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NON-LINEAR SET COVERING PROBLEM

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In this paper an algorithm to solve a non-linear set covering problem is developed. It is divided into three sections. In section I a BI NON-LINEAR Set Covering Problem is discussed. In section II a Quadratic Fractional Set Covering Problem is discussed. Both the objectives are shown to be quasi-concave and hence the optimal solution is at an extreme point of the convex polyhedron formed by the feasible set. The problems are solved by forming the related linear (linear fractional) set covering problem and by enumerating the extreme points. It is illustrated with the help of examples given in section III.

Key Words: Non-Linear Fractional; Binonlinear; Quasi-Concave; Set Covering; Extreme Point

1. INTRODUCTION

The set covering problem with linear objective function has been studied by Garfinkel and Nemhauser[1], Roth[2], Lemke, Salkin and Speilberg[3], Bellmore and Ratliff[4], Chavatal[5] and Balas[6]. They discussed and developed various enumeration techniques for solving the set covering problem, which has many applications such as location of emergency facilities, truck deliveries, political districting, air line crew scheduling etc. This work has been further extended with linear fractional function as objective functions by Arora and Puri[7]. They developed an enumeration technique using branch and bound method to solve this problem. Arora and Saxena[8] developed a cutting plane technique for the multi-objective set covering problem with linear fractional objective function. Gupta and Puri[9] developed an extreme point technique for solving a quadratic programming problem.

In this paper an enumerative technique is developed to solve the aforementioned quadratic fractional set covering problem. For solving this problem extreme points of the convex polytope are ranked in a non decreasing order of the values of the quadratic fractional objective function. A linear fractional set covering problem related to the main problem is formulated and its extreme points are scanned in a systematic manner till an optimal solution of the given problem is obtained.

THEORETICAL DEVELOPMENT

Consider a set $I = [1, 2, ..., m]$ and a set $P = [P_1, P_2, ..., P_n]$ where $P_j \subseteq I$, $j \in J = [1, 2, ..., n]$.

A subset J^* of J is said to be a cover of I if $\bigcup_{j \in J^*} P_j = I$.

Let a cost $c_j > 0$ be associated with every $j \in J$. The total cost of the cover J^* is equal to $\sum_{j \in J^*} c_j$.

The set covering problem (CP) is to find a cover of minimum cost and can be written as

$$\text{Min } z = \sum_{j=1}^{n} c_j x_j$$

subject to $$\sum_{j=1}^{n} a_{ij} x_j \geq 1 \,;\, i \in I \qquad \dots (1)$$

$$x_j = 0 \text{ or } 1;\; j \in J \qquad \dots (2)$$

where

$$x_j = \begin{cases} 1 & \text{if } P_j \text{ is in the cover} \\ 0 & \text{otherwise} \end{cases}$$

$$a_{ij} = \begin{cases} 1 & \text{if } i \in P_j \\ 0 & \text{otherwise} \end{cases}$$

Definitions:

1. Cover Solution: A solution X which satisfies (1) and (2) is said to be a cover solution.

2. Redundant cover : For any cover J', a column $j^* \in J'$ is said to be redundant, if $J' - \{j^*\}$ is also a cover. If a cover contains one or more redundant column it is called a redundant cover. Column j^* is redundant with respect to the cover J' iff.

$$\sum_{j \in J'} a_{ij} \geq 2, \text{ for all } i \in P_{j^*}.$$

3. Prime Cover: A cover that is not redundant is called a prime cover. Column $\hat{j} \in J'$ is not redundant iff.

$$I(\hat{j}) = \left\{ i : \sum_{j \in J'} a_{ij} = 1,\;\; i \in P_{\hat{j}} \right\} \neq \phi.$$

4. Quasi-Concave Function: A function $F : E^n \to E^1$ is called quasi-concave if given $x^1, x^2 \in E^n, F[\lambda x^1 + (1-\lambda) x^2] \geq \min \{F(x^1), F(x^2)\}$ for $0 \leq \lambda \leq 1$.

If $F(x^1) > F(x^2)$ then $F[\lambda x^1 + (1-\lambda) x^2] \geq F(x^2)$ for $0 \leq \lambda \leq 1$.

Notations

x_j = jth component of x.

$\hat{F}_1$ = Optimal objective function value of the problem (BCP).

$\hat{G}_i$ = The value of $G(x)$ at an ith best extreme point solution of the problem (CP) obviously

$i = 1, 2, \ldots, N$ where $\hat{G}_N = \underset{X \in S'}{\text{Max}} \; G(x)$.

$\hat{S}_i = \{X_i^1\}$, is the set of all the ith best extreme point solutions of (CP).

$$\hat{T}^s = \bigcup_{i=1}^{s} \hat{S}_i, \; s = 1, 2, \ldots, N.$$

SECTION I : BINONLINEAR SET COVERING PROBLEM

The mathematical model of the Binonlinear set covering problem (BCP) is

$$\text{Min } F(x) = (CX + X^T \in X + \alpha)\,(DX + \beta)$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij} x_j \geq 1 \,;\, i \in I$$

$$x_j = 0 \text{ or } 1; \quad j \in J$$

$C^T, D^T \in R^n$, $\alpha, \beta \in R : E$ is an $n \times n$ negative semidefinite matrix.

It is assumed that $CX + X^T EX + \alpha \geq 0$ and $DX + \beta > 0 \;\; \forall X \in S'$

where

$$S' = \left\{ X = (x_1, x_2, \ldots, x_n); \; \sum_{j=1}^{n} a_{ij} x_j \geq 1, x_j = 0 \text{ or } 1, i \in I, j \in J \right\}$$

$F(X)$ is called Binonlinear function.

The related Binonlinear programming problem (BCP′) is obtained by replacing the condition $x_j = 0$ or 1 by $x_j \geq 0 \;\; \forall j \in J$.

$$\min_{x \in S''} \; F(x) = (CX + X^T E X + \alpha)\,(DX + \beta)$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij} x_j \geq 1 \,;\, i \in I$$

$$x_j \geq 0; \quad j \in J$$

where

$$S'' = \left\{ x = (x_1, x_2, \ldots, x_n); \; \sum_{j=1}^{n} a_{ij} x_j \geq 1, x_j \geq 0, \; i \in I, j \in J \right\}.$$

It is assumed that the feasible set S'' is regular.

Theorem 1.1 — *The function $F(X) = (CX + X^T EX + \alpha)(DX + \beta)$ is quasi-concave over S'', where E is a negative semidefinite matrix and $CX + X^T EX + \alpha \geq 0$ and $DX + \beta > 0$ $\forall X \in S''$.*

PROOF : $f(X) = CX + X^T EX + \alpha$ is a concave function because E is negative semidefinite matrix $g(X) = DX + \beta$ is both concave as well as convex.

$$\therefore f[\lambda X_1 + (1-\lambda) X_2] \geq \lambda f(X_1) + (1-\lambda) f(X_2)$$

$$\forall X_1, X_2 \in S', \; 0 \leq \lambda \leq 1.$$

$$g[\lambda X_1 + (1-\lambda) X_2] \geq \lambda g(X_1) + (1-\lambda) g(X_2)$$

Suppose

$$(fg)(X_1) \geq (fg)(X_2)$$

i.e.,

$$f(X_1) g(X_1) \geq f(X_2) g(X_2)$$

Now

$$(fg)[\lambda X_1 + (1-\lambda) X_2] = f[\lambda X_1 + (1-\lambda) X_2] \; g[\lambda X_1 + (1-\lambda) X_2]$$

$$\geq [\lambda f(X_1) + (1-\lambda) f(X_2)] \; [\lambda g(X_1) + (1-\lambda) g(X_2)]$$

$$= \lambda^2 f(X_1) g(X_1) + \lambda (1-\lambda) [f(X_1) g(X_2) + f(X_2) g(X_1)]$$

$$+ (1-\lambda)^2 f(X_2) g(X_2)$$

$$\geq \lambda^2 f(X_2) g(X_2) + (1-\lambda)^2 f(X_2) g(X_2)$$

$$+ \lambda (1-\lambda) \left[2 \left\{ f(X_1) g(X_1) f(X_2) g(X_2) \right\}^{1/2} \right] \quad [\because A.M. \geq G.M.]$$

$$\geq \lambda^2 f(X_2) g(X_2) + (1-\lambda)^2 f(X_2) g(X_2)$$

$$+ 2\lambda (1-\lambda) f(X_2) g(X_2)$$

$$= [\lambda + (1-\lambda)]^2 f(X_2) g(X_2) = f(X_2) g(X_2)$$

$$\therefore (fg)[\lambda X_1 + (1-\lambda) X_2] \geq f(X_2) g(X_2)$$

$$= \text{Min } [(fg)(X_1), (fg)(X_2)] \; \forall 0 \leq \lambda \leq 1.$$

Hence $F = fg$ is a quasi-concave function.

Cor 1.1 — Optimal feasible solution of the problem (BCP′) lies at one of the extreme points of S''

Theorem 1.2[1] — *There exists a prime cover which is optimal for the set covering problem.*

Theorem 1.3[1] — *If $J' = \{j : X_j = 1\}$ be any prime cover solution, then $X = \{x_j\}$ is an extreme point solution of the set covering problem.*

In the process of finding an optimal solution of the problem (BCP), an integer solution of (BCP′) is calculated. This is done by constructing a related linear set covering problem, which provides the linear bounds on the optimal value of the binonlinear function $F(X)$. The extreme points of the convex polyhedron are ranked in the nondecreasing order of the values of the objective function $G(X)$ of the related linear set covering problem (CP) until we get an integer solution of (BCP′) and hence that of problem (BCP).

The bounding linear set covering problem (CP) related to (BCP) is formulated as

$$\text{(CP)} \qquad \min_{X \in S'} \; G(X) = LX + \delta$$

where

$L = W + \alpha D + \beta(C + U)$ is an n-component row vector

$\delta = \alpha\beta \in R$

U_j = the jth component of U^T

$$= \min_{X \in S'} \; X^T E_j; \; j = 1, 2, \ldots, n$$

E_j being the jth column of E

W_j = the jth component of W^T

$$= \min_{X \in S'} \; X^T \hat{Q}_j \, ; \; j = 1, 2, \ldots, n.$$

$\hat{Q}_j$ being the jth column of an $n \times n$ symmetric matrix $\hat{Q}$ which represents the quadratic form $(CX + UX)(DX)$

i.e., $$(CX + UX)(DX) = X^T \hat{Q} X$$

If we replace the condition $x_j = 0$ or 1 by $x_j \geq 0$ the related problem (CP′) is

$$\text{(CP')} \qquad \min_{X \in S'} \; G(x) = Lx + \delta$$

Theorem 1.4 — $F(X) \geq G(X) \quad \forall X \in S''$

PROOF : For $X \in S''$

$$F(X) = (CX + X^T EX + \alpha)(DX + \beta)$$

$$= \left[CX + \sum_{J=1}^{n} (X^T E_j) X_j + \alpha \right] (DX + \beta)$$

$$\geq (CX + UX + \alpha)(DX + \beta)$$

$$\left\{ \because \; U_j = \min_{X \in S'} X^T E_j; j = 1, 2, ..., n \right\}$$

$$= (CX + UX)(DX) + \alpha (DX) + \beta (CX + UX) + \alpha \beta$$

$$= X^T \hat{Q} X + \alpha (DX) + \beta (CX + UX) + \alpha \beta$$

$$\geq WX + \alpha (DX) + \beta (CX + UX) + \alpha \beta$$

$$\left\{ \because \; W_j = \min_{X \in S'} X^T \hat{Q}_j; j = 1, 2, ..., n \right\}$$

$$= [W + \alpha D + \beta (C + U)] X + \alpha \beta$$

$$= LX + \delta = G(X)$$

Thus

$$F(X) \geq G(X) \quad \forall X \in S''. \qquad \text{... (1.0)}$$

The following Theorem (1.5) gives the characterization of an optimal feasible solution of the problem (BCP).

***Theorem* 1.5** — *If $\hat{G}_k \geq Min \{F(X) : X \in \hat{T}^k\} = F(\hat{X})$ (say). Then $\hat{X}$ is an optimal solution of the problem (BCP).*

PROOF : $F(\hat{X}) = \text{Min} \{F(X) : X \in \hat{T}^k\}$

$$\Rightarrow F(X) \geq F(\hat{X}); \; X \in \hat{S}_i \;\; i = 1, 2, ..., k \qquad \text{... (1.1)}$$

As $\hat{G}_k$ is the value of $G(X)$ at the kth best extreme point solution of (CP).

$$\hat{G}_w > \hat{G}_k, \; w \geq k+1$$

Also $F(Xw_j) \geq G(Xw_j) = \hat{G}_w, Xw_j \in \hat{S}_w, \; w \geq k+1$ [using (1.0)]

Therefore,

$$F(Xw_j) \geq \hat{G}_w \geq \hat{G}_k \geq \text{Min} \{F(X) : X \in \hat{T}^k\} = F(\hat{X}) \qquad \text{... (1.2)}$$

i.e.,

$$F(Xw_j) > F(\hat{X}), \; Xw_j \in \hat{S}_w; \;\; w \geq k+1.$$

(1.1) and (1.2) imply that $F(\hat{X})$ is the least compared to the values of the function 'F' at all the extreme points of S'. Thus $\hat{X}$ is an optimal extreme point solution of (BCP') and $F(\hat{X}) = \hat{F}_1$.

Cor 1.2 — If $\hat{G}_1 = \text{Min}\ \{F(X) : X \in \hat{T}^1\} = F(\hat{X})$ (say). Then $\hat{X}$ is an optimal solution of the problem (BCP′).

Cor 1.3 — If the solution $\hat{X}$ is of the form 0 or 1, then it will be the optimal solution of (BCP). If not, we apply Gomory Cut to find the integer solution.

Remark 1.1 : If $\hat{G}_k = \text{Min}\ \{F(X) : X \in \hat{T}^k\}$, $k \geq 1$ then $\hat{G}_k < \hat{F}_1 \leq \text{Min}\ \{F(X) : X \in \hat{T}^k\}$ i.e. the current lower bound and upper bound on $\hat{F}_1$ are respectively $\hat{G}_k$ and min $\{F(X) : X \in \hat{T}^k\}$.

Remark 1.2 : Suppose in the worst case $\hat{G}_N$ is reached and $\hat{F}_1$ is not yet known then $\hat{F}_1 = \text{Min}\ \{F(X) : X \in \hat{T}^N\}$.

Remark 1.3 : The proposed solution technique can also work if E is positive semidefinite matrix and $CX + X^T EX + \alpha \leq O$ and $DX + \beta < O$ $\forall X \in S'$ because then $(CX + X^T EX + \alpha)(DX + \beta)$ will again be a quasi-concave function.

SECTION II : QUADRATIC FRACTIONAL SET-COVERING PROBLEM (QUASI-CONCAVE)

The quadratic fractional set-covering problem studied in this section is stated as:

$$\text{(QFCP)} \quad \underset{X \in S'}{\text{Min}}\ F(X) = \frac{CX + X^T EX + \alpha}{DX + X^T FX + \beta}$$

$$\text{subject to} \sum_{J=1}^{n} a_{ij} x_j \geq 1\ ; i = 1, 2, \ldots, m$$

$$x_j = 0 \text{ or } 1,\ j = 1, 2, \ldots, n$$

$C^T, D^T \in R^n$, $\alpha, \beta \in R : E$ is an $n \times n$ negative semi definite matrix, F is an $n \times n$ positive semi definite matrix. It is assumed that S' is regular and

$$CX + X^T EX + \alpha \leq O,\ DX + X^T FX + \beta > 0\ \ \forall X \in S'$$

where

$$S' = \left\{ X = (x_1, x_2, \ldots, x_n);\ \sum_{j=1}^{n} a_{ij} x_j \geq 1, x_j = 0 \text{ or } 1\ \ i \in I, j \in J \right\}$$

The related problem (QFCP′) is obtained by replacing the condition $x_j = 0$ or 1 by $x_j \geq 0$.

$$\text{(QFCP}'\text{)} \quad \underset{X \in S'}{\text{Min}} \; F(X) = \frac{CX + X^T EX + \alpha}{DX + X^T FX + \beta}$$

$$\text{subject to} \sum_{j=1}^{n} a_{ij} x_j \geq 1 \; ; \; i \in I$$

$$x_j \geq 0 \; ; \; j \in J$$

where

$$S'' = \left\{ X = (x_1, x_2, ..., x_n); \sum_{j=1}^{n} a_{ij} x_j \geq 1, x_j \geq 0, \; i \in I, j \in J \right\}.$$

It is assumed that the feasible set S'' is regular.

The following theorem established that $F(X)$ is a quasi-concave function over S''.

Theorem 2.1 — *The function* $F(X) = \dfrac{CX + X^T EX + \alpha}{DX + X^T FX + \beta}$ *is quasi-concave over the set* S'' *under the assumptions of the problem (QFCP).*

PROOF : Since E is negative semidefinite

$$\therefore f(X) = CX + X^T EX + \alpha \text{ is a concave function.}$$

Since F is positive semidefinite

$$\therefore g(X) = DX + X^T FX + \beta \text{ is a convex function.}$$

Since f is concave

$$\Rightarrow f[\lambda X_1 + (1 - \lambda) X_2] \geq \lambda f(X_1) + (1 - \lambda) f(X_2) \quad \text{for } 0 \leq \lambda \leq 1. \quad \text{... (2.1)}$$

and g is convex

$$\Rightarrow g[\lambda X_1 + (1 - \lambda) X_2] \leq \lambda g(X_1) + (1 - \lambda) g(X_2) \quad \text{for } 0 \leq \lambda \leq 1. \quad \text{... (2.2)}$$

Dividing (2.1) by (2.2), we get

$$\frac{f[\lambda X_1 + (1 - \lambda) X_2]}{g[\lambda X_1 + (1 - \lambda) X_2]} \geq \frac{\lambda f(X_1) + (1 - \lambda) f(X_2)}{\lambda g(X_1) + (1 - \lambda) g(X_2)} \quad \text{... (2.3)}$$

Suppose

$$\frac{f(X_1)}{g(X_1)} \geq \frac{f(X_2)}{g(X_2)} \Rightarrow f(X_1) \geq \frac{g(X_1)}{g(X_2)} f(X_2) \quad \text{... (2.4)}$$

Using (2.4) in (2.3) we have

$$\frac{f[\lambda X_1 + (1-\lambda) X_2]}{g[\lambda X_1 + (1-\lambda) X_2]} \geq \frac{\lambda \frac{g(X_1)}{g(X_2)} f(X_2) + (1-\lambda) f(X_2)}{\lambda g(X_1) + (1-\lambda) g(X_2)}$$

$$= \frac{\frac{f(X_2)}{g(X_2)} [\lambda g(X_1) + (1-\lambda) g(X_2)]}{[\lambda g(X_1) + (1-\lambda) g(X_2)]} = \frac{f(X_2)}{g(X_2)}$$

$$\frac{f[\lambda X_1 + (1-\lambda) X_2]}{g[\lambda X_1 + (1-\lambda) X_2]} \geq \frac{f(X_2)}{g(X_2)} = \text{Min} \left\{ \frac{f(X_1)}{g(X_1)}, \frac{f(X_2)}{g(X_2)} \right\} \text{ for } 0 \leq \lambda \leq 1$$

Hence $F(X) = \frac{f(X)}{g(X)} = \frac{CX + X^T EX + \alpha}{DX + X^T FX + \beta}$ is a quasi-concave function.

As the function $F(X)$ is quasi-concave, it is sufficient to investigate only the points of S'' for solving the problem (QFCP). For this purpose the following bounding linear fractional set covering problem (FCP) is constructed.

$$\text{(FCP)} \quad \underset{X \in S'}{\text{Min}} \quad G(X) = \frac{CX + UX + \alpha}{DX + VX + \beta}$$

where

$$U_j = \text{the } j\text{th component of } U^T$$

$$= \underset{X \in S'}{\text{Min}} \; E_j^T X, j = 1, 2, \ldots, n$$

$$V_j = j\text{th component of } V^T$$

$$= \underset{X \in S'}{\text{Max}} \; F_j^T X, j = 1, 2, \ldots, n.$$

If we replace the condition $x_j = 0$ or 1 by $x_j \geq 0$. Then the related problem (FCP′) is

$$\text{(FCP)} \quad \underset{X \in S'}{\text{Min}} \quad G(X) = \frac{CX + UX + \alpha}{DX + VX + \beta}$$

Clearly

$$F(X) \geq G(X) \quad \forall X \in S''.$$

ALGORITHM - I

The algorithm comprises of the following steps:

Initial Step ($k = 1$)

For $j = 1, 2, \ldots, n$ find $U_j = \underset{X \in S'}{\text{Min}} \; X^T E_j$

Construct $\hat{Q}$ such that $(CX+UX)(DX)=X^T\hat{Q}X$.

Find

$$W_j = \min_{X \in S'} X^T \hat{Q}_j, \; j = 1, 2, ..., n.$$

Determine the vector L and δ and construct the linear set covering problem (CP) and (CP′). Solve the problem (CP′) to find $\hat{S}_1 = \{X^1_1\}$ and $\hat{G}_1$. Find $F(X)$, $X \in \hat{S}_1$.

If $\hat{G}_1 = \text{Min}\ \{F(X) : X \in \hat{T}^1\} = F(\hat{X})$ say.

Then set $\hat{F}_1 = F(\hat{X})$ and $\hat{X}$ is an optimal solution of the problem and the process terminates.

If $\hat{G}_1 \in \min\ \{F(X) : X \in \hat{T}^1\}$, then by Theorem (1.5), the optimal solution is not yet reached. Go to general step for$k = 2$.

General Step $(k \geq 2)$

Find $\hat{S}_k = \{X^1_k\}$ and compute $\hat{G}_k$, $k \geq 2$. If $\hat{G}_k \geq \min\ \{F(X) : X \in \hat{T}^k\} = F(\hat{X})$ (say).

Then $\hat{F}_1 = F(\hat{X})$ and $\hat{X}$ is an optimal solution of the problem and the process terminates.

.If $\hat{G}_k < \text{Min}\ \{F(X) : X \in \hat{T}^k\} = F(\hat{X})$, then the current incumbent is $\hat{X}$ and repeat this step for the next higher value of k. Continue likewise until an optimal solution of the problem is obtained.

ALGORITHM - II

The algorithm comprises of the following steps:

Initial step ($k = 1$). Find U_j and V_j as given in Algorithm I, then construct the linear fractional set covering problem (FCP) and (FCP′). Solve the problem (FCP′) to find $\hat{S}_1 = \{X^1_1\}$ and $\hat{G}_1$ and proceed as in Algorithm I.

Remark 1.4 — Since the algorithm moves from one extreme point to another extreme point of the convex polyhedron, which are finite in number and none of them is repeated, therefore, the algorithm converges in a finite number of steps.

SECTION III : NUMERICAL ILLUSTRATION

Example 1 — Consider the following set covering problem.

$$\text{(BCP)} \quad \min_{X \in R^4} f(x) = (2x_1 + x_2 + x_3 + 3x_4 - x_1^2 - x_2^2 - x_1 x_2 - x_1 x_3 - x_4^2 + 5)$$

$$(3x_1 + x_2 + x_3 + 1)$$

subject to

$$x_1 + x_2 \geq 1$$

$$x_2 + x_3 + x_4 \geq 1$$

$$x_1 + x_3 + x_4 \geq 1$$

$$x_1, x_2, x_3, x_4 = 0 \text{ or } 1$$

Solution procedure

In this problem $I = [1, 2, 3]$, J] $[1, 2, 3, 4]$,

$$P_1 = [1, 3],\ P_2 = [1, 2],\ P_3 = [2, 3],\ P_4 = [2, 3].$$

Relaxing the condition $x_j = 0$ or 1 by $x_j \geq 0$ for $j \in J$. The problem reduces to

$$\text{(BCP}') \quad \underset{X \in R^4}{\text{Min}} \ F(x) = (2x_1 + x_2 + x_3 + 3x_4 - x_1^2 - x_2^2 - x_1 x_2 - x_1 x_3 - x_4^2 + 5)$$

$$(3x_1 + x_2 + x_3 + 1)$$

subject to

$$x_1 + x_2 \geq 1$$

$$x_2 + x_3 + x_4 \geq 1$$

$$x_1 + x_3 + x_4 \geq 1$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Here

$$E = \begin{pmatrix} -1 & -1/2 & -1/2 & 0 \\ -1/2 & -1 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Find

$$U_1 = \underset{X \in S'}{\text{Min}} \ X^T E_1 = \underset{X \in S'}{\text{Min}} \ - x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 = -2$$

$$U_2 = \underset{X \in S'}{\text{Min}} \ X^T E_2 = \underset{X \in S'}{\text{Min}} \ - \frac{1}{2}x_1 - x_2 = -3/2$$

$$U_3 = \underset{X \in S'}{\text{Min}} \ X^T E_3 = \underset{X \in S'}{\text{Min}} \ - \frac{1}{2}x_1 = -1/2$$

$$U_4 = \underset{X \in S'}{\text{Min}} \ X^T E_4 = \underset{X \in S'}{\text{Min}} \ - x_4 = -1$$

$$X^T \hat{Q} X = (CX + UX)(DX)$$

$$= \left(2x_1 + x_2 + x_3 + 3x_4 - 2x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 - x_4 \right)(3x_1 + x_2 + x_3)$$

$$= \left(-\frac{1}{2}x_2 + \frac{1}{2}x_3 + 2x_4 \right)(3x_1 + x_2 + x_3)$$

$$= (x_1, x_2, x_3, x_4) \begin{pmatrix} 0 & -3/4 & 3/4 & 3 \\ -3/4 & -1/2 & 0 & 1 \\ 3/4 & 0 & 1/2 & 1 \\ 3 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$W_1 = \min_{X \in S'} X^T \hat{Q}_1 = \min_{X \in S'} -\frac{3}{4}x_2 + \frac{3}{4}x_3 + 3x_4 = -3/4$$

$$W_2 = \min_{X \in S'} X^T \hat{Q}_2 = \min_{X \in S'} -\frac{3}{4}x_1 - \frac{1}{2}x_2 + x_4 = -5/4$$

$$W_3 = \min_{X \in S'} X^T \hat{Q}_3 = \min_{X \in S'} \frac{3}{4}x_1 - \frac{1}{2}x_3 + x_4 = 1/2$$

$$W_4 = \min_{X \in S'} X^T \hat{Q}_4 = \min_{X \in S'} 3x_1 + x_2 + x_3 = 1$$

$$LX = [W + \alpha D + \beta(C + U)]X$$

$$= \left[-\frac{3}{4}x_1 - \frac{5}{4}x_2 + \frac{1}{2}x_3 + x_4 + 5(3x_1 + x_2 + x_3) - \frac{1}{2}x_2 + \frac{1}{2}x_3 + 2x_4 \right]$$

$$= \frac{57}{4}x_1 + \frac{13}{4}x_2 + 6x_3 + 3x_4$$

$$\delta = \alpha\beta = 5 \times 1 = 5.$$

Thus the related linear set covering problem (CP) is

$$\text{(CP)} \qquad \min_{x \in R^4} G(X) = \frac{57}{4}x_1 + \frac{13}{4}x_2 + 6x_3 + 3x_4 + 5$$

subject to

$$x_1 + x_2 \geq 1$$

$$x_2 + x_3 + x_4 \geq 1$$

$$x_1 + x_3 + x_4 \geq 1$$

$$x_1, x_2, x_3, x_4 = 0 \text{ or } 1$$

Relaxing the condition $x_j = 0$ or 1 by $x_j \geq 0$ then our problem (CP′) is

$$(\text{CP}') \quad \underset{x \in R^4}{\text{Min}} \; G(X) = \frac{57}{4}x_1 + \frac{13}{4}x_2 + 6x_3 + 3x_4 + 5$$

subject to

$$x_1 + x_2 \geq 1$$

$$x_2 + x_3 + x_4 \geq 1$$

$$x_1 + x_3 + x_4 \geq 1$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Compute

$$\hat{S}_1 \; \{X_1^1\}, \; X_1^1 = (0, 1, 0, 1) \; \hat{G}_1 = \frac{45}{4}$$

$$F(X_1^1) = \hat{F}_1 = 14, \quad \hat{T}^1 = \hat{S}_1$$

$$\hat{G}_1 = \frac{45}{4} < \text{Min } \{F(X) : X \in \hat{T}^1\} = \hat{F}_1 = 14$$

therefore go to general step.

General step $(k \geq 2)$

Compute

$$\hat{S}_2 = \{X_2^1\}, \; X_2^1 = (0, 1, 1, 0) \; \hat{G}_2 = \frac{57}{4}$$

$$F(X_2^1) = \hat{F}_2 = 18, \quad \hat{T}^2 = \bigcup_{i=1}^{2} \hat{S}_i$$

$$\hat{G}_2 = \frac{57}{4} > \text{Min } \{F(X) : X \in \hat{T}^2\} = \hat{F}_1 = 14$$

Hence the optimal value of $F(X)$ is $\hat{F}_1 = F(X_1^1) = 14$ and the optimal solution is

(0, 1, 0, 1) i.e. $x_1 = 0, x_2 = 1, \quad x_3 = 0$ and $x_4 = 1$.

Example 2 —

$$(\text{QFCP}) \quad \underset{x \in R^4}{\text{Min}} \; \frac{\left(3x_1 + 2x_2 + x_3 + 3x_4 - x_1^2 - x_2^2 - x_1 x_2 - x_1 x_3 - x_4^2\right)}{\left(-5x_1 - 4x_2 - 3x_3 - 2x_4 + 3x_1^2 + x_2^2 + x_2^2 + x_4^2 - 2x_1 x_2 - 2x_1 x_3 - 2x_1 x_4 + 14\right)}$$

subject to

$$x_1 + x_2 \geq 1$$

$$x_2 + x_3 + x_4 \geq 1$$

$$x_1 + x_3 + x_4 \geq 1$$

$$x_1, x_2, x_3, x_4 = 0 \text{ or } 1$$

Solution Procedure : Here P_1, P_2, P_3, P_4 are same as given in Example 1.

Relaxing the condition $x_j = 0$ or 1 by $x_j \geq 0$ for $j \in J$. The problem reduces to

$$\text{(QFCP')} \quad \underset{x \in R^4}{\text{Min}} \frac{\left(3x_1 + 2x_2 + x_3 + 3x_4 - x_1^2 - x_2^2 - x_1 x_2 - x_1 x_3 - x_4^2\right)}{\left(-5x_1 - 4x_2 - 3x_3 - 2x_4 + 3x_1^2 + x_2^2 + x_3^2 + x_4^2 - 2x_1 x_2 - 2x_1 x_3 - 2x_1 x_4 + 14\right)}$$

subject to

$$x_1 + x_2 \geq 1$$

$$x_2 + x_3 + x_4 \geq 1$$

$$x_1 + x_3 + x_4 \geq 1$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Here

$$E = \begin{pmatrix} -1 & -1/2 & -1/2 & 0 \\ -1/2 & -1 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ and } F = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Find

$$U_1 = \underset{x \in S'}{\text{Min}} \; X^T E_1 = \underset{x \in S'}{\text{Min}} \; -x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 = -2$$

$$U_2 = \underset{x \in S'}{\text{Min}} \; X^T E_2 = \underset{x \in S'}{\text{Min}} \; -\frac{1}{2}x_1 - x_2 = -3/2$$

$$U_3 = \underset{x \in S'}{\text{Min}}\ X^T E_3 = \underset{x \in S'}{\text{Min}}\ -\frac{1}{2}x_1 = -1/2$$

$$U_4 = \underset{x \in S'}{\text{Min}}\ X^T E_4 = \underset{x \in S'}{\text{Min}}\ -x_4 = -1$$

$$V_1 = \underset{x \in S'}{\text{Max}}\ X^T F_1 = \underset{x \in S'}{\text{Max}}\ 3x_1 - x_2 - x_3 - x_4 = 2$$

$$V_2 = \underset{x \in S'}{\text{Max}}\ X^T F_2 = \underset{x \in S'}{\text{Max}}\ -x_1 + x_2 = 1$$

$$V_3 = \underset{x \in S'}{\text{Max}}\ X^T F_3 = \underset{x \in S'}{\text{Max}}\ -x_1 + x_3 = 1$$

$$V_4 = \underset{x \in S'}{\text{Max}}\ X^T F_4 = \underset{x \in S'}{\text{Max}}\ -x_1 + x_4 = 1$$

Thus the related bounding linear fractional set covering problem (FCP) is

$$\text{(FCP)}\quad \underset{x \in S'}{\text{Min}}\ G(X) = \frac{CX + UX + \alpha}{DX + VX + \beta}$$

$$= \frac{2x_1 + x_2 + x_3 + 4x_4}{-6x_1 - 6x_2 - 4x_3 - 2x_4 + 28}$$

subject to

$$x_1 + x_2 \geq 1$$

$$x_2 + x_3 + x_4 \geq 1$$

$$x_1 + x_3 + x_4 \geq 1$$

$$x_1, x_2, x_3, x_4 = 0 \text{ or } 1$$

Relaxing the condition $x_j = 0$ or 1 by $x_j \geq 0$ for $j = 1, 2, 3, 4$ then our problem (FCP′) is

$$\text{(FCP}'\text{)}\quad \underset{x \in S'}{\text{Min}}\ G(X) = \frac{CX + UX + \alpha}{DX + VX + \beta}$$

$$= \frac{2x_1 + x_2 + x_3 + 4x_4}{-6x_1 - 6x_2 - 4x_3 - 2x_4 + 28}$$

subject to

$$x_1 + x_2 \geq 1$$

$$x_2 + x_3 + x_4 \geq 1$$

$$x_1 + x_3 + x_4 \geq 1$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Compute

$$\hat{S}_1 = \{X_1^1\},\ X_1^1 = (0,\ 1,\ 1,\ 0)\ \hat{G}_1 = \frac{1}{9}$$

$$F(X_1^1) = \hat{F}_1 = \frac{2}{9},\ \hat{T}^1 = \hat{S}_1$$

$$\hat{G}_1 = \frac{1}{9} < \text{Min}\ \{F(X) : X \in \hat{T}^1\} = \hat{F}_1 = \frac{2}{9}$$

therefore go to general step.

General Step $(k \geq 2)$

Compute

$$\hat{S}_2 = \{X_2^1\},\ X_2^1 = (1,\ 0,\ 1,\ 0)\ \hat{G}_2 = \frac{1}{6}$$

$$F(X_2^1) = \hat{F}_2 = \frac{1}{4},\ \ \hat{T}^2 = \bigcup_{i=1}^{2} \hat{S}_i$$

$$\hat{G}_2 = \frac{1}{6} < \text{Min}\ \{F(X) : X \in \hat{T}^2\} = \hat{F}_1 = \frac{2}{9}$$

Compute

$$\hat{S}_3 = \{X_3^1\},\ X_3^1 = (1,\ 1,\ 0,\ 0)\ \hat{G}_3 = \frac{3}{16}$$

$$F(X_3^1) = \hat{F}_3 = \frac{2}{7},\ \ \hat{T}^3 = \bigcup_{i=1}^{3} \hat{S}_i$$

$$\hat{G}_3 = \frac{3}{16} < \text{Min}\ \{F(X) : X \in \hat{T}^3\} = \hat{F}_1 = \frac{2}{9}$$

Compute

$$\hat{S}_4 = \{X_4^1\},\ X_4^1 = (1,\ 0,\ 0,\ 2)\ \hat{G}_4 = \frac{3}{10}$$

$$F(X_4^1) = \hat{F}_4 = \frac{4}{9},\ \ \hat{T}^4 = \bigcup_{i=1}^{4} \hat{S}_i$$

$$\hat{G}_4 = \frac{3}{10} > \text{Min}\ \{F(X) : X \in \hat{T}^4\} = \hat{F}_1 = \frac{2}{9}.$$

Hence the optimal value of $F(X)$ is $\hat{F}_1 = F(X_1^1) = 2/9$ and the optimal solution is (0, 1, 1, 0) i.e. $x_1 = 0, x_2 = 1, x_3 = 1$ and $x_4 = 0$.

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