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ON THE ORDER OF APPROXIMATION OF FUNCTIONS BY GENERALIZED BASKAKOV OPERATORS

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The object of this paper is to establish the order of approximation using modulus of continuity of f' and asymptotic behaviour of f over $[0, +\infty)$ for linear positive operator $B_n^a(f; x)$ introduced by Mihesan. Also a direct estimate is proved using Ditzian-Totik modulus of smoothness.

Key Words: Approximation by Linear Positive Operators; Pointwise Estimate; Voronovskaja Theorem; Ditzian-Totik Modulus of Smoothness

1. INTRODUCTION

In 1998 Vasile Mihesan[4] introduced a sequence of Linear Positive Operators on $[0, +\infty)$ defined as:

$$B_n^a(f; x) = \sum_{k=0}^{\infty} p_{n,k}(x, a) \cdot f\left(\frac{k}{n}\right), \quad a \geq 0, x \geq 0,$$

$$k = 0, 1, 2, \ldots, \quad n = 1, 2, 3, \ldots \qquad \ldots (1.1)$$

where,

$$p_{n,k}(x, a) = e^{\frac{-ax}{1+x}} \cdot \frac{p_k(x, a)}{k!} \cdot \frac{x^k}{(1+x)^{n+k}}, \qquad \ldots (1.2)$$

such that

$$\sum_{k=0}^{\infty} p_{n,k}(x, a) = 1, \qquad \ldots (1.3)$$

and

$$p_k(n, a) = \sum_{i=0}^{k} {}^kC_i \, (n)_i \, a^{k-i} \qquad \ldots (1.4)$$

$$\text{with } (n)_0 = 1; \quad (n)_i = n(n+1)(n+2)\ldots(n+i-1), \quad \text{for } i \geq 1. \qquad (1.5)$$

For $a = 0$, $B_n^a(f; x)$ reduces to Baskakov operator[1]. Mihesan proved the following lemma and theorems in his paper[4].

Lemma 1.1 — For $x \geq 0$, $n = 1, 2, \ldots$, we have,

$$B_n^a(1;x)=1;\quad B_n^a(t;x)=x+\frac{ax}{n(1+x)}, \qquad \text{... (1.6)}$$

$$B_n^a((t-x)^2;x) = \frac{x(1+x)}{n}+\frac{1}{n^2}\cdot\frac{ax}{1+x}\cdot\frac{(a+1)x+1}{1+x}. \qquad \text{... (1.7)}$$

He also obtained the following two estimations in order to prove Lemma (1.1):

$$\sum_{k=0}^{\infty}\frac{p_{k+1}(n,a)}{k!}\cdot\frac{x^k}{(1+x)^{n+k}} = e^{\frac{ax}{(1+x)}}[a+n(1+x)] \qquad \text{... (1.8)}$$

$$\sum_{k=0}^{\infty}\frac{p_{k+2}(n,a)}{k!}\cdot\frac{x^k}{(1+x)^{n+k}} = e^{\frac{ax}{(1+x)}}[n(1+x)^2+\{a+n(1+x)\}^2]. \qquad \text{... (1.9)}$$

The following theorem proved by Mihesan[4] yields uniform convergence for functions with exponential growth on the positive x-axis.

Theorem 1.1 — *If* $f\in C[0,+\infty)$, $|f(x)|\le e^{Ax}$, $x\ge 0$, *for some finite number A, then for a* ≥ 0, $a/n\to 0$:

$$\underset{n\to\infty}{Lt}\; B_n^a(f;x)=f(x) \qquad \text{... (1.10)}$$

holds uniformly on [0, b] for each b > 0.

Popoviciu[5] proved the following theorem which shows the manner in which the Bernstein operator $B_n(f;x)$ tends to $f(x)$:

Theorem 1.2 — *If* $f(x)$ *is continuous and* $\omega(\delta)$ *the modulus of continuity of* $f(x)$, *then,*

$$\left|f(x)-B_n(f;x)\right| \le \frac{5}{4}\,\omega\left(f;\frac{1}{\sqrt{n}}\right).$$

Let E be the set of all functions $f:[0,+\infty)\to R$ which are continuous and differentiable on $[a,b]$, $b>0$ s.t. $|f(x)|\le e^{Ax}$ for some finite number A.

Mihesan[4] established the order of approximation in terms of modulus of continuity for $f\in E$:

Theorem 1.3 — *Let* $a\ge 0$, $a/n\to 0$ *and* B_n^a *defined as in (1.1), then for every* $f\in E, x\in[0,b]$ *the following estimation*

$$\left|f(x)-B_n^a(f;x)\right| \le \left\{1+\sqrt{x(1+x)+\frac{1}{n}\cdot\frac{ax}{1+x}\cdot\frac{(a+1)x+1}{1+x}}\right\}\omega\left(f;\frac{1}{\sqrt{n}}\right) \qquad \text{... (1.11)}$$

holds.

Mihesan[4] also proved that for $a, x \geq 0$, $k = 0, 1, 2, \ldots$, the following inequality holds:

$$\sum_{k=0}^{\infty} p_{n,k}(x, a) \mid x - t \mid \leq \sqrt{B_n^a((t-x)^2; x)}. \qquad \ldots (1.12)$$

2. Estimations and Lemmas

To investigate some of the characteristics of the operator (1.1), we can easily find out from[4] the following identities for $x \geq 0$, $n = 1, 2, \ldots, a \geq 0$,

$$\sum_{k=0}^{\infty} = p_{n,k}(x, a) = \frac{ax}{1+x} + nx, \qquad \ldots (2.1)$$

$$\sum_{k=0}^{\infty} k(k-1)\, p_{n,k}(x, a) = \frac{\left[a^2 + n(1+n)(1+x)^2\right]x^2}{(1+x)^2} + \frac{2anx^2}{1+x}. \qquad \ldots (2.2)$$

Also we will prove following lemma for later use.

Lemma 2.1 : For $a, x \geq 0$, $n = 1, 2, \ldots$, if $a = a/n \to 0$ as $n \to \infty$, we have,

$$B_n^a((t-x); x) = \frac{ax}{n(1+x)} \leq 0 \qquad \ldots (2.3)$$

and

$$B_n^a(t^2; x) = \frac{x^2}{n} + \frac{x}{n} + x^2 + \frac{a^2x^2}{n^2(1+x)^2} + \frac{2ax^2}{n(1+x)} + \frac{ax}{n^2(1+x)}. \qquad \ldots (2.4)$$

Proof . Taking $f(t) = t - x$ in (1.1) we get,

$$B_n^a((t-x); x) = \sum_{k=0}^{\infty} p_{n,k}(x, a)(t-x)$$

$$= e^{\frac{-ax}{1+x}} \sum_{k=0}^{\infty} \frac{p_k(n, a)}{k!} \cdot \frac{x^k}{(1+x)^{n+k}}(t-x)$$

$$= e^{\frac{-ax}{1+x}} \sum_{k=0}^{\infty} \frac{p_k(n, a)}{k!} \cdot \frac{x^k}{(1+x)^{n+k}}\, t$$

$$- e^{\frac{-ax}{1+x}} \sum_{k=0}^{\infty} \frac{p_k(n, a)}{k!} \cdot \frac{x^k}{(1+x)^{n+k}}\, x$$

$$= B_n^a(t; x) - xB_n^a(1; x).$$

Using (1.6), we get,

$$B_n^a((t-x);x) = x+\frac{ax}{n(1+x)}-x=\frac{ax}{n(1+x)},$$

which tends to 0 as n tends to ∞.

Now, taking $f(t)=t^2$ in (1.1), we have,

$$B_n^a(t^2;x) = \sum_{k=0}^{\infty} p_{n,k}(x,a) \,.\, t^2 = \sum_{k=0}^{\infty} p_{n,k}(x,a) \,.\left(\frac{k}{n}\right)^2$$

$$= \frac{1}{n^2}\sum_{k=0}^{\infty} p_{n,k}(x,a)\,[k(k-1)+k],$$

$$B_n^a(t^2;x) = \frac{1}{n^2}\left\{\sum_{k=0}^{\infty} k(k-1)\,p_{n,k}(x,a) + \sum_{k=0}^{\infty} k\,p_{n,k}(x,a)\right\},$$

from (2.1) and (2.2), we obtain,

$$= \frac{1}{n^2}\left\{\frac{\left[a^2+n(1+n)(1+x)^2\right]x^2}{(1+x)^2} + \frac{2anx^2}{1+x}+\frac{ax}{1+x}+nx\right\},$$

$$= \frac{x^2}{n}+\frac{x^2}{n}+\frac{a^2x^2}{n^2(1+x)^2}+\frac{2ax^2}{n(1+x)}+\frac{ax}{n^2(1+x)}.$$

3. Order of Approximation

Assuming $f\in E$, we shall obtain the difference $\left|f(x)-B_n^a(f;x)\right|$ in terms of modulus of continuity of f' with $\delta=\frac{1}{\sqrt{n}}$, i.e., $\omega\left(f';\frac{1}{\sqrt{n}}\right)=\omega_1(f;\delta)$, which is an improvement over the Theorem (1.2) of Mihesan[4]

Theorem 3.1 — *Let* $a\geq 0, a/n\to 0$. *Then for every* $f\in E$, $x\in[0,b], b>0$, *such that* f *is continuous on* $[0,+\infty)$ *and* $\omega_1(\delta)$ *is the modulus of continuity of* $f'(x)$, *we have*

$$\left|B_n^a(f;x)-f(x)\right| \leq \sqrt{B_n^a((t-x)^2;x)}$$

$$\left\{1+\sqrt{n}\,.\sqrt{B_n^a((t-x)^2;x)}\right\}\ \omega_1\left(\frac{1}{\sqrt{n}}\right), \qquad \ldots (3.1)$$

where,

$$B_n^a((t-x)^2;x) = \frac{x(1+x)}{n}+\frac{1}{n^2}\cdot\frac{ax}{1+x}\cdot\frac{(a+1)x+1}{1+x}.$$

Proof : By Mean-Value Theorem of differential calculus, we have, for $x_1, x_2\in[a,b]$.

$$f(x_1)-f(x_2) = (x_1-x_2)f'(\xi)$$

$$= (x_1-x_2)f'(x_1)+(x_1-x_2)\,[f'(\xi)-f'(x_1)] \qquad \ldots (3.2)$$

where $x_1<\xi<x_2$ such that

$$\left|(x_1-x_2)\,[f'(\xi)-f'(x_1)]\right| \le \left|x_1-x_2\right| (\lambda+1)\ \omega_1(\delta),$$

$$\lambda=\lambda(x_1, x_2; \delta) \qquad \ldots (3.3)$$

Therefore, from above relation, we obtain,

$$\left|B_n^a(f; x)-f(x)\right| = \left|e^{\frac{-ax}{1+x}}\sum_{k=0}^{\infty}\frac{p_k(n, a)}{k!}\cdot\frac{x^k}{(1+x)^{n+k}}f\left(\frac{k}{n}\right)-f(x)\right|$$

and using eq. (1.2), we get,

$$= \left|\sum_{k=0}^{\infty} p_{n,k}(x, a)\, f\left(\frac{k}{n}\right)-f(x)\right|$$

Since, $\sum_{k=0}^{\infty} p_{n,k}(x, a) = 1$, we have

$$\left|B_n^a(f; x)-f(x)\right| = \left|\sum_{k=0}^{\infty} p_{n,k}(x, a)\, f\left(\frac{k}{n}\right) - \sum_{k=0}^{\infty} p_{n,k}(x, a) f(x)\right|$$

$$= \left|\sum_{k=0}^{\infty} p_{n,k}(x, a)\left[f\left(\frac{k}{n}\right)-f(x)\right]\right|$$

Taking $t=\frac{k}{n}$, the above relation changes to

$$\left|B_n^a(f; x)-f(x)\right| = \left|\sum_{k=0}^{\infty} p_{n,k}(x, a)\ [f(x)-f(t)]\right|$$

Applying (3.2) and for $x<\xi<t$, we have,

$$\left|B_n^a(f; x)-f(x)\right| = \left|\sum_{k=0}^{\infty} p_{n,k}(x, a)\ [(x-t)f'(\xi)]\right|$$

$$= \left| \sum_{k=0}^{\infty} p_{n,k}(x,a) \left\{ (x-t) f'(x) + (x-t) [f'(\xi) - f'(x)] \right\} \right|$$

$$\leq \left| \sum_{k=0}^{\infty} p_{n,k}(x,a) \ (x-t) f'(x) \right|$$

$$+ \left| \sum_{k=0}^{\infty} p_{n,k}(x,a) \ (x-t) [f'(\xi) - f'(x)] \right| .$$

Using (3.3), above inequality reduces to

$$\left| B_n^a(f;x) - f(x) \right| \leq \left| \sum_{k=0}^{\infty} p_{n,k}(x,a) \ (x-t) f'(x) \right|$$

$$+ \sum_{k=0}^{\infty} \left| p_{n,k}(x,a) \right| \ | x-t | \ (\lambda+1) \ \omega_1(\delta). \qquad \text{... (3.4)}$$

First term on the right hand side of (3.4) vanishes because of Lemma (2.1). Now, we have,

$$| f'(\xi) - f'(x) | \leq \omega_1 (| \xi - x|) = \omega_1 \left(\frac{1}{\delta} | \xi - x | . \delta \right)$$

$$\leq \left(1 + \frac{1}{\delta} | \xi - x | \right) \omega_1(\delta)$$

$$\leq \left(1 + \frac{1}{\delta} | t - x | \right) \omega_1(\delta), \qquad (x < \xi < t). \qquad \text{... (3.5)}$$

Comparing (3.3) and (3.5), we get,

$$(1+\lambda) \ \omega_1(\delta) = \left(1 + \frac{1}{\delta} | t - x | \right) \omega_1(\delta) \qquad \text{... (3.6)}$$

Using (3.6) in (3.4), we get

$$\left| B_n^a(f;x) - f(x) \right| = \sum_{k=0}^{\infty} p_{n,k}(x,a) \ | x-t | \cdot \left(1 + \frac{1}{\delta} | t - x | \right) \omega_1(\delta)$$

$$\leq \left\{ \sum_{k=0}^{\infty} p_{n,k}(x,a) \ | x-t | + \frac{1}{\delta} \sum_{k=0}^{\infty} p_{n,k}(x,a) \ (t-x)^2 \right\} \omega_1(\delta)$$

$$= \left\{ \sum_{k=0}^{\infty} p_{n,k}(x,a) \ | x-t | + \frac{1}{\delta} B_n^a((t-x)^2; x) \right\} \omega_1(\delta).$$

In view of inequality (1.12), eq. (1.7) and taking $\delta = \frac{1}{\sqrt{n}}$, above inequality changes to

$$\left| B_n^a(f; x) - f(x) \right| \leq \left\{ \sqrt{B_n^a((t-x)^2; x)} + \sqrt{n} \;.\; B_n^a((t-x)^2; x) \right\} \omega_1\left(\frac{1}{\sqrt{n}}\right)$$

$$\leq \sqrt{B_n^a((t-x)^2; x)} \;\left\{ 1 + \sqrt{n} \;.\; \sqrt{B_n^a((t-x)^2; x)} \right\} \omega_1\left(\frac{1}{\sqrt{n}}\right).$$

This is the required relation (3.1).

4. Asymptotic Relation

Later on the equation arose about the speed with which $B_n^a(f; x)$ tends to $f(x)$. An answer to this question has been given in different directions. One direction is that in which $f(x)$ is supposed to be atleast twice differentiable at a point x of $[0, +\infty)$.

We establish a similar asymptotic relation for the operator $B_n^a(f; x)$ (1.1) as proved by Voronovskaja[7] for Bernstein Operator.

Theorem 4.1 — *If* $f \in C[0, +\infty)$ *s.t.* $|f(x)| \leq e^{Ax}, x \geq 0$, *for some finite number A and supposing that* $f''(x)$ *exists at a certain point of* $[0, +\infty)$, *then*

$$B_n^a(f; x) - f(x) = \frac{1}{2}\left\{ \frac{x(1+x)}{n} + \frac{1}{n^2} \cdot \frac{ax}{1+x} \cdot \frac{(a+1)x+1}{1+x} \right\} . f''(x)$$

$$- 2x^2 \frac{\varepsilon_n^{(a)}(x)}{n} \qquad \ldots (4.1)$$

where

$$\varepsilon_n^{(a)}(x) \to 0 \text{ as } n \to \infty.$$

Proof : Let $t \in [0, +\infty)$. It is well known that if $f(t)$ has finite second order derivative at a point $x \in [0, +\infty)$, then $\exists$ a function $g(t)$ defined on $[0, +\infty)$ so that as $t -> x$, $g(t) \to 0$ and $f(t)$ can be expanded by Taylor's formula,

$$f(t) = f(x) + (t-x) f'(x) + (t-x)^2 \left[\frac{1}{2} f''(x) + g(t) \right].$$

Multiplying both sides by $p_{n,k}(x, a)$ and summing over k, we have,

$$\sum_{k=0}^{\infty} p_{n,k}(x, a) f(t) = f(x) \sum_{k=0}^{\infty} p_{n,k}(x, a) + f'(x) \sum_{k=0}^{\infty} p_{n,k}(x, a)(t-x)$$

$$+ \frac{1}{2} f''(x) \sum_{k=0}^{\infty} p_{n,k}(x, a)(t-x)^2 + \sum_{k=0}^{\infty} p_{n,k}(x, a) g(t)(t-x)^2.$$

From relations (1.1), (1.3), (2.3) and (1.7), above relation reduces to:

$$B_n^a(f;x)-f(x)$$

$$= \frac{1}{2} f''(x) \left\{ \frac{x(1+x)}{n} + \frac{1}{n^2} \cdot \frac{ax}{1+x} \cdot \frac{(a+1)x+1}{1+x} \right\} + \rho_n^{(a)}(x) \qquad \text{... (4.2)}$$

where,

$$\rho_n^{(a)}(x) = \sum_{k=0}^{\infty} p_{n,k}(x,a)(t-x)^2 g(t).$$

Furthermore,

$$\left| \rho_n^{(a)}(x) \right| \leq \sum_{|t-x| \leq \delta} p_{n,k}(x,a)(t-x)^2 |g(t)|$$

$$+ \sum_{|t-x| > \delta} p_{n,k}(x,a)(t-x)^2 |g(t)|. \qquad \text{... (4.3)}$$

Now, since $g(t) \to 0$ as $t \to x$, this implies for all $\varepsilon > 0, \exists$ another number δ such that $|t-x| < \delta \Rightarrow |g(t)| < \varepsilon$, and also for $t, x \in [0, +\infty)$, $(t-x)^2 |g(t)| < M$, we can write (4.3) as

$$\left| \rho_n^{(a)}(x) \right| \leq \varepsilon \sum_{|t-x| \leq \delta} p_{n,k}(x,a)(t-x)^2$$

$$+ M \sum_{|t-x| > \delta} p_{n,k}(x,a)(t-x)^2 .$$

Applying relation (1.7), we have,

$$\left| \rho_n^{(a)}(x) \right| \leq \varepsilon \cdot \left\{ \frac{x(1+x)}{n} + \frac{1}{n^2} \cdot \frac{ax}{1+x} \cdot \frac{(a+1)x+1}{1+x} \right\}$$

$$+ M \sum_{|t-x| > \delta} p_{n,k}(x,a)(t-x)^2 . \qquad \text{... (4.4)}$$

Given positive numbers $\delta, \eta, \exists$ positive integer $N_{(\delta,\eta)}$ such that $n > N_{(\delta,\eta)}$ implies that

$$\left\{ \begin{array}{l} \displaystyle\sum_{|t-x| > \delta} p_{n,k}(x,a)(t-x)^2 < \eta \\ \\ \displaystyle\frac{M}{\delta^2} \sum_{k=0}^{\infty} p_{n,k}(x,a)(t-x)^2 < M\eta \end{array} \right\} \qquad \text{... (4.5)}$$

Using (4.5) in (4.4), we have,

$$\left| \rho_n^{a}(x) \right| \leq \varepsilon \cdot \left\{ \frac{x(1+x)}{n} + \frac{1}{n^2} \cdot \frac{ax}{1+x} \cdot \frac{(a+1)x+1}{1+x} \right\} + M\eta$$

$$\leq \varepsilon \cdot \left\{ \frac{x(1+x)}{n} + \frac{1}{n} \cdot \frac{ax}{1+x} \cdot (ax+x+1) \right\} + M\eta$$

$$\leq \frac{\varepsilon}{n} \cdot \left\{ x(1+x) + \frac{ax}{1+x} \cdot (ax+x+1) \right\} + M\eta$$

$$\leq \frac{\varepsilon x}{n} \cdot \left\{ \frac{x^2+(a^2+a+2)\,x+(a+1)}{1+x} \right\} + M\eta.$$

Choosing,

$$\eta = \varepsilon x \cdot \left\{ \frac{x^2-(a^2+a)\,x-(a+1)}{n(1+x)\,M} \right\},$$

we have,

$$\left| \rho_n^{(a)}(x) \right| \leq \frac{\varepsilon x}{n} \cdot \left\{ \frac{x^2+(a^2+a+2)\,x+(a+1)}{1+x} \right\}$$

$$+ \frac{\varepsilon x}{n} \left\{ \frac{x^2-(a^2+a)\,x-(a+1)}{(1+x)\,M} \right\} M$$

$$= \frac{\varepsilon x}{n} \cdot \frac{2x\,(x+1)}{(1+x)},$$

therefore,

$$\left| \rho_n^{(a)}(x) \right| \leq 2x^2 \cdot \frac{\varepsilon}{n} \quad \text{or} \quad \frac{n}{2x^2} \left| \rho_n^{(a)}(x) \right| \leq \varepsilon,$$

whenever, x is greater than a certain number $n_0(\varepsilon)$.

Denoting $\frac{n}{2x^2}\, \rho_n^{(a)}(x) \leq \varepsilon$ by $-\varepsilon_n^{(a)}(x)$, we have,

$$\rho_n^{(a)}(x) = -2x^2 \cdot \frac{\varepsilon_n^{(a)}(x)}{n}$$

where,

$$\varepsilon_n^{(a)}(x) \to 0 \text{ as } n \to \infty.$$

Hence, substituting (4.6) in (4.2), we get (4.1),

$$B_n^a(f;x) - f(x) = \frac{1}{2} \left\{ \frac{x(1+x)}{n} + \frac{1}{n^2} \cdot \frac{ax}{1+x} \cdot \frac{(a+1)\,x+1}{1+x} \right\} f''(x)$$

$$- 2x^2\, \frac{\varepsilon_n^{(a)}(x)}{n}.$$

5. Direct Estimate

Recently, a new pointwise approximation for Bernstein polynomials was proved by Ditzian[2] as:

$$\left| B_n(f;x) - f(x) \right| \le C\,\omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\varphi(x)^{1-\lambda}\right),$$

$$0 \le \lambda \le 1,\ \varphi(x)^2 = x(1-x). \qquad \text{... (5.1)}$$

This result bridges the gap between the classical estimates on approximation obtained by Timan[6] and others ($\lambda = 0$) and those estimated by Ditzian and Totik ($\lambda = 1$)[3].

In this section we will prove following analogous pointwise approximation for the operator (1.1).

Theorem 5.1 — *For $f \in C[0, +\infty)$, we have,*

$$\left| B^a_n(f;x) - f(x) \right| \le \omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\varphi(x)^{1-\lambda}\right)$$

$$\left[C + \frac{1}{n} \cdot \frac{a}{(1+x)^2} \cdot \frac{(a+1)x+1}{1+x} \right],$$

$$0 \le \lambda \le 1,\ \varphi(x)^2 = x(1+x). \qquad \text{... (5.2)}$$

$$\le C\,\omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\varphi(x)^{1-\lambda}\right), \text{ for large } n.$$

Before giving the proof we need the Ditzian-Totik Modulus of smoothness[3] which is defined as:

$$\omega^2_{\varphi^\lambda}(f, \delta) = \sup_{0<h\le\delta} \left\| \Delta^2_{h\varphi(x)^2} f(x) \right\|$$

$$= \sup_{0<h\le\delta}\ \sup_{x \pm h\varphi^\lambda \in [0,+\infty)} \left| f(x - h\varphi^\lambda(x)) - 2f(x) + f(x + h\varphi^\lambda(x)) \right|,$$

where,

$$\varphi(x)^2 = x(1+x). \qquad \text{... (5.3)}$$

Also, Peetre's K-functional[3] is given by

$$K_{\varphi^\lambda}(f, \delta^2) = \inf_g \left(\| f - g \|_{C[0,+\infty)} + \delta^2 \left\| \varphi^{2\lambda} g'' \right\|_{C[0,+\infty)} \right)$$

$$g, g' \in AC_{loc}. \qquad \text{... (5.4)}$$

This K-functional is equivalent to the modulus of smoothness[3], i.e.,

$$C^{-1} K_{\varphi^\lambda}(f, \delta^2) \le \omega^2_{\varphi^\lambda}(f, \delta) \le C K_{\varphi^\lambda}(f, \delta^2). \qquad \text{... (5.5)}$$

Proof : Using (5.4) and (5.5), we can choose $g_n \equiv g_{n,x,\lambda}$ for fixed x and λ s.t.

$$\| f-g \|_{C[0,+\infty)} \le A\,\omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\varphi(x)^{1-\lambda}\right) \quad \text{... (5.6)}$$

$$n^{-1}\varphi(x)^{2-2\lambda}\left\| \varphi^{2\lambda} g'' \right\|_{C[0,+\infty)} \le B\,\omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\varphi(x)^{1-\lambda}\right). \quad \text{... (5.7)}$$

Thus, we have,

$$\left| B_n^a(f;x) - f(x) \right| \le \left| B_n^a(f-g_n;x) - (f-g_n)(x) \right| + \left| B_n^a(g_n;x) - g_n(x) \right|$$

$$\le 2\left| B_n^a(f-g_n;x) \right| + \left| B_n^a(g_n(t)-g_n(x);x) \right|,$$

with $B_n^a(1;x) = 1$ and (5.6), we get,

$$\left| B_n^a(f;x) - f(x) \right| \le 2\left\| f-g_1 \right\|_{C[0,+\infty)} B_n^a(1;x) + \left| B_n^a(g_n(t)-g_n(x);x) \right|$$

$$\le 2A\,\omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\varphi(x)^{1-\lambda}\right) + \left| B_n^a(g_n(t)-g_n(x);x) \right|. \quad \text{... (5.8)}$$

With the application of Taylor's Formula, following the steps as in[3] and using (2.3), we get,

$$\left| B_n^a(g_n(t)-g_n(x);x) \right|$$

$$\le \left| B_n^a\left((g_n'(x)(t-x) + \int_t^x (x-u) g_n''(u)\,du); x \right) \right|$$

$$\le \left| g_n'(x) B_n^a((t-x);x) \right| + B_n^a\left(\int_t^x |x-u| \left| g_n''(u) \right| du; x \right)$$

$$\le B_n^a\left(\frac{|x-k/n|}{\varphi^{2\lambda}(x)} \int_{k/n}^x \varphi^{2\lambda}(u) \left| g_n''(u) \right| du; x \right)$$

$$\le \left\| \varphi^{2\lambda} g_n'' \right\|_{C[0,+\infty)} \frac{1}{\varphi^{2\lambda}(x)} B_n^a((x-k/n)^2;x)$$

$$\le \left\| \varphi^{2\lambda} g_n'' \right\|_{C[0,+\infty)} \frac{1}{\varphi^{2\lambda}(x)} \cdot \left[\frac{x(1+x)}{n} + \frac{1}{n^2} \cdot \frac{ax}{1+x} \cdot \frac{(a+1)x+1}{1+x} \right]$$

$$\le \left\| \varphi^{2\lambda} g_n'' \right\|_{C[0,+\infty)} \varphi^{2-2\lambda}(x)\, n^{-1} \cdot \left[1 + \frac{1}{n} \cdot \frac{a}{(1+x)^2} \cdot \frac{(a+1)x+1}{1+x} \right],$$

using (5.7), it reduces to

$$\left| B_n^a (g_n(t) - g_n(x); x) \right|$$

$$\leq B\, \omega^2_{\varphi^\lambda} \left(f, n^{-1/2} \varphi(x)^{1-\lambda}\right) \left[1 + \frac{1}{n} \cdot \frac{a}{(1+x)^2} \cdot \frac{(a+1)x+1}{1+x} \right]. \qquad \dots (5.9)$$

Using (5.9) in (5.8), we yield,

$$\left| B_n^a (f; x) - f(x) \right| \leq A\, \omega^2_{\varphi^\lambda} \left(f, n^{-1/2} \varphi(x)^{1-\lambda}\right)$$

$$+ B\, \omega^2_{\varphi^\lambda} \left(f, n^{-1/2} \varphi(x)^{1-\lambda}\right) \left[1 + \frac{1}{n} \cdot \frac{a}{(1+x)^2} \cdot \frac{(a+1)x+1}{1+x} \right],$$

choosing $C = \max\, [2A, B]$, we have,

$$\leq \omega^2_{\varphi^\lambda} \left(f, n^{-1/2} \varphi(x)^{1-\lambda}\right) C \left[1 + \frac{1}{n} \cdot \frac{a}{(1+x)^2} \cdot \frac{(a+1)x+1}{1+x} \right].$$

For large value of n above inequality reduces to

$$\left| B_n^a (f; x) - f(x) \right| \leq C\, \omega^2_{\varphi^\lambda} \left(f, n^{-1/2} \varphi(x)^{1-\lambda}\right).$$

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