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ISOMORPHISM THEOREMS OF HYPERRINGS

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In this paper the three isomorphism theorems of ring theory are derived in the context of hyperrings. Then a Jordan-Holder theorem is derived for hyperrings. Finally, we consider the fundamental relation γ^* defined on a hyperring and prove some results and a fundamental theorem in this respect.

Key Words: Hyperring; Hyperideal; Strong Homomorphism; Fundamental Relation

1. INTRODUCTION

Krasner has studied the notion of a hyperring in[8]. Hyperrings are essentially rings, with approximately modified axioms in which addition is a hyperoperation (i.e., $a + b$ is a set). Then this concept has been studied by a variety of authors. The principal notions of hyperring theory are described in[2-5,8-12,14]

In this paper the three isomorphism theorems of ring theory are derived in the context of hyperrings. Then a Jordan-Holder theorem is derived for hyperrings. The methods used are adopted from those in ring theory. Finally, we consider the fundamental relation γ^* defined on a hyperring and prove some results and a fundamental theorem in this respect.

2. BASIC DEFINITIONS AND RESULTS

We will be concerned primarily with a basic non-empty set H. A hyperoperation $\circ$ on H is a mapping of $H \times H$ into the family of non-empty subsets of H. Let $\circ$ be a hyperoperation on H then $(H, \circ)$ is called a hypergroupoid. A hypergroup (in the sense of Marty) is a hypergroupoid $(H, \circ)$, that satisfies:

(1) $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$,

(2) $x \circ H = H \circ x = H$ for all $x \in H$.

A comprehensive review of the theory of hypergroups appears in[1].

Now, we recall the following definition from[8].

Definition 2.1 — A hyperring is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup, i.e.,

(*i*) for every $x, y, z \in R$, $x + (y + z) = (x + y) + z$, (*ii*) for every $x, y \in R$, $x+y=y+x$,

(*iii*) there exists $0 \in R$ such that $0 + x = x$ for all $x \in R$,

(*iv*) for every $x \in R$ there exists a unique element $x \in R$ such that $0 \in x+x'$; (We shall write $-x$ for x' and we call it the opposite of x.)

(*v*) $z \in x+y$ implies $y \in -x+z$ and $x \in z-y$;

(2) Relating to the multiplication, $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element.

(3) The multiplication is distributive with respect to the hyperoperation +.

In the above definition if $A, B \subseteq R$ and $x \in R$ then

$$A+B= \bigcup_{a \in A,\, b \in B} a+b, \quad A+x=A + \{x\} \text{ and } x + B = \{x\} + B.$$

The following elementary facts follow easily from the axioms:

$$-(-x)=x \text{ and } -(x+y)=-x-y,$$

where

$$-A^{-1} = \{-a^{-1} \mid a \in A\}.$$

Example 2.2 — (Krasner[8]). Let $(A, +, \cdot)$ be a ring and N a normal subgroup of its multiplicative semigroup. Then the multiplicative classes $\overline{x}=xN\,(x \in A)$ form a partition of R, and let $\overline{A}=A/N$ be the set of these classes. The product of $\overline{x}, \overline{y} \in \overline{A}$ as subsets of A is again a class (*modN*), and their sum as such subsets is a union of such classes. If we define the product $\overline{x} * \overline{y}$ in $\overline{A}$ of $\overline{x}, \overline{y} \in \overline{A}$ as equal to their product as subsets of A, and their sum $\overline{x} \oplus \overline{y}$ in $\overline{A}$ as the set of all $\overline{z} \in \overline{A}$ contained in their sum as subsets of A, i.e.,

$$\overline{x} \oplus \overline{y} = \{\overline{z} \mid z \in \overline{x}+\overline{y}\}, \quad \text{and} \quad \overline{x} * \overline{y}=\overline{x \cdot y},$$

then the obtained structure is a hyperring.

Definition 2.3 — Let $(R, +, \cdot)$ be a hyperring and A a non-empty subset of R. Then A is said to be subhyperring of R if $(A, +, \cdot)$ is itself a hyperring.

Definition 2.4 — A subhyperring A of a hyperring R is a left (right) hyperideal of R provided that $r \cdot a \in A$ $(a \cdot r \in A)$ for all $r \in P$ $a \in A$. A is called a hyperideal if A is both left and right hyperideal.

It would be useful to have some criterion for deciding whether a given subset of a hyperring is a left (right) hyperideal. This is the purpose of the next lemma.

Lemma 2.5 — A non-empty subset A of a hyperring R is a left (right) hyperideal if and only if (*i*) $a, b \in A$ implies $a - b \subseteq A$,

(*ii*) $a \in A$, $r \in R$ imply $r \cdot a \in A$ $(a \cdot r \in A)$.

Definition 2.6 — The subhyperrıng A of R is normal in R if and only if

$$x + A - x \subseteq A \quad \text{for al} \quad x \in R$$

The following two corollaries are obtained exactly from definitions.

Corollary 2.7 — Let A be a normal hyperideal of R. Then

(*i*) $(A + x) + (A + y) = A + x + y$ for all $x, y \in R$,

(*ii*) $A + x = A + y$ for all $y \in A + x$.

Corollary 2.8 — Let A and B be hyperideals of a hyperrıng R with B normal in R. Then

(*i*) $A \cap B$ is a normal hyperideal of A,

(*ii*) B is a normal hyperideal of $A + B$.

3. The Isomorphism Theorems

In this section, first, the three isomorsm theorems of ring theory are derived in the context of hyperrings.

Definition 3.1 — If A is a normal hyperideal of a hyperring R, then we define the relation

$$x \equiv y \,(modA) \text{ if and only if } x - y \cap A \neq \phi.$$

This relation is denoted by $xA^* y$.

Lemma 3.2 — The relation A^* is an equivalence relation.

PROOF : (*i*) Since $0 \in x - x \cap A$ for all $x \in R$; then $xA^* x$, i.e., A^* is reflexive.

(*ii*) Suppose that $xA^* y$ then there exists $z \in x - y \cap A$ which implies $-z \in y - x$ and $-z \in A$, this means that $yA^* x$, and so A^* is symmetric.

(*iii*) Let $xA^* y$ and $xA^* z$ where $x, y, z \in R$. Then there exist $a \in x - y \cap A$ and $b \in y - z \cap A$. So $x \in a + y$ and $-z \in -y + b$. Hence $x \in y + a$, $-z \in b - y$ which imply $x - z \subseteq y + a + b - y$. Since $a + b \subseteq A$ and A is normal, then $y + a + b - y \subseteq A$. Therefore $x - z \cap A \neq \phi$, which satisfies the condition for $xA^* z$, and so A^* is transitive. □

Let $A^* [x]$ be the equivalence class of the element $x \in R$. Then

Lemma 3.3 — If A is a normal hyperideal of R, then $A + x = A^* [x]$ for all $x \in R$.

PROOF : Suppose $y \in A + x$ then there exists $a \in A$ such that $y \in a + x$, which implies $a \in y - x$ and so $y - x \cap A \neq \phi$. Thus $A + x \subseteq A^* [x]$. Similarly we have $A^* [x] \subseteq A + x$. □

Lemma 3.4 — Let A be a normal hyperideal of R. Then for all $x, y \in R$, we have $A+x+y=A+z$ for all $z \in x+y$.

PROOF : Suppose $z \in x+y$, then it is clear that $A+z \subseteq A+x+y$. Now, let $a \in A+x+y$, by condition (v) of Definition 2.1, we get $y \in -(A+x)+a$ or $y \in A-x+a$, and so $x+y \subseteq x+A-x+a$. Since A is normal, we obtain $x+y \subseteq A+a$. Therefore, for every $z \in x+y$, we have $z \in A+a$ which implies $a \in A+z$. □

The next two corollaries are fundamental.

Corollary 3.5 — For all $x, y \in R$, we have $A^* [A^* [x] + A^* [y]] = A^* [x] + A^* [y]$.

PROOF : The proof follows easily from Lemma 3.4. □

Corollary 3.6 — For all $x, y \in R$, we have $A^* [A^* [x \cdot y]] = A^* [x \cdot y]$.

PROOF : Clearly, we have $A^* [x \cdot y] \subseteq A^* [A^* [x \cdot y]]$. Now let $a \in A^* [A^* [x \cdot y]]$. Then there exists $b \in A^* [x \cdot y]$ with $a \in A^* [b]$. So $aA^* b$ and $bA^* x \cdot y$ which imply $aA^* x \cdot y$. Hence $a \in A^*$ $[x \cdot y]$. □

Proposition 3.7 — Let R be a hyperring. If A is a normal hyperideal of R, then on the set of all classes $[R : A^*] = \{A^* [x] \mid x \in R\}$ we define the hyperoperation $\oplus$ and the multiplication $\odot$ as follows:

$$A^* [x] \oplus A^* [y] = \{A^* [z] \mid z \in A^* [x] + A^* [y]\},$$

$$A^* [x] \odot A^* [y] = A^* [x, y].$$

PROOF : This follows from Corollary 3.5, Corollary 3.6 and Definition 2.1. □

Definition 3.8 — Let R_1 and R_2 be hyperrings. A mapping φ from R_1 into R_2 is said to be a strong homomorphism if for all $a, b \in R_1$,

$$\varphi(a+b) = \varphi(a) + \varphi(b), \quad \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b), \quad \varphi(0) = 0.$$

Clearly, a strong homomorphism φ is an isomorphism if φ is one to one and onto. We write $R_1 \cong R_2$ if R_1 is isomorphic to R_2.

Because R_1 is a hyperring, $\ulcorner \equiv a-a$ for all $a \in R_1$, then we have $\varphi(0) \in \varphi(a) + \varphi(-a)$ or $0 \in \varphi(a) + \varphi(-a)$ which implies $\varphi(-a) \in -\varphi(a) + 0$, therefore $\varphi(-a) = -\varphi(a)$ for all $a \in R_1$. Moreover, if φ is a strong homomorphism from R_1 into R_2, then the kernel of φ is the set *ker* φ $= \{x \in R_1 \mid \varphi(x) = 0\}$. It is trivial that *ker* φ is a hyperideal of R_1, but in general it is not normal in R_1.

Corollary 3.9 — Let φ be a strong homomorphism from R_1 into R_2. Then φ is injective if and only if $ker\,\varphi = \{0\}$.

PROOF : Let $y, z \in R_1$ be such that $\varphi(y) = \varphi(z)$. Then $\varphi(y) - \varphi(y) = \varphi(z) - \varphi(y)$.

It follows that $\varphi(0) \in \varphi(z-y)$, and so there exists $x \in z-y$ such that $0 = \varphi(0) = \varphi(x)$. Thus, if $ker\,\varphi = \{0\}$, $x = 0$, whence $y = z$.

Now, let $x \in ker\,\varphi$. Then $\varphi(x) = 0 = \varphi(0)$. Thus, if φ is injective, we conclude that $x = 0$. □

The first isomorphism theorem comes next.

Theorem 3.10 — *(First Isomorphism Theorem). Let φ be a strong homomorphism from R_1 into R_2 with kernel K such that K is a normal hyperideal of R_1, then $[R_1 : K^*] \cong Im\,\varphi$.*

PROOF : We define $\rho : [R_1 : K^*] \to Im\,\varphi$ by setting $\rho(K^*[x]) = \varphi(x)$ for all $x \in R_1$. We first prove that ρ is well-defined. Suppose xK^*y, then $x - y \cap K \neq \phi$. So there exists $z \in x - y \cap K$. Consequently, $\varphi(z) = 0$ and $\varphi(z) \in \varphi(x) - \varphi(y)$. Thus $\varphi(x) = \varphi(y)$. Clearly ρ is onto. To show that ρ is one to one, suppose $\varphi(x) = \varphi(y)$. Then $0 \in \varphi(x-y)$, and so there exists $z \in x - y$ with $z \in ker\,\varphi$. Therefore, $x - y \cap K \neq \phi$ which implies $K^*[x] = K^*[y]$, and so ρ is one to one.

Moreover,

$$\begin{aligned}
\rho(K^*[x] \oplus K^*[y]) &= \rho(\{K^*[z] \mid z \in K^*[x] + K^*[y]\}) \\
&= \{\varphi(z) \mid z \in K^*[x] + K^*[y]\}) \\
&= \varphi(K^*[x]) + \varphi(K^*[y]) = \varphi(x) + \varphi(y) \\
&= \rho(K^*[x]) + \rho(K^*[y]), \\
\rho(K^*[x] \odot K^*[y]) &= \rho(K^*[x \cdot y]) = \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) \\
&= \rho(K^*[x]) \cdot \rho(K^*[y]),
\end{aligned}$$

and $\rho(K^*[0]) = \varphi(0) = 0$. Therefore ρ is a isomorphism. □

Using Lemma 3.3, we can put $[R : A^*] = \{A + x \mid x \in R\}$ and then we have

Corollary 3.12 — If A is a normal hyperideal of R, then

$$(A+x) \oplus (A+y) = \{A + z \mid z \in x + y\}.$$

PROOF : Using Corollary 2.8 and Lemma 3.4, we have

$$\begin{aligned}
(A+x) \oplus (A+y) &= \{A + c \mid c \in A + x + A + y\} = \{A + c \mid c \in A + x + y\} \\
&= \{A + c \mid c \in A + z, z \in x + y\} = \{A + z \mid z \in x + y\}.
\end{aligned}$$

□

We are now in a position to state and prove the second and third isomorphism theorems in hyperrings.

Theorem 3.13 — *(Second Isomorphism Theorem). If A and B are hyperideals of a hyperring R, with B normal in R, then* $[A:(A\cap B)^*\cong[A+B:B^*]$.

PROOF : Clearly, B is a normal hyperideal of $A+B$; Consequently $[A+B:B^*]$ is defined. Define $\rho:A\to[A+B:B^*]$ by $\rho(a)=B+a$. ρ is a strong homomorphism. Consider any $B+y\in[A+B:B^*]$ $y\in A+B$. Now $y\in A+B$ given $y\in a+b$ for some $a\in A, b\in B$. Thus, by Lemma 3.4, $B+y=B+a+b=B+a=\rho(a)$. This shows that ρ is also onto. If we can establish that $ker\,\rho=A\cap B$, since $A\cap B$ is a normal hyperideal of A, we shall get that $[A:(A\cap B)^*]\cong[A+B:B^*]$. For any $a\in A, a\in ker\,\rho \Leftrightarrow \rho(a)=B\Leftrightarrow B+a=B\Leftrightarrow a\in B\Leftrightarrow a\in A\cap B$ (since $a\in A$). This yields $ker\,\rho=A\cap B$. Hence that results follows. □

Theorem 3.14 — *(Third Isomorphism Theorem). If A and B are normal hyperideals of hyperrings R such that* $A\subseteq B$, *then* $[B:A^*]$ *is a normal hyperideal of* $[R:A^*]$ *and* $[[R:A^*]:[B:A^*]]\cong[R:B^*]$.

PROOF : We leave it to reader to verify that $[B:A^*]$ is a normal hyperideal of $[R:A^*]$. Further $\rho:[R:A^*]\to[R:B^*]$ defined by $\rho(A+x)=B+x$ is a strong homomorphism of $[R:A^*]$ onto $[R:B^*]$ such that $ker\,\rho=[B:A^*]$. □

Definition 3.15 — Let R be a hyperring. A finite chain of $n+1$ normal hyperideals of R

$$R=A_0\supset A_1\supset\ldots\supset A_n=0$$

is called a composition series of length n for R provided that $[A_{i-1}:A_i^*]$ is simple ($i=1,\ldots,n$), i.e., provided each term in the chain is maximal in its predecessor.

We will consider a generalization of the Jordan-Holder Theorem for hyperrings. The methods used are adopted from those in ring theory.

Theorem 3.16 — *(The Jordan-Holder Theorem). If a hyperring R has a composition series, then every pair of composition series for R are equivalent.*

PROOF : If R has a composition series, then denote by $c(R)$ the minimum length of such series for R. We shall induct on $c(R)$. Clearly, if $c(R)=1$, there is no challenge. So assume that $c(R)=n>1$ and that any hyperring with a composition series of smaller length has all of its composition series equivalent.

Let

$$R=A_0\supset A_1\supset\ldots\supset A_n=0 \qquad \ldots \text{(I)}$$

be a composition series of minimal length for R and

$$R = B_0 \supset B_1 \supset \ldots \supset B_m = 0 \qquad \ldots \text{(II)}$$

be a second composition series for R. If $A_1 = B_1$, then by induction hypothesis, since $c(A_1) \le n-1$, the two series are equivalent. So we may assume that $A_1 \neq B_1$. Then since A_1 is a maximal hyperideal of R, we have $A_1 + B_1 = R$, so

$$[R : A_1^*] = [A_1 + B_1 : A_1^*] \cong [B_1 : (A_1 \cap B_1)^*],$$

and

$$[R : B_1^*] = [A_1 + B_1 : B_1^*] \cong [A_1 : (A_1 \cap B_1)^*].$$

Thus $A_1 \cap B_1$ is maximal in both A_1 and B_1. Now $A_1 \cap B_1$ has a composition series

$$A_1 \cap B_1 = C_0 \supset C_1 \supset \ldots \supset C_k = 0$$

So

$$A_1 \supset C_0 \supset C_1 \supset \ldots \supset C_k = 0 \text{ and } B_1 \supset C_0 \supset C_1 \supset \ldots \supset C_k = 0,$$

are composition series for A_1 and B_1. Since $c(A_1) < n$, every two composition series for A_1 are equivalent, so the two series

$$R = A_0 \supset A_1 \supset \ldots \supset A_n = 0 \text{ and } R = A_0 \supset A_1 \supset C_0 \supset \ldots \supset C_k = 0,$$

are equivalent. In particular, $k < n-1$, so clearly $c(B_1) < n$. Thus by our induction hypothesis, every two composition series for B_1 are equivalent. Thus the two series

$$R = B_0 \supset B_1 \supset \ldots \supset B_m = 0 \text{ and } R = B_0 \supset B_1 \supset C_0 \supset \ldots \supset C_k = 0,$$

are equivalent. But as we denote

$$[R : A_1^*] \cong [B_1 : C_0^*] \quad \text{and} \quad [R : B_1^*] \cong [A_1 : C_0^*];$$

thus the series (*I*) and (*II*) are equivalent, and we are done. □

Let $(R, +, \cdot)$ be a hyperring. We define the relation γ^* as the smallest equivalence relation on R such that the quotient R/γ^*, the set of all equivalence classes, is a ring. In this case γ^* is called the fundamental equivalence relation on R and R/γ^* is called the fundamental ring.

Suppose $\gamma^*(a)$ is the equivalence class containing $a \in R$. Then both the sum $\uplus$ and the product $\otimes$ in R/γ^* are defined as follows:

$$\gamma^*(a) \uplus \gamma^*(b) = \gamma^*(c) \text{ for all } c \in \gamma^*(a) + \gamma^*(b),$$

$$\gamma^*(a) \otimes \gamma^*(b) = \gamma^*(c) \text{ for all } c \in \gamma^*(a) + \gamma^*(b).$$

This relation was introduced and studied by Vougiouklis[13,14]

Let $\mathcal{U}_R$ be the set of all finite sums of products of elements of R. We define the relation γ as follows:

$$a \gamma b \text{ if and only if } \{a, b\} \subseteq u \text{ for some } u \in \mathcal{U}_R.$$

Also, we can define the relation β^*_+ as the smallest equivalence relation such that the quotient R/β^*_+ is a group. The equivalence relation β^*_+ was introduced by Koskas[7] on hypergroups and was studied mainly by Corsini[1]. We will denote by β_+ the relation defined in R as follows:

$$a \beta_+ b \text{ if and only if there exists } (c_1, ..., c_n) \in R^n \text{ such that } \{a, b\} \subseteq c_1 + \ldots + c_n.$$

Freni proved in[6] that for hypergroups we have $\beta^*_+ = \beta_+$. Since all hyperrings in this paper are additive, we have the following theorem:

Theorem 3.17 — *(Theorem 3[14]) For all hyperrings we have* $\gamma^* = \beta^*_+$.

The kernel of the canonical map $\varphi: R \to R/\gamma^*$ is called the core of R and is denoted by ω_R. Here we also denote by ω_R the zero element of R/γ^*. We have the following statements (see[3]),

(*i*) $\omega_R = \gamma^*(0)$,

(*ii*) $\gamma^*(-x) = -\gamma^*(x)$ for all $x \in R$.

Theorem 3.18 — (Theorem 3.3[3]). *Let* γ^*_1, γ^*_2 *and* γ^* *be fundamental equivalence relations on hyperrings* R_1, R_2 *and* $R_1 \times R_2$ *respectively, then*

$$(R_1 \times R_2)/\gamma^* \cong R_1/\gamma^*_1 \times R_2/\gamma^*_2.$$

Lemma 3.19 — If A, B are normal hyperideals of R_1, R_2 respectively, then

$$[(R_1 \times R_2) : (A \times B)^*] \cong [R_1 : A^*] \times [R_2 : B^*].$$

PROOF : The proof is straightforward and we omit it. □

Corollary 3.20 — If A, B are normal hyperideals of R_1, R_2 respectively, and γ^*_1, γ^*_2 and γ be fundamental equivalence relations on $[R_1 : A^*], [R_2 : B^*]$ and $[(R_1 \times R_2) : (A \times B)^*]$ respectively, then

$$[(R_1 \times R_2) : (A \times B)^*/\gamma^* \cong [R_1 : A^*]/\gamma^*_1 \times [R_2 : B^*]/\gamma^*_2 .$$

PROOF : The proof is obtained exactly from Theorem 3.18 and Lemma 3.19. □

Definition 3.21 — Let f be a strong homomorphism from R_1 into R_2 and let γ_1^*, γ_2^* be fundamental relations on R_1, R_2 respectively, then we define

$$\overline{kerf} = \{\gamma_1^*(x) \mid x \in R_1,\ \gamma_2^*(f(x)) = \omega_{R_2}\}.$$

Lemma 3.22 — $\overline{kerf}$ is an ideal of the fundamental ring R_1/γ_1^*.

PROOF : Assume that $\gamma_1^*(x), \gamma_1^*(y) \in \overline{kerf}$ then for every $z \in x - y$ we have $\gamma_1^*(z) = \gamma_1^*(x) \uplus \gamma_1^*(-y)$. On the other hand, we have

$$\gamma_2^*(f(z)) = \gamma_2^*(f(x) + f(-y)) = \gamma_2^* f(x)) \uplus \gamma_2^*(f(-y))$$

$$= \gamma_2^*(f(x)) \uplus (-\gamma_2^*(f(y)))$$

$$= \omega_{R_2} \uplus \omega_{R_2} = \omega_{R_2}.$$

Therefore $\gamma_1^*(z) \in \overline{kerf}$. For $\gamma_1^*(r) \in R_1/\gamma_1^*$ and $\gamma_1^*(x) \in \overline{kerf}$ we have

$$\gamma_2^*(f(r \cdot x)) = \gamma_2^*(f(x) \cdot f(r)) = \gamma_2^*(f(r)) \otimes \gamma_2^*(f(x))$$

$$= \gamma_2^*(f(r)) \otimes \omega_{R_2} = \omega_{R_2},$$

and so $\gamma_1^*(r) \otimes \gamma_1^*(x) \in \overline{kerf}$. Therefore $\overline{kerf}$ is an ideal of R_1/γ^*. □

Theorem 3.23 — *Let R be a hyperring, A, B two normal hyperideals of R with $A \subseteq B$ and $\phi : [R : A^*] \to [R : B^*]$ canonical map. Suppose γ_A^*, γ_B^* be the fundamental equivalence relations on $[R : A^*]$, $[R : B^*]$ respectively, then*

$$([R : A^*]/\gamma_A^*)/\overline{ker\phi} \cong [R : B^*]/\gamma_B^*.$$

PROOF : We define the map

$$\rho : [R : A^*]/\gamma_A^* \to [R : B^*]/\gamma_B^*$$

by

$$\rho \ \ \gamma_A^*(A+x) \mapsto \gamma_B^*(B+x) \text{ for all } (x \in R).$$

We must check that ρ is well-defined, i.e., that if $x, y \in R$ and $\gamma_A^*(A+x) = \gamma_A^*(A+y)$ then $\gamma_B^*(B+x) = \gamma_B^*(B+y)$. Using Theorem 3.17, we have $\gamma_A^* = (\beta_+^*)_A$ and $\gamma_B^* = (\beta_+^*)_B$. Now $(\beta_+^*)_A(A+x) = (\beta_+^*)_A(A+y)$ if and only if there exists $(A+x_1, A+x_2, ..., A+x_n) \in [R : A^*]^n$ such that $\{A+x, A+y\} \subseteq \oplus \sum_{i=0}^{n} A+x_i$. By Corollary 3.12, we have

$$\oplus \sum_{i=1}^{n} A+x_i = \left\{ A+z \mid z \in \sum_{i=1}^{n} x_i \right\}.$$

Therefore for some $z_1 \in \sum_{i=1}^{n} x_i$, $z_2 \in \sum_{i=1}^{n} x_i$, we have $A+x = A+z_1$ and $A+y = A+z_2$. So there exist $a \in x - z_1 \cap A$ and $b \in y - z_2 \cap A$, then $x \in a + z_1$ and $y \in b + z_2$. Hence $B+x \in (B+a) \oplus (B+z_1)$ and $B+y \in (B+b) \oplus (B+z_2)$. Since $a, b \in A \subseteq B$, then $B+a = B, B+b = B$. Since $B \oplus (B+z_1) = B+z_1$ and $B \oplus (B+z_2) = B+z_2$, we have $B+x = B+z_1$ and $B+y = B+z_2$.

From $\left\{B+z_1, B+z_2\right\} \subseteq \left\{ B+z \mid z \in \sum_{i=1}^{n} x_i \right\}$, we get

$$\left\{B+x, B+y\right\} \subseteq \left\{ B+z \mid \in \sum_{i=1}^{n} x_i \right\} = \oplus \sum_{i=1}^{n} (B+x_i).$$

Therefore $(\beta_+^*)_B(B+x) = (\beta_+^*)(B+y)$. This follows that ρ is well-defined. Moreover, ρ is a strong homomorphism, for if $x, y \in R_1$ then

$$\begin{aligned} \rho(\gamma_A^*(A+x) \uplus \gamma_A^*(A+y)) &= \rho(\gamma_A^*(A+x+y)) = \gamma_B^*(B+x+y) \\ &= \gamma_B^*(B+x) \uplus \gamma_B^*(B+y) \\ &= \rho(\gamma_A^*(A+x)) \uplus \rho(\gamma_A(A+y)), \end{aligned}$$

$$\rho(\gamma_A^*(A+x) \otimes \gamma_A^*(A+y)) = \rho(\gamma_A^*(A+xy)) = \gamma_B^*(B+xy)$$

$$= \gamma_B^*(B+x) \otimes \gamma_B^*(B+y)$$

$$= \rho(\gamma_A^*(A+x)) \otimes \rho(\gamma_A^*(A+y)),$$

and $\rho(\omega_{[R.A^*]}) = \rho(\gamma_A^*(A)) = \gamma_B^*(B) = \omega_{[R:B^*]}$. Clearly, ρ is onto. Now, we show that $ker\,\rho = \overline{ker\,\phi}$. We have

$$ker\,\rho = \left\{ \gamma_A^*(A+x) \mid \rho(\gamma_A^*(A+x)) = \omega_{[R:B^*]} \right\}$$

$$= \left\{ \gamma_A^*(A+x) \mid \gamma_B^*(B+x)) = \omega_{[R:B^*]} \right\}$$

$$= \left\{ \gamma_A^*(A+x) \mid \beta_B^*(\phi(A+x)) = \omega_{[R:B^*]} \right\}$$

$$= \overline{ker\,\phi}.$$

□

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