}^{\infty} \int_0^{\infty} e^{i\xi x} \sqrt{yt}\, J_\mu(yt)\, \phi(x, y)\, dx\, dy \qquad \text{... (2.1)}$$

Here $J_\mu(yt)$ is the Bessel function of first kind of order μ, where μ is real and $\mu \geq -1/2$.

3. Test Function Spaces and their Duals

Let $\{a_k\}$ and $\{b_q\}$ be two arbitrary sequences of positive real numbers. We shall impose some of the following constrains on these sequences so that the resulting space of test functions may be non-quasi analytic and closed under certain algebraic, differential and integral operations.

$$b_q^2 \leq b_{q-1}\, b_{q+1} \quad \text{for all} \quad q \,\varepsilon\, \mathbb{N}_0 \qquad \text{... (C.1)}$$

An immdeiate consequence of this inequality is

$$b_p\, b_q \leq b_0\, b_{p+q},\ p, q \,\varepsilon\, \mathbb{N}_0; \qquad \text{... (C.2)}$$

and

$$\sum_{q=0}^{\infty} b_{q-1}\, b_q < \infty \qquad \text{... (C.3)}$$

Further there are constants $R, R_1 > 0$ and $H, H_1 > 1$ such that

$$b_p \leq RH^p \min_{0 \leq q \leq p} b_q\, b_{p-q},\ p \,\varepsilon\, \mathbb{N}_0 \qquad \text{... (C.4)}$$

and

$$a_p \leq R_1\, H_1^p \min_{0 \leq q \leq p} a_q\, a_{p-q},\ p \,\varepsilon\, \mathbb{N}_0 \qquad \text{... (C.5)}$$

We now construct certain test function spaces on which F-H transformation can be studied systematically.

The test function spaces:

$$FH_{\mu, ak, A, ak', A'},\ FH_\mu^{bq, B, bq', B'},\ FH_{\mu, ak, A, ak', A'}^{bq, B, bq', B'}.$$

Let ϕ be an infinitely differentiable function defined on the open set

$$I = (-\infty, \infty) \times (0, \infty)$$

$$\phi \,\varepsilon\, FH_{\mu, ak, A, ak', A'}^{bq, B, bq', B'} \text{ iff } \left| X^k D_x^q y^{k'} (y^{-1} D_y)^{q'} y^{-\mu - 1/2} \phi(x, y) \right|$$

$$\leq C^\mu (A + \delta)^k (B + \rho)^q (A' + \delta')^{k} (B' + \rho')^{q} a_k b_q a'_k b'_q$$

for all $k, k', q, q' \,\varepsilon\, \mathbb{N}_0$ where δ, δ' and $\rho, \rho' > 0$ are arbitrary small numbers and C^μ, A, B, A', B' are certain positive constants depending on ϕ and $\{a_k\}$, $\{b_q\}$ are arbitrary sequences of positive numbers satisfying the conditions (C.1)-(C.5) for ascertaining that the resultant space of test functions is non-quasi analytic and closed under certain algebraic differential and integral operations.

In this space, the system of norms is introduced as follows:

$$\| \phi \|^{\mu}_{\delta, \delta', \rho, \rho'} = \text{Sup} \frac{| x^k D_x^1 y^{k'} (y^{-1} D_y)^{q'} y^{-\mu - 1/2} \phi(x, y) |}{(A + \delta)^k (B + \rho)^q (A' + \delta')^{k'} (B' + \rho')^{q'} a_k b_q a'_k b'_q} \quad \text{... (3.1)}$$

where Sup is over all

$$(x, y) \in (-\infty, \infty) \times (0, \infty), k, k', q, q' \in \mathbb{N}_0.$$

Here

$$q, q', \delta, \delta', \rho, \rho' = 1, 1/2, \dots .$$

Here we note that

$$\| \phi \|^{\mu}_{1/n} \leq \| \phi \|^{\mu}_{1/(n+1)}, \quad n = 1, 2, \dots$$

Similarly the other spaces $FH_{\mu, ak, A, ak', A'}$ $FH_{\mu}{}^{bq, B, bq', B'}$ can be defined and corresponding norms on each of them, as follows:

$$\phi \,\varepsilon\, FH_{\mu, ak, A, ak', A'} \text{ iff } \left| X^k D_x^q y^{k'} (y^{-1} D_y)^{q'} y^{-\mu - 1/2} \phi(x, y) \right|$$

$$\leq C^{\mu} (A + \delta)^k (A' + \delta')^{k'} a_k a_k$$

and

$$\phi \,\varepsilon\, FH_{\mu}{}^{bq, B, bq', B'} \text{ iff } \left| X^k D_x^q y^{k'} (y^{-1} D_y)^{q'} y^{-\mu - 1/2} \phi(x, y) \right|$$

$$\leq C^{\mu} (B + \delta)^q (B' + \rho')^{q'} b_q b_{q'}.$$

For $a_k = k^{k\alpha}$, $a_{k'} = k'^{k'\alpha}$ and $b_q = q^{q\beta}$, $b_q' = q'^{q'\beta}$, $\alpha, \alpha', \beta, \beta' \geq 0$, it can be seen that the spaces $FH_{\mu, ak, A, ak', A'}$, $FH_{\mu}{}^{bq, B, bq', B'}$, $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$ reduce to $FH_{\mu, \alpha, \alpha', A, A'}$, $FH_{\mu}{}^{\beta, \beta, B, B'}$ $FH_{\mu, \alpha, \alpha', A, A'}{}^{\beta, \beta, B, B'}$ respectively, similar to the those studies by Lee (1974)[3].

If b_q, b'_q satisfy the condition (C.1), then the space $D\{b_q, b'_q, (-\infty, \infty) \times (0, \infty)\}$ is a subspace of $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$ and the convergence in

$D\{b_q, b'_q, (-\infty, \infty) \times (0, \infty)\}$ implies convergence in $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$

Now, following Gel'fand and Shilov, [1, pp 179-181)] we prove the following theorem.

Theorem 3.1 — *Let a_k, a_k satisfy (C.5). Then $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$ is a complete countably normed perfect space. The dual space is also complete.*

Let $\phi(x, y)$ be an infinitely differentiable function defined on:

$$-\infty < x < \infty, \ 0 < y < \infty.$$

$$\phi(x, y) \, \varepsilon \, FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'} \quad \text{iff (3.1) holds.}$$

We assert that with the system of norms (3.1), the space $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$ becomes a complete countably normed space.

All that we need here is to establish that for every Cauchy sequence $\{\phi_v(x, y)\}$ in $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$, $\{D^k \ \phi_v(x, y)\}$ converged uniformly on every compact subset of $R \times I$ to smooth function $D^k \ \phi(x, y)$, for each $k = 1, 2, \ldots,$ where $\phi(x, y) \in FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$.

Now, the convergence of $\{\phi_v(x, y)\}$ can be defined as follows:

Definition 3.1 — A sequence of an infinitely differentiable function $\{\phi_v(x, y)\}$ is said to be correctly convergent to the function $\phi(x, y)$ if for any q, q', the function

$$x^k D_x^q y^{k'} (y^{-1} D_y)^{q'} y^{-\mu - 1/2} \ \phi_v(x, y) \text{ converges uniformly to}$$

$$x^k D_x^q y^{k'} (y^{-1} D_y)^{q'} y^{-\mu - 1/2} \ \phi_v(x, y) \text{ in any bounded interval.}$$

The proof of the theorem which runs parallel to that of one given by Pathak[4] [pp. 286-289] is broken into several steps:

(*a*) If the sequence $\{\phi_v(x, y)\}$ converges correctly to some function $\phi(x, y)$ and for some $\delta, \rho, \delta', \rho'$,

$$\| \phi_v \|_{\delta', \rho'}^{\delta, \rho} \leq C^{\mu}, C^{\mu} > 0,$$

then the norm $\| \ \|_{\delta', \rho'}^{\delta, \rho}$ exists even for some function $\phi(x, y)$

and

$$\| \phi_v \|_{\delta', \rho'}^{\delta, \rho} \leq C^{\mu}$$

Now for $-a < x < a, \ 0 \leq y < b$,

$$\sup_{x, y} \frac{| x^k D_x^q y^{k'} (y^{-1} D_y)^{q'} y^{-\mu - 1/2} \phi(x, y) |}{(A + \delta)^k (B + \rho)^q (A' + \delta')^{k'} (B' + \rho')^{q'} a_k b_q a'_k b'_q}$$

$$k, k' \leq q, q' \leq p$$

$$= \lim_{v \to \infty} \sup_{x, y} \frac{| x^k D_x^q y^{k'} (y^{-1} D_y)^{q'} y^{-\mu - 1/2} \phi(x, y) |}{(A + \delta)^k (B + \rho)^q (A' + \delta')^{k'} (B' + \rho')^{q'} a_k b_q a'_k b'_q}$$

$$\leq \| \phi_v \|_{\delta', \rho'}^{\delta, \rho}$$

$$\leq C^{\mu}$$

Now, we take the limit $a \to \infty, b \to \infty, p \to \infty, l \to \infty$ and obtain

$$\operatorname*{Sup}_{x, y} \frac{| x^k D_x^q y^{k'} (y^{-1} D_y)^{q'} y^{-\mu - 1/2} \phi(x, y) |}{(A+\delta)^k (B+\rho)^q (A'+\delta')^{k'} (B'+\rho')^{q'} a_k b_q a_k' b_q'} \leq C^{\mu}.$$

$k, k' \leq q, q'$.

(*b*) If the sequence $\{\phi_v(x, y)\}$ converges to zero at each point and is fundamental in the norm $\| \ \|_{\delta', \rho'}^{\delta, \rho}$, then $\| \phi_v \|_{\delta', \rho'}^{\delta, \rho} \to 0$.

As the sequence $\{\phi_v\}$ is fundamental, it converges correctly to zero and hence the sequence $\{\phi_v - \phi_\mu\}$ converges correctly to ϕ_v as $\mu \to \infty$.

Thus, for given $\varepsilon > 0$ there exists a sufficiently large v such that

$$\| \phi_v \|_{\delta', \rho'}^{\delta, \rho} \leq \operatorname*{Sup}_{\mu \geq v} \| \phi_v - \phi_\mu \|_{\delta', \rho'}^{\delta, \rho} < \varepsilon.$$

(*c*) The space $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$ be a fundamental sequence in each of the norms $\| \ \|_{\delta', \rho'}^{\delta, \rho}$. Then, according to (*a*) each of the norms $\| \ \|_{\delta', \rho'}^{\delta, \rho}$ exists for limit function $\phi(x, y)$; hence

$$\phi(x, y) \ \varepsilon \ FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$$

Also, according to (*b*), the difference $\{\phi - \phi_v\}$ converges correctly to zero and is bounded in each of the norms.

Hence, we have

$$\| \phi - \phi_v \|_{\delta', \rho'}^{\delta, \rho} \to 0 \qquad \text{for any } q, q'$$

Thus, the space $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$ is complete.

(*d*) The norms $\| \ \|_{\delta', \rho'}^{\delta, \rho}$ are pairwise consistent.

Let $\eta > 0$, δ, δ' and $\rho, \rho' > 0$ be given and choose arbitrarily $\delta'' < \delta, \rho'' < \rho', \delta' < \delta, \rho' < \rho$

Let $\{\phi(x, y)\} \ \varepsilon \ FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$ be fundamental in $\| \ \|_{\delta'', \rho''}^{\delta, \rho}$. Since $\phi_v(x, y)$ is bounded with respect to $\| \ \|_{\delta'', \rho''}^{\delta, \rho}$, for any k, k', q, q' and x, y, we have

$$| x^k D_x^q y^{k'} (y^{-1} D_y)^{q'} y^{-\mu - 1/2} \phi(x, y) |$$

$$(A + \delta)^k (B + \rho)^q (A' + \delta')^{k'} (B' + \rho')^{q'} a_k b_q a_k' b_q'.$$

For sufficiently large $k > k_0, k' > k_0'$,

the inequality

$$(A + \delta')^k (A' + \delta')^{k'} \leq (\eta / C_1^{\mu}) (A + \delta)^k (A' + \delta')^{k'} \text{ holds.}$$

Consequently, for any q, q', x, y and $k \geq k_0, k' \geq k_0'$,

$$| x^k D_x^q y^{k'} (y^{-1} D_y)^{q'} y^{-\mu - 1/2} \phi_v(x, y) |$$

$$\leq \eta (A + \delta)^k (B + \rho)^q (A' + \delta')^{k'} (B' + \rho')^{q'} a_k b_q a_k' b_q'. \quad \text{... (3.2}$$

Next, using boundedness of $\phi_v(x, y)$ with respect to $\| \|_{\delta'', \rho''}^{\delta, \rho}$ we arrive at (3.2), for any k, k', x, y and $q \geq q_0, q' \geq q_0'$.

We now examine the remaining case when $k < k_0$ and

$$k' < k_0', q < q_0, q' < q_0'.$$

For $k < k_0, k' < k_0', | x | > 1, | y | > 1$, we have for any q, q' and x, y by virtue of (3.2),

$$| x^k D_x^q y^{k'} (y^{-1} D_y)^{q'} y^{-\mu - 1/2} \phi_v(x, y) |$$

$$= \frac{| x |^{k_0} | y |^{k_0'}}{| x |^{k_0 - k} | y |^{k_0' - k'}} \quad | D_x^q (y^{-1} D_y)^{q'} y^{-\mu - 1/2} \phi_v(x, y) |$$

$$\leq \frac{1}{| x | | y |} \eta (A + \delta)^{k_0} (B + \rho)^q (A' + \delta')^{k_0'} (B' + \rho')^{q'} a_{k0} b_q a_{k0}' b_q'.$$

For sufficiently large $| x |$, say $| x | > x_0$, and $| y |$, say $| y | > y_0$, we obtain

$$\frac{(A + \delta)^{k0}}{| x |} \frac{(A' + \delta')^{k0'}}{| y |} a_{k0} b_q a_{k0}' b_q' \leq (A + \delta)^k (A' + \delta')^{k'} a_k b_q a_k' b_q'$$

$$(k' = 0, 1, 2, ..., k_0' \quad 1, q' = 0, 1, 2, ..., q_0' - 1)$$

and therefore for $q < q_0, q' < q_0', k < k_0, k' < k_0'$, the inequality (3.2) is satisfied.

Finally, if $k < k_0, k' < k_0'$ then for fixed $\delta, \delta', \rho, \rho'$ constants

$$(A + \delta)^k (B + \rho)^q (A' + \delta')^{k'} (B' + \rho')^{q'} a_k b_q a_k' b_q' \text{ are bounded by some number } C_2.$$

Since the sequence $\{|D_x^q (y^{-1} D_y)^{q'} y^{-\mu-1/2} \phi_v(x, y)|\}$ tends to zero for $-x_0 \le x \le x_0$, $0 \le y \le y_0$ as $v \to \infty$ for given $\eta > 0$ there exists v_0 sufficiently large such that for $v > v_0$, the inequality (3.2) holds. Then, for $v > v_0$, the inequality (3.2) is satisfied for all x, y, k, k', q, q'.

Consequently, for $v > v_0$, $\| \phi_v \|_{\delta', \rho'}^{\delta, \rho} \le \eta$, from which it also follows that the sequence $\{\phi_v\}$ tends to zero in the topology of the space $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$ as $v \to \infty$.

(*e*) If the sequence $\{\phi_v(x, y)\}$ is bounded in each of the norms $\| \phi_v \|_{\delta', \rho'}^{\delta, \rho}$, and converges correctly to zero, it tends to zero in the topology of the space $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$.

Let $\delta, \delta', \rho, \rho'$ and an arbitrary $\eta > 0$ be given.

Choose $\delta' < \delta, \rho' < \rho, \delta'' < \delta', \rho'' < \rho'$.

The numbers $\| \phi_v \|_{\delta', \rho'}^{\mu, \delta, \rho}$ are bounded by the constant $C_{\delta', \rho''}^{\mu, \delta, \rho}$.

For sufficiently large q, q', k, k' say $q_0 \ge q_0', k_0 \ge k_0'$ respectively, the inequality

$$\frac{(A + \delta')^k (B + \rho')^q (A' + \delta'')^{k'} (B' + \rho'')^{q'}}{(A + \delta)^k (B + \rho)^q (A' + \delta')^{k'} (B' + \rho')^{q'}} < \frac{\eta}{C_{\delta', \rho'}^{\mu, \delta, \rho}} \text{ holds.}$$

Hence, for $k \le k_0, k' \le k_0', \ q \le q_0 \, q' \ge q_0'$, we have

$$|x^k D_x^q y^{k'} (y^{-1} D_y)^{q'} y^{-\mu-1/2} \phi_v(x, y)|$$

$$\le \eta (A + \delta)^k (B + \rho)^q (A' + \delta')^{k'} (B' + \rho')^{q'} a_k b_q a_k' b_q'.$$

For $k \le k_0, k' \le k_0', \ q \le q_0 \, q' \ge q_0'$ respectively and

$$|x||y| > H_1^{k_0 + k_0'} (C_{\delta', \rho''}^{\mu, \delta, \rho} / \eta),$$

where $$C_{\delta', \rho''}^{\mu, \delta, \rho} = a_1 R_1 H_1 C_{\delta', \rho''}^{\mu, \delta, \rho} (A + \delta)(A' + \delta')$$

We have,

$$|x^k D_x^q y^{k'} (y^{-1} D_y)^{q'} y^{-\mu-1/2} \phi(x, y)|$$

$$= \frac{1}{|x||y|} \frac{|x^{k+1} D_x^q y^{k+1} (y^{-1} D_y)^{q'} y^{-\mu-1/2} \phi_v(x, y)|}{(A + \delta)^{k+1} (B + \rho)^q (A' + \delta')^{k'+1} (B' + \rho')^{q'} a_{k+1} b_q}$$

$$X (A + \delta)^{k+1} (B + \rho)^q (A' + \delta')^{k'+1} (B' + \rho')^{q'} a_{k+1} b_q$$

$$\leq a_1 R_1 \underset{|x||y|}{H_1^{k+k'+2}} \| \phi \|_{\delta', \rho'}^{\mu, \delta, \rho} (A+\delta)^k (B+\rho)^q (A'+\delta')^{k'} (B'+\rho')^{q'} a_k b_q a_k' b_q'$$

$$\leq \eta (A+\delta)^k (B+\rho)^q (A'+\delta')^{k'} (B'+\rho')^{q'} a_k b_q a_k' b_q'.$$

Finally, if $k < k_0, k' < k_0', q < q_0, q' < q_0'$ and

$$|x||y| \leq H_q^{k_0+k_0} (C_{\delta', \rho''}^{\mu, \delta, \rho} / \eta)$$

then by virtue of uniform convergence of the sequence

$$\{| D_x^q (y^{-1} D_y)^{q'} y^{-\mu-1/2} \phi_v (x, y) |\}$$

the inequality

$$| x^k D_x^q y^{k'} (y^{-1} D_y)^{q'} y^{-\mu-1/2} \phi_v (x, y) |$$

$$\leq \eta (A+\delta)^k (B+\rho)^q (A'+\delta')^{k'} (B'+\rho')^{q'} a_k b_q a_k' b_q' \quad .. (3.3)$$

will alsohold for sufficiently larve $v, v \geq v_0$

Therefore, for $v \geq v_0$, the inequality (3.3) holds for all x, y, k, k', q, q'.

For $v \geq v_0$,

$$\| \phi_v \|_{\delta', \rho'}^{\delta, \rho} = \underset{\substack{x, y \\ q, q' \\ k, k'}}{\text{Sup}} \frac{| x^k D_x^q y^{k'} (y^{-1} D_y)^{q'} y^{-\mu-1/2} \phi (x, y) |}{(A+\delta)^k (B+\rho)^q (A'+\delta')^{k'} (B'+\rho')^{q'} a_k b_q a_k' b_q'} < \eta,$$

from which it follows that $\phi_v (x, y) \to 0$ in the topology of $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$

(*f*) If the sequence $\{\phi_v (x, y)\}$ is bounded in each of the norms $\| \|_{\delta', \rho'}^{\delta, \rho}$, and converges correctly to some function $\phi (x, y)$ then

$$\phi_v (x, y) \quad \varepsilon \quad FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$$

and $Q(x, y)$ is the limit of the sequence $\{\phi_v (x, y)\}$ in the topology of the space $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$

Now, $\phi_v (x, y) \quad \varepsilon \quad FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$ by virtue of (*a*).

The difference $\{\phi (x, y) - \phi_v (x, y)\}$ is bounded in all the norms and converges to zero; according to (*b*) the difference converges to zero in the topology of the space $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$.

This completes the proof.

Similarly the other spaces can also be shown to be the complete countably normed spaces. Moreover, by invoking the theorem due to Zemanian[6] [pp. 21-23], we infer that the corresponding dual spaces are also complete.

Now, we define a countable union space as follows:

Let $A_1 < A_2$ and $B_1 < B_2$; then the space $FH_{\mu, ak, A1, ak', A1'}^{bq, B_1, bq', B_1'}$ is a subspace of $FH_{\mu, ak, A2, ak', A2'}^{bq, B_2, bq', B_2}$.

Further, the convergence of a sequence $\{\phi_\nu (x, y)\}$ in

$FH_{\mu, ak, A1, ak', A1'}^{bq, B_1, bq', B_1'}$ implies the convergence in $FH_{\mu, ak, A2, ak', A2'}^{bq, B_2, bq', B_2'}$.

Hence, we may construct the union of countably normed spaces $FH_{\mu, ak, A, ak', A'}^{bq, B, bq', B'}$, for all indices A, B = 1, 2, ...

This union coincides with the space $FH_{\mu, ak, A, ak', A2'}^{bq, B_2, bq', B_2'}$.

$$FH_{\mu, ak, ak'}^{bq, B, bq', B'} = \bigcup_{A, B = 1}^{\infty} FH_{\mu, ak, A, ak', A'}^{bq, B, bq', B'}.$$

Similarly, we define

$$FH_{\mu, ak, ak'} = \bigcup_{A = 1}^{\infty} FH_{\mu, ak, A, ak', A'}$$

and

$$H_\mu^{bq, B, bq'} = \bigcup_{B = 1}^{\infty} H_\mu^{bq, B, bq', B'}.$$

The elements of the spaces $FH_{\mu, ak, ak'}$, $FH_{\mu, ak, A, ak', A'}$, $H_\mu^{bq, B, bq', B'}$, $H_\mu^{bq, B, bq'}$, $FH_{\mu, ak, ak'}^{bq, B, bq', B'}$ are called ultra-differentiable functions and those of corresponding dual spaces are called ultra-distributions.

4. Some Differential and Integral Operators

Following Zemanian[6] we define the following operators.

$$N_\mu \phi (x, y) = y^{\mu + 1/2} D y^{-\mu - 1/2} \phi (x, y)$$

$$M_\mu \phi (x, y) = y^{-\mu - 1/2} D y^{\mu + 1/2} \phi (x, y)$$

$$N_\mu \phi(x, y) = y^{\mu+1/2} \int_\infty^y t^{-\mu-1/2} \phi(x, t)\, dt$$

From which follows

$$S_\mu = N_\mu M_\mu = D^2 - (4\mu^2 - 1)/4x^2$$

We now study these operators on the above spaces.

***Theorem* 4.1** — *The operation $\phi \to N_\mu \phi$ is a continuous linear mapping $FH_{\mu, ak, A, ak', A'}$ into $FH_{\mu+1, ak, A, ak', A'}$. If b_q' satisfies (C.1) then the operation $\phi \to N_\mu \phi$ is a continuous linear mapping $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$ into $FH_{\mu+1, ak, A, ak', A'}{}^{bq, B, bq', B', H'}$.*

PROOF : $\phi \in FH_{\mu, ak, A, ak', A'}$, we have

$$| x^k D_x^q y^{k'} (y^{-1} D_y)^{q'} y^{-\mu-1/2} (N_\mu \phi(x, y)) |$$

$$= | x^k D_x^q y^{k'} (y^{-1} D_y)^{q'+1} y^{-\mu-1/2} \phi(x, y) |$$

$$\leq C C_q^\mu (A + \delta)^k (A' + \rho')^{k'} a_k a_{k'}.$$

The proof of the remaining part is similar.

***Theorem* 4.2** — *(a) Let a_k' satisfy (C.5) and b_q' satisfy (C.1) then the operation $\phi \to M_\mu \phi$ is a continuous linear mapping from $FH_{\mu+1, ak, A, ak', A'}$*

$$\left(FH_{\mu+1}{}^{bq, B, bq', B'} \right) \text{ into } FH_{\mu, ak, A, ak', A', H1'} \left(FH_\mu{}^{bq, B, bq', B' H'} \right).$$

(b) Let a_k' satisfy (C.5) and b_q' satisfy (C.1) then the operation $\phi \to M_\mu \phi$ is a continuous linear mapping from $FH_{\mu+1, ak, A, ak', A'}{}^{bq, B, bq', B'}$ into $FH_{\mu+1, ak, A, ak', A'}{}^{bq, B, bq', B', H'}$.

PROOF : $\phi \in FH_{\mu+1, ak, A, ak', A'}$, we have

$$| x^k D_x^q y^{k'} (y^{-1} D_y)^{q'} y^{-\mu-1/2} (M_\mu \phi(x, y)) |$$

$$= | x^k D_x^q y^{k'} (y^{-1} D_y)^{q'+1} y^{-\mu-1/2} \phi(x, y) |$$

$$\leq C C_q^\mu (A + \delta)^k (A' + \rho')^{k'} a_k a_{k'}.$$

Hence, the result follows.

The proof of (*b*) is similar.

***Theorem* 4.3** — *The operation $\phi \to N_\mu \phi$ is a continuous linear mapping from $FH_{\mu+1, ak, A, ak', A'}$ into $FH_{\mu, ak, A, ak', A' H1'}$. If b_q' satisfies (C.1) then the operation $\phi \to N_\mu^{-1} \phi$ is a continuous linear mapping $FH_{\mu+1}{}^{bq, B, bq', B'}$*

$$\left(FH_{\mu+1, ak, A, ak', A'}{}^{bq, B, bq', B'} \right)$$

into $FH_\mu{}^{bq, B, bq', B'} \left(FH_{\mu+1, ak, A, ak', A'}{}^{bq, B, bq', B'} \right)$.

PROOF : We prove the last part of the theorem; the other two parts can be similarly proved.

$$\phi \in FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'},$$

we have

$$| x^k D_x^q y^{k'} (y^{-1} D_y)^{q'} y^{-\mu-1/2} (N_\mu^{-1} \phi(x, y)) |$$

$$= | x^k D_x^q y^{k'} (y^{-1} D_y)^{q'-1} y^{-(\mu+1)-1/2} \phi(x, y) |$$

$$\le C C_{q'}^{\mu+1} (A+\delta)^k (B+\rho)^q (A'+\rho')^{k'} (B'+\rho')^{q'-1} a_k b_q a_k' b_{q-1}'$$

$$\le C C_{q'}^{\mu} (A+\delta)^k (B+\rho)^q (A'+\rho')^{k'} (B'+\rho')^{q'} a_k b_q a_k' b_q'.$$

Theorems 4.1 and 4.2 imply that if a_k' satisfies (C.5) and b_q' satisfies (C.1) then the operation $S_\mu = N_\mu M_\mu$ is a continuous linear mapping from

$$FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'} \text{ into } FH_{\mu, ak, A, ak', A' H'}{}^{bq, B, bq', B' H'}.$$

Similar results hold for other two spaces also.

Operations in Dual spaces:

In the dual spaces $FH'_{\mu, ak, A, ak', A'}$, $FH_{\mu,}{}^{bq, B, bq', B'}$, $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$, N_μ is defined as the adjoint of $-M_\mu$ and M_μ is defined as the adjoint of $-N_\mu$. More precisely, N_μ is defined as a generalized differential operator on the above dual spaces by

$\langle N_\mu f, \phi \rangle = \langle f - M_\mu \phi \rangle$, wehre ϕ belongs to $FH_{\mu+1, ak, A, ak', A'}$ or $FH_{\mu+1}{}^{bq, B, bq', B'}$, or $FH_{\mu+1, ak, A, ak', A'}{}^{bq, B, bq', B'}$ and f belongs to $FH'_{\mu, ak, A, ak', A' H1'}$ or $FH_{\mu,}{}^{bq, B, bq', B'}$ or $FH_{\mu, ak, A, ak', A' H1'}{}^{bq, B, bq', B' H'}$.

On the other hand, M_μ is defined as a generalized differential operator on the dual spaces by

$\langle M_\mu f, \phi \rangle = \langle f - N_\mu \phi \rangle$, wehre ϕ belongs to $FH_{\mu, ak, A, ak', A'}$ or $FH_\mu{}^{bq, B, bq', B'}$, or $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$ and f belongs to $FH'_{\mu+1, ak, A, ak', A' H1'}$ or $FH_{\mu+1}{}^{bq, B, bq', B' H'}$ or $FH_{\mu, ak, A, ak', A' H1'}{}^{bq, B, bq', B' H'}$.

Now, invoking theorem due to Zemanian[6] [p. 21-23], and Theorems 4.2 and 4.1 to the above definitions, we get.

***Theorem* 4.4** — *(a) The operation $f \to N_\mu f$ is a continuous linear mapping $FH_{\mu, ak, A, ak', A' H1'}$ into $FH_{\mu+1, ak, A, ak', A'}$ if a_k satisfies (C.5), if b'_q satisfies (C.1).*

$FH_\mu{}^{bq, B, bq', B' H'}$ into $FH_{\mu+1}{}^{bq, B, bq', B'}$ if b_q satisfies (C.4), and of $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B' H'}$ into $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$ if a_k satisfies (C.5) and b_q satisfies (C.4).

(b) The operation $f \to M_\mu f$ is a continuous linear mapping $FH_{\mu+1, ak, A, ak', A'}$ onto $FH_{\mu, ak, A, ak', A'}$, $FH_{\mu+1}{}^{bq, B, bq', B' H'}$ into $FH_\mu{}^{bq, B, bq', B'}$ if b_q satisfies (C.4), and if a_k satisfies (C.5), if b'_q satisfies (C.1), $FH_\mu{}^{bq, B, bq', B' H}$ into.

$FH_{\mu+1, ak, A, ak', A'}{}^{bq, B, bq', B' H'}$ into $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$ if b_q satisfies (C.4).

(c) The operation $f \to S_\mu f$ is a continuous linear mapping of $FH_{\mu, ak, A, ak', A' H1'}$ onto $FH_{\mu, ak, A, ak', A'}$ if a_k satisfies (C.5), $FH_\mu{}^{bq, B, bq', B' H'}$ into $FH_\mu{}^{bq, B, bq', B'}$ if b_q satisfies (C.4), and if a_k satisfies (C.5), if b'_q satisfies (C.4).

$FH_{\mu, ak, A, ak', A' H1}{}^{bq, B, bq', B' H'}$ into

$FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$ if b'_q satisfies (C.4).

5. Fourier-Hankel Transformation of Test Functions

Now, in this section we consider the mapping of the aforesaid ultra-differentiable functions by fh_μ. It is easily seen that the Fourier-Hankel transform $fh_\mu \phi$ exists for each test function ϕ in $FH_{\mu, ak, ak'}$, $FH_{\mu, ak, A, ak', A'}$, $FH_\mu{}^{bq, B, bq', B'}$, $FH_\mu{}^{bq, B, bq'}$, $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$ when $\mu \geq -1/2$.

***Theorem* 5.1** — *If a_k, a'_k satisfy the condition (C.5) then for $\mu \geq -1/2$, the conventional Fourier-Hankel transform fh_μ defined by (2.1), is a continuous linear mapping from the space $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$ into the space*

$$FH_{\mu, ak, A, ak', A'}{}^{q_q, a_q, 2 B'_1} \text{ where } A'_1 = A' B' H'^2_q \text{ and } B'_q = A'^2 H'^6_1.$$

PROOF : Let K be a bounded st in $FH_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$. Then every ϕ in K satisfies the inequality

$$| x^k D_x^q y^{k'} (y^{-1} D_y)^{q'} y^{-\mu - 1/2} \phi(x, y) |$$

$$\leq C^{\mu} (A + \delta)^k (B + \rho)^q (A' + \delta')^{k'} (B' + \rho')^{q'} a_k b_q a'_k b_q$$

for all $q, q' \varepsilon \mathbb{N}_0$ and $k, k' = 0, 1.2, \ldots$.

Let $\Phi(\xi, t) = fh_{\mu} \phi(x, y)$.

For any pair of non-negative integers k' and q', from Zemanian[6] [p. 139],

$N_{\mu + q' + k' - 1} \cdots N_{\mu + q'} \phi(x, y) = y^{q'} N_{\mu + q' - 1} \cdots N_{\mu} \phi(x, y)$ and using q'-times

$fh_{\mu + 1}(-y\,\phi) = N_{\mu} fh_{\mu} \phi$ we get

$$N_{\mu + q' + k' - 1} \cdots N_{\mu} \Phi = (-1)^{q'} fh_{\mu + q'} (y^{q'} \phi(x, y))$$

Next apply k' times, $fh_{\mu}(N_{\mu} \phi) = -(t) fh_{\mu} \phi$,

we get

$$N_{\mu + q' + k' - 1} \cdots N_{\mu} \Phi = (-1)^{q'} (-1/t)^{k'} fh_{\mu + q' + k' - 1} \cdots N_{\mu} \phi(x, y))$$

So that

$$(-t)^{k'} N_{\mu + q' + k' - 1} \cdots N_{\mu} \Phi(\xi, t) = \int_{-\infty}^{\infty} \int_{0}^{\infty} (-y)^{q'} [N_{\mu + q' + k' - 1} \cdots N_{\mu} \phi(x, y)$$

$$X e^{i \xi x} \sqrt{y}\, J_{\mu + q + k}(y, x)\, dx\, dy.$$

Now, the proof is similar to the proof of Theorem 4.1.1 of Lee[3].

Thus, we can write:

$$(-1)^{k' + q'} \xi^k D_{\xi}^q t^{k'} (t^{-1} D_t)^{q'} t^{-\mu - 1/2} \Phi(\xi, t)$$

$$= \int_{-\infty}^{\infty} \int_{0}^{\infty} (y)^{2\mu + 1 + k' + 2q'} x^k D_x^q y^{k'} (y^{-1} Dy)^{q'} y^{-\mu - 1/2} \phi(x, y)$$

$$X e^{i \xi x} yt^{-\mu - q'} J_{\mu + q' + k'}(y\, t)\, dx\, dy.$$

Now, we assume that v, v' are positive integers such that $v' \geq 2\mu + 1$ set $n = v + 2q + k$ and use the fact that $\left| (z)^{-\mu - q'} J_{\mu + q' + k'}(z) \right| \leq C$, and the estimate from[4] [p. 107] and the Theorem 3.11.1 from[4] [p. 107],

$$\left| \xi^k D_\xi^q \Phi(\xi, t) \right| \le C A^q (1+\delta)^{q+2} (2B)^k a_q b_k \text{ for all } k, q \ge \nu,$$

we obtain the following estimate.

$$\left| (-1)^{k'+q'} \xi^k D_\xi^q t^{k'} (t^{-1} D_t)^{q'} t^{-\mu-1/2} \Phi(\xi, t) \right|$$

$$\le C A^q (1+\delta)^{q+2} (2B)^k a_q b_k C_1^\mu (B'+\rho')^{q'} q_q'$$

$$[(A'+\rho')^{k'+2q'} a_{k'+2q'}' + (A'+\delta')^{n+2} a_{n+2}]$$

$$\le C A^q (1+\delta)^{q+2} (2B)^k a_q b_k C_1^\mu (B'+\rho')^{q'} q_q'$$

$$(H_1' (A'+\delta'))^{k'+2q'} a_{k'+2q'}$$

$$\times [1 + R_q' (A+\delta')^{k'} H_q'^{\nu,+2q'} a_{\nu'+2q'}]$$

$$\le C A^q (1+\delta)^{q+2} (2B)^k a_q b_k C_2^\mu (B'+\rho')^{q'} q_q'$$

$$(H_1' (A'+\delta'))^{k'+2q'} a_{k'+2q'}$$

$$\le C A^q (1+\delta)^{q+2} (2B)^k a_q b_k C_2^\mu R_q'^2$$

$$(A' B' H_1'^2 + \rho')^{k'} a_{k'} b_{q'} (A'^2 H_1'^6 + \delta')^{q'} a_{q'}^2.$$

This completes the proof.

***Theorem* 5.2** — *If $a_k, a_{k'}$ satisfy the condition $a_p \le R_q H_q^p$ min $a_q a_{p-q}$ $p \in \mathbb{N}_0$ $0 \le q \le p$ then for $\mu \ge -1/2, Fh_{\mu'}$ is continuous linear mapping from the space*

$FH_{\mu, ak, A, ak', A'}$ into the space $FH_\mu^{a_q, a_q^{-1}, B_1'}$, where $B_q' = A'^2 H_q'^6$.

PROOF : Following the procedure of the proof of the above theorem we get

$$\left| (-1)^{k'+q'} \xi^k D_\xi^q t^{k'} (t^{-1} D_t)^{q'} \tau^{-\mu-1/2} \Phi(\xi, t) \right|$$

$$\le C A^q (1+\delta)^{q+2} (2B)^k a_q b_k C_{k'}^\mu (A^2 H_1^6 + \delta') q' a_{q'}^2.$$

6. Generalised Fourier-Hankel Transformation of the Ultra-Distributions and its Inversion

For $\mu \ge -1/2$ we define the generalized F-H transformation fh_μ' on each of the dual spaces $FH_{\mu, ak, ak'}'$, $FH_{\mu, ak, A, ak', A'}$, $FH_\mu^{bq, B, bq', B'}$, $FH_\mu^{bq, B, bq'}$, and $FH_{\mu, ak, A\, ak' A'}'^{\,bq, B, bq', B'}$ as follows:

$$\langle F, \Phi \rangle = \langle f, \phi \rangle \qquad \dots (6.1)$$

where $\Phi = fh_\mu \phi$, $F = fh'_\mu \phi$, ϕ belongs to FH_μ and f belongs to the corresponding dual space.

The generalized Fourier transform of $f \varepsilon D'$ is defined to be the element $F \varepsilon Z'$ such that the generalized Parseval relation

$$\langle F, \psi \rangle = (2\pi)^n \langle f, \phi^\nu \rangle$$

where $f \varepsilon FH'_\mu, \phi^\nu \varepsilon FH_\mu$

$$\Phi = fh_\mu, \phi, \quad F = fh'_\mu, f$$

and

$$\Phi \varepsilon FH_\mu, \phi^\nu \varepsilon FH_\mu.$$

Thus, fh_μ on FH'_μ is the adjoint of the mapping $\Phi \to (2\pi)^n \phi^\nu$

Since $f^{-1}[f] = (2\pi)^{-n} [f(f^\nu)]$

and

$h'_\mu = h_\mu^{-1}$, we also have

$$\langle fh'_\mu f, \Phi \rangle = (2\pi)^{-1} \langle f, fh'_\mu \Phi \rangle \qquad \dots (6.2)$$

The inverse Fourier-Hankel transform can therefore be defined as:

$$f = (fh'_\mu)^{-1} F, \; \mu \geq -1/2.$$

Now, applying theorem due to Zemanian[6] [pp. 21-23] to Theorems 5.1 and 5.2 above and in view of definition (6.2) above, we can state the following theorems.

Theorem 6.1 — *Let $\mu \geq -1/2$. If a_k, a'_k satisfy the condition C.5, then the generalized Fourier-Hankel transform fh'_μ is a continuous linear mapping from the dual space*

$$FH'_{\mu, bk, B'}{}^{a_q, q_q, 2, B_1^2} \text{ into } FH'_{\mu, ak, A}{}^{bq, B, bq', B'},$$

where $A'_1 = A' B' H_1'^2$ and $B'_1 = A'^2 H_1'^6$.

Theorem 6.2 — *Let $\mu \geq -1/2$. If a_k, a'_k satisfy the condition C.5, then the generalized Fourier-Hankel transform fh'_μ is a continuous linear mapping from the dual space*

$FH_{\mu, bk, B, ak', bk' A1'}{}^{a_q, a_q'^2, B'_1}$ into $FH'_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$, where $A'_1 = A' B' H_1'^2$ and $B'_1 = A'^2 H_1'^6$.

7. An Operational Calculus

The distributional Fourier-Hankel transform generates an operational calculus by means of which certain differential equations involving generalized functions can be solved. We now consider the differential equation:

$$P(D_x^k, S_{\mu,y}^{k'})\,u = g \qquad \text{... (7.1)}$$

where $P(x, y)$ is a polynomial having no zeros on $-\infty < x, y < 0$, g is a given member of $FH'_{\mu, ak, ak'}$ or $FH'_{\mu, ak, A\,ak'A'}$ or $FH'_{\mu}{}^{bq, B, bq', B'}$, or $FH'_{\mu}{}^{bq, B, bq'}$ or $FH'_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$, P is a polynomial such that

$$P((-i\xi), (-t^2) \neq 0$$

and u is an unknown generalized function which is to be determined.

Using

$$fh'_{\mu}\,((D_x^k, S_{\mu,y}^{pk'})f) = (-i\xi)^k (-t^2)^{k'} fh'_{\mu}(f) \text{ and applying } fh'_{\mu} \text{ to (7.1),}$$

we obtain

$$(P((-i\xi)^k\ (-t^2)^{k'})\ U\ (\xi, t) = G(\xi, t) \qquad \text{... (7.2)}$$

where U and G are distributional Fourier-Hankel transforms of u and g respectively.

Since $P((-i\xi)^k, (-t^2)^{k'})$ is a {italic mu}ltiplier in $FH_{\mu, ak, ak'}$, $FH_{\mu, ak, A\,ak'A'}$, $FH'_{\mu}{}^{bq, B, bq' B'}$, $FH'_{\mu}{}^{bq, B, bq'}$, $FH'_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$, $1/P((-i\xi)^k, (-t^2)^{k'})$ is a multiplier in the corresponding dual space for a_k, a'_k satisfying the condition (C.5) and b_q, b'_q satisfying the condition (C.1).

Therefore,

$$U(\xi, t) = G(\xi, t)/P((-i\xi)^k, (-t^2)^{k'})$$

By taking the generalized inverse Fourier-Hankel transform $\left(fh'_{\mu}\right)^{-1}$, the solution is given by

$$u(x, y) = \left(fh'_{\mu}\right)^{-1} F(\xi, t)\,/P((-i\xi)^k, (-t^2)^{k'})$$

This means that for each testing function ϕ belonging to one of the spaces $FH_{\mu, ak, ak'}$ or $FH_{\mu, ak, A\,ak'A'}$, or $FH'_{\mu}{}^{bq, B, bq' B'}$, $FH'_{\mu}{}^{bq, B, bq'}$, $FH'_{\mu, ak, A, ak', A'}{}^{bq, B, bq', B'}$, P is polynomial such that $P(-i\xi, -t^2) \neq 0$, the unknown u belonging to the corresponding dual space is given by

$$\langle u, \left(fh'_{\mu}\right)^{-1} \Phi\rangle = \langle F(\xi, t)/P((-i\xi)^k, (-t^2)^{k'}), \Phi(\xi, t)\rangle$$

$$= \langle F(\xi, t)\, \Phi(\xi, t)/P((-i\xi)^k, (-t^2)^{k'})\rangle,$$

where $\Phi = fh_\mu \phi.$

REFERENCES

1. I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Vol. 2, Academic Press, New York, (1968).
2. H. Komatsu, *J. Fac. Sci. Uni. Tokyo Sec.*, IA **20** (1973), 25-105.
3. W. Y. Lee, *Siam. J. Math. Analysis*, **5**(2) (1974), 336-48.
4. R. S. Pathak, *Integral Transformations of Generalized Functions and their Applications*, Gordon and Breach Science Publishers, Amsterdam, (1997).
5. C. Roumieu, *J.d' Analyse Math.*, **10** (1962-63), 153-92.
6. A. H. Zemanian, *Generalized Integral Transformations*, Interscience Publ., New York, (1968).