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ON THE UNICITY OF MEROMORPHIC FUNCTIONS THAT SHARE FOUR VALUES*

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The uniqueness of meromorphic functions that share three values IM and a fourth value CM is investigated, and the open question "if two nonconstant meromorphic functions share three values IM and a fourth value CM, then do the functions share all four values CM?" is partly resolved.

Key Words: Meromorphic Functions; Shared-Value

1. INTRODUCTION

It is assumed that the reader is familiar with Nevanlinna's theory of meromorphic functions and its basic notations, as well as its fundamental results (see Hayman[1]). Let $f(z)$ be a meromorphic function in the complex plane, we denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ for $r \to \infty$ except possibly a set of r of finite linear measure. We say that two nonconstant meromorphic functions f and g share the value $c (c = \infty$ is allowed) provided that $f(z) = c$ if and only if $g(z) = c$. Usually, we will state whether a shared value is by CM (counting multiplicities) or IM (ignoring multiplicities). We denote by $\overline{N}_E(r, f = c = g)$ or $\overline{N}_E(r, c)$ the counting function of those c-points where $f(z)$ and $g(z)$ have same multiplicity (counting each point only once), while by $\overline{N}_D(r, f = c = g)$ or $\overline{N}_D(r, c)$ the counting function of those c-points where f and g have different multiplicities (counting each point only once).

Nevanlinna (see[3]) proved the following two theorems:

***Theorem* A** — *Let f and g be non-constant meromorphic functions. If they share five distinct values $a_1, ..., a_5$ IM, then $f \equiv g$.*

***Theorem* B** — *If f and g are distrinct nonconstant meromorphic functions that share four distinct values a_1, a_2, a_3, a_4 CM, then f is a Möbius transformation of g; two of the values, say, a_1 and a_2, are Picard values, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

In 1976, Rubel asked the following question: whether CM can be replaced by IM in Theorem B with the same conclusion or not? Gundersen[4] gave a negative answer for this question by the following counterexample.

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$$f(z)=\frac{e^{h(z)}+b}{(e^{h(z)}-b)^2},\quad g(z)=\frac{(e^{h(z)}+b)^2}{8b^2\,(e^{h(z)}-b)}, \qquad \ldots (1.1)$$

where $h(z)$ is a non-constant entire function and $b\,(\neq 0)$ a finite value. It is easy to verify that f and g share $0, \infty, \frac{1}{b}, -\frac{1}{8b}$ IM but not CM. In fact, f and g share these four values with the property that f and g have different multiplicities at any of their zeros, poles, $\frac{1}{b}$-points, and $-\frac{1}{8b}$-points. And f is not a Möbius transformation of g.

On the other hand, Gundrsen showed (see[5]) an improvement of Theorem B.

Theorem C — *If two nonconstant meromorphic functions share two values IM, and share two other values CM, then f and g share all four values CM.*

However the so called "1CM + 3IM question" that "If two nonconstant meromorphic functions share three values IM and share a fourth value CM, then do the functions necessarily share all four values CM ?" remains open.

Let f and g be nonconstant meromorphic functions sharing the value a IM. Define

$$\tau(a) = \begin{cases} \liminf\limits_{r\to\infty} \dfrac{\overline{N}_E(r,a)}{\overline{N}(r,a)} & \text{if } \overline{N}(r,a) \not\equiv 0, \\ 1 & \text{if } \overline{N}(r,a) \equiv 0. \end{cases}$$

Mues proved the following partial result on this question.

Theorem D[7] — *Let f and g be nonconstant meromorphic functions that share four distinct values a_1, a_2, a_3, a_4. If a_1 is shared CM and $\tau(a_2) > 2/3$, then f and g share all four values CM.*

Gundersen[6] obtained another partial result on this question as follows:

Theorem E — *Let f and g be nonconstant meromorphic functions that share a_1, a_2, a_3 IM and a_4 CM. Suppose that there exist some real constant $\lambda > 4/5$ and some set $I \subset (0, \infty)$ that has infinite linear measure such that*

$$\frac{N(r, a_4, f)}{T(r, f)} \geq \lambda \qquad \ldots (1.2)$$

for all $r \in I$. Then f and g share all four values CM.

Wang[8] proved the following theorem, which is of the same nature as Theorem D:

Theorem F — *Let f and g be nonconstant meromorphic functions that share four distinct values a_1, a_2, a_3, a_4 IM. If $\tau(a_1) > 4/5$, $\tau(a_2) > 4/5$, then f and g share all four values CM.*

Recently, Yi and Zhou[9] got a further result, which gives Theorems D and F as Corollaries:

Theorem G — *Let f and g be nonconstant meromorphic functions that share four distinct values a_1, a_2, a_3, a_4 IM. If*

$$\tau(a_1) > 2/3, \quad \tau(a_2) > \frac{2\tau(a_1)}{5\tau(a_1)-2}, \qquad \ldots (1.3)$$

then f and g share all four values CM.

In this paper, a new partial result on the "1CM + 3IM question" is obtained:

Theorem 1 — *Let f and g be nonconstant meromorphic functions that share four distinct values a_1, a_2, a_3, a_4. If a_4 is shared CM and*

$$\min\{\tau(a_j), j = 1, 2, 3\} > 1/2,$$

then f and g share all four values CM.

And an inequality is established to include the above theorems from Theorem D to G:

Theorem 2 — *Let f and g be nonconstant meromorphic functions that share four distinct values a_1, a_2, a_3, a_4 IM. Then either the functions share all four values CM or else for every $i \in \{1, 2, 3, 4\}$, the relation*

$$\overline{N}_E(r, a_i) \leq 2\overline{N}_D(r, a_i) + 2\overline{N}_D(r, a_k) + S(r, f)$$

holds for $k \in \{1, 2, 3, 4\} \backslash \{i\}$.

Moreover, by Theorem 2, we obtain the following results:

Theorem 3 — *Let f and g be nonconstant meromorphic functions that share a_1, a_2, a_3 IM and a_4 CM. If*

$$\overline{N}(r, a_1, f) + \overline{N}(r, a_2, f) \leq \mu T(r, f) + S(r, f)$$

holds for some $\mu < 2/3$, then f and g share all four values CM.

Theorem 4 — *Let f and g be nonconstant meromorphic functions that share three distinct values a_1, a_2, a_3 IM and a fourth value a_4 CM. Suppose that there exist some real constant $\lambda > \frac{4}{2+r}$, i.e. where $\tau = \sum_{j+1}^{3} \frac{1}{1-\tau(a_j)}$ ($\tau = \infty$ if $\tau(a_j) = 1$ for some $j \in \{1, 2, 3\}$), and some set $I \subset (0, \infty)$ which has infinite linear measure such that*

$$\frac{N(r, a_4, f)}{T(r, f)} \geq \lambda \qquad \text{... (1.4)}$$

for all $r \in I$. Then f and g share all four values CM.

Remark 1 : Theorem 4 is an improvement of Theorem E.

Remark 2 : Both Theorem E and G are implied in Theorem 2. In fact, if f and g satisfy the assumption of Theorem E, then by Theorem 2, either a_1, a_2, a_3, a_4 are all shared CM, or else for every $i \in \{1, 2, 3, 4\}$ the inequality

$$\overline{N}_E(r, a_1) + 2\overline{N}_D(r, a_i) + 2\overline{N}_D(r, a_k) + S(r, f)$$

holds for $k \in \{1, 2, 3, 4\} \backslash \{i\}$. So we assume

$$\overline{N}_E(r, a_4) \le 2\overline{N}_D(r, a_k) + S(r,f), \qquad k = 1, 2, 3,$$

since $\overline{N}_D(r, a_4) = 0$. It follows that

$$3\,\overline{N}_E(r, a_4) \le 2 \sum_{j=1}^{3} \overline{N}(r, a_j) + S(r,f).$$

Hence

$$5\,\overline{N}_E(r, a_4) \le 2 \sum_{j=1}^{3} \overline{N}(r, a_j) + S(r,f)$$

since by Lemma 1 below we have $N(r, a_4) = \overline{N}_E(r, a_4) + S(r,f)$. Furthermore, by Lemma 1, we have

$$5\,N(r, a_4) = 4T(r,f) + S(r,f)$$

which contradicts (1.2). Thus f and g share all four values CM.

To show that Theorem G is also a consequence of Theorem 2, we are proceeding with the assumption (1.3), from which it follows that, for any given positive numbers λ and μ such that

$$2/3 < \lambda < \tau(a_1) \quad \text{and} \quad \frac{2\lambda}{5\lambda - 2} < \mu < \tau(a_2), \qquad \dots (1.5)$$

the inequalities

$$\overline{N}_E(r, a_1) > \overline{N}(r, a_1), \qquad \text{and} \quad \overline{N}_E(r, a_2) > \mu\,\overline{N}(r, a_2)$$

hold for sufficiently large r, namely

$$\lambda\,\overline{N}_D(r, a_1) < (1 - \lambda)\,\overline{N}_E(r, a_1), \qquad \dots (1.6)$$

$$\mu\,\overline{N}_D(r, a_2) < (1 - \mu)\,\overline{N}_E(r, a_2). \qquad \dots (1.7)$$

On the other hand, by Theorem 2, we only need to consider the case that the following two inequalities hold:

$$\overline{N}_E(r, a_1) \le 2\overline{N}_D(r, a_1) + 2\overline{N}_D(r, a_2) + S(r,f), \qquad \dots (1.8)$$

$$\overline{N}_E(r, a_2) \le 2\overline{N}_D(r, a_2) + 2\overline{N}_D(r, a_1) + S(r,f). \qquad \dots (1.9)$$

From (1.6)~(1.9), we have

$$\lambda\,\overline{N}_D(r, a_1) < 2(1 - \lambda)\,\overline{N}_D(r, a_1) + 2(1 - \lambda)\,\overline{N}_D(r, a_2) + S(r,f),$$

$$\mu\,\overline{N}_D(r, a_2) < 2(1 - \mu)\,\overline{N}_D(r, a_2) + 2(1 - \mu)\,\overline{N}_D(r, a_1) + S(r,f)$$

or

$$(3\lambda - 2)\,\overline{N}_D(r, a_1) < 2(1 - \lambda)\,\overline{N}_D(r, a_2) + S(r,f), \qquad \dots (1.10)$$

$$(3\mu-2)\,\overline{N}_D(r,a_2) < 2(1-\mu)\,\overline{N}_D(r,a_1) + S(r,f). \qquad \dots (1.11)$$

Substituting (1.11) into (1.10) yields

$$(3\lambda-2)(3\mu-2)\overline{N}_D(r,a_1) < (2-2\lambda)(2-2\mu)\,\overline{N}_D(r,a_1) + S(r,f),$$

which implies

$$\overline{N}_D(r,a_1) = S(r,f),$$

since (1.5) holds. Similarly, by substituting (1.10) into (1.11), we can deduce

$$\overline{N}_D(r,a_2) = S(r,f).$$

Thus both a_1 and a_2 are shared CM* (where the terminology "two nonconstant meromorphic functions share the value a CM*" means a is shared by f and g and furthermore, $\overline{N}(r,a) = \overline{N}_E(r,a) + S(r,f)$. This leads to that f and g share all four values CM by the following result which is a slight generalization of Theorem C:

***Theorem* C*[6]** — *If two nonconstant meromorphic functions share two values IM, and share two other values CM*, then f and g share all four values CM.*

2. Lemmas

For proving the theorems, we need the following lemmas.

Lemma 1[2,4,5,10] : Let f and g be distinct nonconstant meromorphic functions that share four values a_1, a_2, a_3, a_4 IM. Then the following statements hold:

(*i*) $T(r,f) = T(r,g) + S(r,f),\ T(r,g) = T(r,f) + S(r,g)$;

(*ii*) $\displaystyle\sum_{j=1}^{4} \overline{N}\left(r, \frac{1}{f-a_j}\right) = 2T(r,f) + S(r,f)$;

(*iii*) $N_0\left(\dfrac{1}{f'}\right) = S(r,f),\ N_0\left(\dfrac{1}{g'}\right) = S(r,g)$,

where $N_0\left(\dfrac{1}{f'}\right)$ and $N_0\left(\dfrac{1}{g'}\right)$ are respectively the counting functions of the roots of $f' = 0$ and $g' = 0$ that refer only to those points z such that $f(z) \neq a_i$ and $g(z) \neq a_i$ for $i = 1, 2, 3, 4$.

(*iv*) $\displaystyle\sum_{j=1}^{4} N^*(r,a_j) = S(r,f)$,

where $N^*(r,a_j)$ is the counting function for common multiple zeros of $f(z) = a_j$ and $g(z) = a_j$, counting the smaller one of the two multiplicities at each of the points.

Lemma 2 — Let f be a nonconstant meromorphic function and let $b_1, b_2, \dots, b_q$ be q constants. Then for any polynomial $P(f)$ of degree $p (p < q)$ in f with constant coefficients, the equality

$$m\left(\frac{P(f)f'}{(f-b_1)(f-b_2)\cdots(f-b_q)}\right)=S(r,f)$$

holds.

PROOF : It is easy to see

$$\frac{P(f)}{(f-b_1)(f-b_2)\cdots(f-b_q)}=\sum_{j=1}^{q}\frac{A_j}{f-b_j},$$

where $A_j\,(j = 1, 2, ..., q)$ are constants. Thus

$$m\left(r,\frac{P(f)f'}{(f-b_1)(f-b_2)\cdots(f-b_q)}\right)=m\left(r,\sum_{j=1}^{q}\frac{A_jf'}{f-b_j}\right)$$

$$\leq\sum_{j=1}^{q}m\left(r,\frac{f'}{f-b_j}\right)+O(1)$$

$$=S(r,f).$$

Lemma 3 — Let f and g be distinct nonconstant meromorphic functions that share four values a_1, a_2, a_3, ∞ IM. Then the function

$$\psi(z)=\frac{f'g'(f-g)^2}{(f-a_1)(f-a_2)(f-a_3)(g-a_1)(g-a_2)(g-a_3)}$$

is an entire function and satisfies

$$T(r,\psi(z))=S(r,f).$$

PROOF : Let z_0 be a point such that $f(z_0)=a$ with multiplicity p and $g(z_0)=a$ with multiplicity q, where $a\in\{a_1, a_2, a_3, \infty\}$. Then

$$\psi(z)=((z-z_0)^{2min(p,q)^{-2}}).$$

Hence $\psi(z)$ is an entire function, and so

$$N(r,\psi)=0. \qquad \text{... (2.1)}$$

By Lemma 2 and Lemma 1 (*i*), we have

$$m(r,\psi(z))\leq\left(r,\frac{f^2f'g'}{(f-a_1)(f-a_2)(f-a_3)(g-a_1)(g-a_2)(g-a_3)}\right)$$

$$+m\left(r,\frac{2ff'gg'}{(f-a_1)(f-a_2)(f-a_3)(g-a_1)(g-a_2)(g-a_3)}\right)$$

$$+m\left(r,\frac{f'g^2g'}{(f-a_1)(f-a_2)(f-a_3)(g-a_1)(g-a_2)(g-a_3)}\right)+O(1)$$

$$+m\left(r,\frac{f'f^2}{(f-a_1)(f-a_2)(f-a_3)}\right)+m\left(r,\frac{ff'}{(f-a_1)(f-a_2)(f-a_3)}\right)$$

$$+m\left(r,\frac{g'g}{(g-a_1)(g-a_2)(g-a_3)}\right)+m\left(r,\frac{g^2g'}{(g-a_1)(g-a_2)(g-a_3)}\right)$$

$$+\ S(r,f)+S(r,g)$$

$$=\ S(r,f)+S(r,g)=S(r,f).$$

By this and (2.1), we get

$$T(r,\psi(z))=m(r,f)+N(r,f)=S(r,f).$$

Lemma 4 — Let f and g be nonconstant meromorphic functions that share four distinct values a_1, a_2, a_3, a_4 IM. Then either the functions share all four values CM or else the inequality

$$\overline{N}(r,a_i)\le \overline{N}_D(r,a_k)+\overline{N}_D(r,a_m)+S(r,f),$$

holds for distinct $i, j, k, m \in \{1, 2, 3, 4\}$.

PROOF : Assume $f\not\equiv g$ and without loss of generality, $a_4=\infty$. Set

$$\eta_2=\frac{f'(f-a_1)}{(f-a_2)(f-a_3)}-\frac{g'(g-a_1)}{(g-a_2)(g-a_3)} \qquad \text{... (2.2)}$$

$$\eta_3=\frac{f'(f-a_2)}{(f-a_1)(f-a_3)}-\frac{g'(g-a_2)}{(g-a_1)(g-a_3)} \qquad \text{... (2.3)}$$

$$\eta_1=\frac{f'(f-a_3)}{(f-a_1)(f-a_2)}-\frac{g'(g-a_3)}{(g-a_1)(g-a_2)} \qquad \text{... (2.4)}$$

If $\eta_1\equiv 0$, then a_1, a_2, a_3, a_4 are shared CM by f and g. And we have the same conclusion when η_2 or η_3 vanishes identically. Therefore, we assume $\eta_1\eta_2\eta_3\not\equiv 0$.

From (2.2), Lemma 1 (*i*) and Lemma 2, we have

$$\overline{N}(r,a_1)\le N\left(r,\frac{1}{\eta_1}\right)\le T(r,\eta_1)+O(1)$$

$$\le N(r,\eta_1)+S(r,f)+S(r,g)$$

$$\le \overline{N}_D(r,a_2)+\overline{N}_D(r,a_3)+\overline{N}_D(r,f)+S(r,f). \qquad \text{... (2.5)}$$

Similarly, considering η_2 and η_3, we have

$$\overline{N}(r,a_2)\le \overline{N}_D(r,a_1)+\overline{N}_D(r,a_3)+\overline{N}_D(r,f)+S(r,f), \qquad \text{... (2.6)}$$

$$\overline{N}(r,a_3)\le \overline{N}_D(r,a_1)+\overline{N}_D(r,a_2)+\overline{N}_D(r,f)+S(r,f). \qquad \dots (2.7)$$

Set

$$F=\frac{1}{f-a_1},\ G=\frac{1}{g-a_1}.$$

Then F and g share b_1, b_2, b_3, b_4 IM, where $b_1=\infty, b_2=\frac{1}{a_2-a_1},\ b_3=\frac{1}{a_3-a_1},\ b_4=0.$

Put

$$\eta_4=\frac{F'(F-b_4)}{(F-b_2)(F-b_3)}-\frac{G'(G-b_4)}{(G-b_2)(F-b_3)} \qquad \dots (2.8)$$

If $\eta_4\equiv 0$, then b_1, b_2, b_3, b_4 are shared CM by F and G. Thus a_1, a_2, a_3, a_4 are shared CM by f and g. Now we suppose $\eta_4\not\equiv 0$. Since $T(r,F)=T(r,f)+O(1)$, $T(r,G)=T(r,g)+O(1)$, from (2.8), Lemma 1(i) and Lemma 2 we deduce that

$$\overline{N}(r,a_4,f)\le\overline{N}(r,b_4,F)\le N\left(r,\frac{1}{\eta_4}\right)\le T(r,\eta_4)+O(1)$$

$$\le N(r,\eta_4)+S(r,F)+S(r,G)$$

$$\le \overline{N}_D(r,b_2,F)+\overline{N}_D(r,b_3,F)+\overline{N}_D(r,F)+S(r,F)+S(r,G)$$

$$\le\overline{N}_D(r,a_2)\le \overline{N}_D(r,a_3)+\overline{N}_D(r,a_1)+S(r,f), \qquad \dots (2.9)$$

By (2.5, (2.6), (2.7) and (2.9), we complete the proof of Lemma 4.

3. Proof of Theorem 1

Let

$$\psi(x)=\frac{2(1-x)}{3-2x},\ \psi^{-1}=\frac{2-3x}{2(1-x)}. \qquad \dots (3.1)$$

We state their two behaviours below:

(A) both of the functions $\psi(x)$ and $\psi^{-1}(x)$ are decreasing in the interval $[0, 1)$;

(B) if $x\in[0,1)$, then the relation $\psi(x)\le\psi^{-1}(x)$ is equivalent to each of the four inequalities: $x\le\psi(x), x\le\psi^{-1}(x), x\le\frac{1}{2}$ and $\psi(x)\ge\frac{1}{2}$.

Now assume that $\frac{1}{2}<\tau(a_1)\le\tau(a_2)\le\tau(a_3)$. Take

$$\mu_j<\tau(a_j) \quad\text{and}\quad \frac{1}{2}<\mu_1<\mu_2<\mu_3. \qquad \dots (3.2)$$

For sufficiently large r, we have

$$\overline{N}_E(r, a_j) > \mu_j \overline{N}(r, a_j), j = 1, 2, 3. \quad \text{... (3.3)}$$

By Lemma 4, we may assume

$$\overline{N}(r, a_1) \le \overline{N}_D(r, a_2) + \overline{N}_D(r, a_3) + S(r, f), \quad \text{... (3.4)}$$

$$\overline{N}(r, a_2) \le \overline{N}_D(r, a_1) + \overline{N}_D(r, a_3) + S(r, f), \quad \text{... (3.5)}$$

$$\overline{N}(r, a_3) \le \overline{N}_D(r, a_1) + \overline{N}_D(r, a_2) + S(r, f), \quad \text{... (3.6)}$$

It follows from (3.4) and (3.5) that

$$\overline{N}_E(r, a_1) + \overline{N}_E(r, a_2) \le 2\overline{N}_D(r, a_3) + S(r, f), \quad \text{... (3.7)}$$

From (3.3), we deduce

$$\overline{N}_D(r, a_3) < (1 - \mu_3)\overline{N}(r, a_3). \quad \text{... (3.8)}$$

Combining (3.7) and (3.8) yields

$$\overline{N}_E(r, a_1) + \overline{N}_E(r, a_2) \le 2(1 - \mu_3)\overline{N}(r, a_3) + S(r, f). \quad \text{... (3.9)}$$

Since $\overline{N}_D(r, a_1) < (1 - \mu_1)\overline{N}(r, a_1), \overline{N}_D(r, a_2) < (1 - \mu_2)\overline{N}(r, a_2)$, by substituting (3.6) into (3.9), we have

$$\overline{N}_E(r, a_1) + \mu\overline{N}_E(r, a_2) < 2(1 - \mu_3)$$
$$\{(1 - \mu_1)\overline{N}(r, a_1) + (1 - \mu_2)\overline{N}(r, a_2)\} + S(r, f).$$

From this and (3.3) we get

$$\mu_1\overline{N}(r, a_1) + \mu_2\overline{N}_E(r, a_2) < 2(1 - \mu_3)$$
$$\{(1 - \mu_1)\overline{N}(r, a_1) + (1 - \mu_2)\overline{N}(r, a_2)\} + S(r, f),$$

or

$$\{\mu_1 - 2(1 - \mu_3)(1 - \mu_1)\overline{N}(r, a_1)\}$$
$$+ \{\mu_2 - 2(1 - \mu_3)(1 - \mu_2)\overline{N}(r, a_2)\} < S(r, f).$$

That is

$$(\mu_1 - \psi(\mu_3))\overline{N}(r, a_1) + (\mu_2 - \psi(\mu_3))\overline{N}(r, a_2) < S(r, f). \quad \text{... (3.10)}$$

which implies $\overline{N}(r, a_1) = S(r, f)$, and $\overline{N}(r, a_2) = S(r, f)$ since from (3.2) and (A) or (B) we know $\psi(\mu_3) < 1/2$. By Theorem C* we see a_1, a_2, a_3, a_4 are all shared CM by $f(z)$ and $g(z)$.

4. Proof of Theorem 2

Now we come to prove Theorem 2. The proof of Theorem 2 below has similarities to the proof of Theorem E in [6]

If $f \equiv g$, then there is nothing to prove. So we assume $f \not\equiv g$. Picking an integer $i \in \{1, 2, 3, 4\}$, say $i = 4$, we shall estimate $\overline{N}(r, a_4)$ by considering two cases.

Case 1 : $a_4 = \infty$.

Put

$$\psi(z) = \left(\frac{f' g' (f-g)^2}{(f-a_1)(f-a_2)(f-a_3)(g-a_1)(g-a_2)(g-a_3)} \right) \quad \text{... (4.1)}$$

$$\eta = \frac{f''}{f'} - \frac{f'}{f-a_1} - \frac{f'}{f-a_2} - \frac{f'}{f-a_3}$$

$$-\left(\frac{g''}{g'} - \frac{g'}{g-a_1} - \frac{g'}{g-a_2} - \frac{g'}{g-a_3} \right) \quad \text{... (4.2)}$$

$$T(r, \psi) = S(r, f). \quad \text{... (4.3)}$$

From the proof of Lemma 3, we know ψ is an entire function and satisfies

It is obvious that $m(r, \eta) = S(r, f)$ from the fundamental estimate of the logarithmic derivative and Lemma 1(*i*). By considering residues in (4.2), we deduce that η is analytic at any a-point $(a \in \{a_1, a_2, a_3\})$ as well as at those poles where $f(z)$ and $g(z)$ have the same multiplicities. And it is obvious that η has a simple pole when $f = a_4$ and $g = a_4$ with different multiplicities. Thus from (4.2) and Lemma 1(*iii*) we obtain that $N(r, \eta) = \overline{N}_D(r, a_4) + S(r, f)$. Hence

$$T(r, \eta) = \overline{N}_D(r, a_4) + S(r, f). \quad \text{... (4.4)}$$

Now consider the following functions:

$$H_1 = \frac{f'}{f-a_1} - \frac{g'}{g-a_1} \quad \text{... (4.5)}$$

$$H_2 = \frac{f'}{f-a_2} - \frac{g'}{g-a_2} \quad \text{... (4.6)}$$

$$H_3 = \frac{f'}{f-a_3} - \frac{g'}{g-a_3} \quad \text{... (4.7)}$$

From the fundamental estimate of the logarithmic derivative and Lemma 1(*i*), we have

$$m(r, H_j) = S(r, f), j = 1, 2, 3. \quad \text{... (4.8)}$$

From (4.5), (4.6), (4.7) and (4.2), we have

$$N(r, 3H_j + \eta) = \overline{N}_D(r, a_j) + \overline{N}_D(r, a_4) + S(r, f), \; j = 1, 2, 3. \quad \text{... (4.9)}$$

and

$$m(r, 3H_j + \eta) = S(r, f), \; j = 1, 2, 3.$$

Hence

$$T(r, 3H_j + \eta) = \overline{N}_D(r, a_j) + \overline{N}_D(r, a_4) + S(r,f),\ j = 1, 2, 3. \qquad \text{... (4.10)}$$

Let z_1 be a common simple pole of f and g. Assume that

$$f(z) = (z - z_1)^{-1}(b_0 + b_1(z - z_1) + b_2(z - z_1)^2 + \cdots)$$

$$g(z) = (z - z_1)^{-1}(c_0 + c_1(z - z_1) + c_2(z - z_1)^2 + \cdots)$$

An elementary calculation gives that

$$H_1(z_1) = \frac{b_1}{b_0} - \frac{c_1}{c_0} - a_1\left(\frac{1}{b_0} - \frac{1}{0}\right) \qquad \text{... (4.11)}$$

$$\eta(z_1) = -3\left(\frac{b_1}{b_0} - \frac{c_1}{c_0}\right) + (a_1 + a_2 + a_3)\left(\frac{1}{b_0} - \frac{1}{c_0}\right) \qquad \text{... (4.12)}$$

$$\psi(z_1) = \left(\frac{1}{b_0} - \frac{1}{c_0}\right)^2. \qquad \text{... (4.13)}$$

From (4.11), (4.12) and (4.13) we obtain

$$(3H_1(z_1) + \eta(z_1))^2 = (2a_1 - a_2 - a_3)^2\,\psi(z_1) \qquad \text{... (4.14)}$$

If

$$(3H_1 + \eta)^2 \equiv (2a_1 - a_2 - a_3)^2\,\psi$$

then $3H_1 + \eta$ has no poles since ψ is an entire function. Thus $N(r, 3H_1 + \eta) = \overline{N}_D(r, a_1) + \overline{N}_D(r, a_4) + S(r,f) = 0$, which implies a_1 and $a_4 (= \infty)$ must be shared CM by f and g. Thus f and g share all four values CM by Theorem C. Now we suppose

$$(3H_1 + \eta)^2 \not\equiv (2a_1 - a_2 - a_3)^2\,\psi.$$

Then from Lemma 1(*iv*), (4.14) and (4.3), we can deduce that

$$\overline{N}_E(r, a_4) \le N(r, 0, (3H_1 + \eta)^2 - (2a_1 - a_2 - a_3)^2\,\psi) + S(r,f)$$

$$\le 2T(r, 3H_1 + \eta) + T(r, \psi) + S(r,f)$$

$$\le 2T(r, 3H_1 + \eta) + S(r,f).$$

It follows from (4.10), that

$$\overline{N}_E(r, a_4) \le 2\overline{N}_D(r, a_1) + 2\overline{N}_D(r, a_4) + S(r,f). \qquad \text{... (4.15)}$$

Similarly, considering H_2 and H_3, we can obtain that either f and g share all four values CM or else

$$\overline{N}_E(r, a_4) \le 2\overline{N}_E(r, a_2) + 2\overline{N}_E(r, a_4) + S(r,f). \qquad \text{... (4.16)}$$

and

$$\overline{N}_E(r, a_4) \le 2\overline{N}_D(r, a_3) + 2\overline{N}_D(r, a_4) + S(r,f). \qquad \text{... (4.17)}$$

hold. It is shown from (4.15), (4.16) and (4.17) that in the case $a_4 = \infty$ the conclusion of Theorem 2 is valid.

Case 2 : $a_4 \ne \infty$.

Set

$$F = \frac{1}{f - a_4}, \quad \frac{1}{g - a_4}$$

Then F and G share b_1, b_2, b_3, b_4 IM, where $b_j = \dfrac{1}{a_j - a_4}, j = 1, 2, 3$; $b_4 = \infty$. Since $T(r, F) = T(r,f) + O(1)$, $\overline{N}_E(r, b_j) = \overline{N}_E(r, a_j)$ and $\overline{N}_D(r, b_j) = \overline{N}_D(r, a_j)$, $j = 1, 2, 3, 4$, treating $\overline{N}_E(r, b_4)$ in the same way as in Case 1, we still obtain that either f and g share all four values CM or else (4.15), (4.16) and (4.17) hold. Thus Theorem 2 is proved.

5. THE PROOF OF THEOREM 3

Assume that each of the values a_1, a_2, a_3 is not shared CM by $f(z)$ and $g(z)$, then the following inequalities hold by Lemma 4 and Theorem 2.

$$\overline{N}(r, a_3) \le \overline{N}_D(r, a_1) + \overline{N}_D(r, a_2) + S(r,f),$$

$$\overline{N}(r, a_4) \le 2\overline{N}_D(r, a_1) + S(r,f)$$

$$\overline{N}(r, a_4) \le 2\overline{N}_D(r, a_2) + S(r,f).$$

It follows that

$$2T(r,f) = \sum_{j=1}^{4} \overline{N}(r, a_j) + S(r,f)$$

$$\le \overline{N}(r, a_1) + \overline{N}(r, a_2) + 2\overline{N}_D(r, a_1) + 2\overline{N}_D(r, a_2) + S(r,f)$$

$$\le 3(\overline{N}(r, a_1) + \overline{N}(r, a_2)) + S(r,f).$$

This is a contradiction since $\overline{N}(r, a_1) + \overline{N}(r, a_2) \le \mu T(r,f) + S(r,f)$, and $\mu < 2/3$. Thus Theorem 3 is proved.

6. THE PROOF OF THEOREM 4

Assume that each of the values a_1, a_2, a_3 is not shared CM by $f(z)$ and $g(z)$. Notice that a_4 is shared CM by $f(z)$ and $g(z)$, by Theorem 2, we have

$$\overline{N}(r, a_i) \le 2\overline{N}_D(r, a_i), \quad i = 1, 2, 3. \qquad \text{... (61)}$$

If $\tau(a_i) > 0$ $(i = 1, 2, 3)$, then we take $0 < \mu_i < \tau(a_i)$. The inequality

$$\overline{N}(r, a_i) \leq (1 - \mu_i)\,\overline{N}(r, a_i), \qquad (i = 1, 2, 3) \qquad \ldots (6.2)$$

holds for sufficiently large r.

If $\tau(a_i) = 0$, then we take $\mu_i = 0$. The inequality (6.2), still holds.

So from (6.1) and (6.2), it follows that

$$\overline{N}(r, a4) \leq 2\,(1 - \mu i)\,\overline{N}(r, ai), \qquad (i = 1, 2, 3) \qquad \ldots (6.3)$$

Hence,

$$\{(1 - \mu_1)\,(1 - \mu_2) + (1 - \mu_2)\,((1 - \mu_3)$$

$$+ (1 - \mu_3)\,(1 - \mu_1)\}\;\overline{N}(r, a_4) \leq 2A \sum_{j=1}^{3} \overline{N}(r, a_j),$$

where

$$A = (1 - \mu_1)\,(1 - \mu_2)\,(1 - \mu_3).$$

That is

$$\{(1 - \mu_1)\,(1 - \mu_2) + (1 - \mu_2)\,((1 - \mu_3)$$

$$+ (1 - \mu_3)\,(1 - \mu_1) + 2A\}\;\overline{N}(r, a_4) \leq 2A \sum_{j=1}^{4} \overline{N}(r, a_j).$$

From this and Lemma 1(*ii*), we derive that

$$\limsup_{\substack{r \to \infty \\ r \notin E}} \frac{\overline{N}(r, a_4)}{T(r, f)} \leq \frac{4}{2 + \sum_{j=1}^{3} \frac{1}{1 - \mu i}},$$

where E is a set of r of finite linear measure. This leads to

$$\limsup_{\substack{r \to \infty \\ r \notin E}} \frac{\overline{N}(r, a_4)}{T(r, f)} \leq \frac{4}{2 + \sum_{j=1}^{3} \frac{1}{1 - \tau(a_j)}},$$

which contradicts the condition (1.4). Thus at least one of a_1, a_2, a_3 must be shared CM by $f(z)$ and $g(z)$. As a_4 is also shared CM, a_1, a_2, a_3 are all shared CM by $f(z)$ and $g(z)$ according to Theorem C. This completes the proof of Theorem 4.

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