Indian J. pure appl. Math., **35**(3) : 373-387, March 2004

SECOND ORDER IMPULSIVE DIFFERENTIAL EQUATIONS OF MIXED MONOTONE TYPE ON UNBOUNDED DOMAINS IN BANACH SPACES*

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(*Received* 11 *November* 2002; *after final revision* 7 *July* 2003; *accepted* 18 *November* 2003)

In this paper, we use a new coupled fixed point theorem for mixed monotone operators to obtain an existence and uniqueness theorem of solutions of initial value problems for nonlinear second order impulsive differential equations of mixed monotone type on unbounded domains in Banach spaces and its application.

Key Words: Initial Value Problem; Impulsive Differential Equation of Mixed Monotone Type; Ordered Banach Space; Measure of Noncompactness

1. INTRODUCTION AND PRELIMINARIES

The theory of nonlinear impulsive differential equations is an important branch of differential equations[1]. In a recent paper[2], in the special case where f does not contain u', the existence and uniqueness theorem of solutions of initial value problems for second order mixed monotone type of impulsive differential equations in Banach spaces was investigated only on a finite interval with a finite number of impulsive times. In this paper, we shall use a new coupled fixed point theorem with weaker compactness condition, i.e., a completely diferent method to discuss the initial value problems (IVP) for the nonlinear second order impulsive differential equations of mixed monotone type on a infinite interval with a infinite number of impulsive times in the Banach space E:

*Project supported by : N FC (19871048) and NSFSP (Z2000A02).

$$\begin{cases} u''=f(t,u,u,u'), & t\geq 0,\ t\neq t_i, \\ \Delta u|_{t=t_i}=I_i(u(t_i),u(t_i)), & \\ \Delta u'|_{t=t_i}=\overline{I}_i(u(t_i),u(t_i)) & (i=1,2,\ldots), \\ u(0)=w_0,\ u'(0)=w_1, & \end{cases} \quad \ldots (1)$$

where $f\in C[J\times E\times E\times E,E]$, $J=[0,\infty)$, $0 < t_1<t_2<\ldots<t_i<\ldots, t_i\to\infty$ as $i\to\infty$, $I_i,\overline{I}_i\in C[E\times E,E]$ $(i = 1, 2, \ldots)$, $w_0,w_1\in E$. $\Delta u|_{t=t_i}$ denotes the jump of $u(t)$ at $t=t_i$, i.e., $\Delta u|_{t=t_i} = u(t_i^+)-u(t_i^-)$, where $u(t_i^+)$ and $u(t_i^-)$ represent the right and left limits of $u(t)$ at $t=t_i$, respectively, and $\Delta u'|_{t=t_i}$ has a similar meaning for $u'(t)$.

Let $PC[J,E] = \{u:J\to E\mid u(t)$ is continuous at $t\neq t_i$, left continuous at $t=t_i$ and $u'(t_i^+)$, $u'(t_i^+)$ exist for $i = 1, 2, \ldots\}$, $PC^1[J,E] = \{\mu\in PC[J,E]\mid u'(t)$ is continuous at $t\neq t_i$ and $u'(t_i^-)$, $u'(t_i^+)$ exist for $i = 1, 2, \ldots\}$. For $u\in PC^1[J,E]$, by virtue of the mean value theorem,

$$u'_-(t_i) = \lim_{h\to 0^+} h^{-1}[u(t_i)-u(t_i-h)]=u'(t_i^-).$$

Throughout this paper $u'(t_i)$ is understood as $u'_-(t_i)$. Evidently, $PC[J,E]$ and $PC^1[J,E]$ both are Banach spaces with norms $\|u\|_{PC}=\sup_{t\in J}\|u(t)\|$ and $\|u\|_{PC^1}=\max\{\|u\|_{PC}, \|u'\|_{PC}\}$, respectively. Let $J'=J\setminus\{t_1,t_2,\ldots,t_i,\ldots\}$. For any $r>0$, we assume that there are finite number k of impulsive times in the interval $[0, r]$ and denote $J_r = [0, r]$, $J_{r0} = [0, t_1]$, $J_{01}=(t_1,t_2]$, ..., $J_{rj}=(t_j,t_{j+1}]$, ..., $J_{rk}=(t_k,r]$, $J_r'=J\setminus\{t_1,t_2,\ldots,t_i,\ldots,t_k\}$. A map $u\in PC^1[J,E]\cap C^2[J',E]$ is called a solution of IVP (1), if it satisfies (1).

Let E be a real Banach space with norm $\|\cdot\|$. The cone P defines a partial ordering in E by $x,y\in x\leq y\Leftrightarrow y-x\in P$. We write order interval $[x,y] = \{u\in E\mid x\leq u\leq y\}$. The properties of the cone and the partial order may be found in [3]. Let $D\subset E$. $A:D\times D\to E$ is called a mixed monotone operator, if $A(x,y)$ is nondecreasing in x and nonincreasing in y. A is called demicontinuous at point $(x_0,y_0)\in D\times D$, if for any $\{x_n\}$, $\{y_n\}\subset D, x_n\to x_0, y_n\to y_0 \Rightarrow A(x_n,y_n)\rightharpoonup A(x_0,y_0)$. If A is demicontinuous at each point of $D\times D$, then A is called demicontinuous on $D\times D$. It is obvious that if A is continuous, then A is demicontinuous. Point $(x^*,y^*)\in D\times D$ is called a coupled fixed point of A, iff $A(x^*,y^*)=x^*$ and $A(y^*,x^*)=y^*$. When $x^*=y^*$, i.e., $A(x^*,x^*)=x^*$, we say that

$x^* \in D$ is called a fixed point of A. Recall that $\to$ and $\rightharpoonup$ denote the strong convergence and the weak convergence, respectively.

Lemma 1 — Let P be a cone of real Banach space $E, u_0, v_0 \in E, u_0 \leq v_0$. $A : [u_0, v_0] \times [u_0, v_0] \to E$ is a mixed monotone operator and demicontinuous, and the following two conditions are satisfied:

(*i*) $u_0 \leq A(u_0, v_0), \quad A(v_0, u_0) \leq v_0$.

(*ii*) For some x_0 and some $y_0 \in [u_0, v_0], C_1, C_2 \subset [u_0, v_0]$ countable and

$$\overline{C}_1 \subset \overline{co}\left\{\left\{x_0\right\} \cup A(C_1, C_2)\right\}, \qquad \overline{C}_2 \subset \overline{co}\left\{\left\{y_0\right\} \cup A(C_2, C_1)\right\}$$

imply C_1, C_2 are relatively compact.

Then A has a minimax coupled fixed point $(x^*, y^*) \in [u_0, v_0] \times [u_0, v_0]$, i.e., for any coupled fixed point $(\overline{x}, \overline{y}) \in [u_0, v_0] \times [u_0, v_0]$ of A, it must be that $x^* \leq \overline{x} \leq y^*, x^* \leq \overline{y} \leq y^*$; moreover, we have $x^* = \lim_{n \to \infty} u_n, y^* = \lim_{n \to \infty} v_n$, where $u_n = A(u_{n-1}, v_{n-1}), v_n = A(v_{n-1}, u_{n-1})$ $(n = 1, 2, ...)$, which satisfy

$$u_0 \leq u_1 \leq \ldots \leq u_n \leq \ldots \leq v_n \leq \ldots \leq v_1 \leq v_0. \qquad \text{... (2)}$$

PROOF : From (*i*), we know $u_0 < v_1 \leq v_1 \leq v_0$. By induction, (2) holds. Let $B_1 = \{u_n \mid n = 0, 1, 2, ...\}$, $B_2 = \{v_n \mid n = 0, 1, 2, ...\}$, then B_1, B_2 both are countable subsets in D. We shall prove

$$\overline{B}_1 \subset \overline{co}\left\{\left\{u_0\right\} \cup A(B_1, B_2)\right\}.$$

In fact, $u_0 \in co\ \{\{u_0\}, A(B_1, B_2)\}$. Suppose that $u_k \in co\ \{\{u_0\}, A(B_1, B_2)\}$ $(k \geq 0)$. Evidently, $A(u_k, v_k) \in A(B_1, B_2) \subset co\ \{\{u_0\}, A(B_1, B_2)\}$. Since $u_{k+1} = 1 \cdot A(u_k, v_k) + 0 \cdot u_k, u_{k+1} \in co\ \{\{u_0\}, A(B_1, B_2)\}$. Hence

$$\overline{B}_1 \subset \overline{co}\left\{\left\{u_0\right\}, A(B_1, B_2)\right\} \subset \overline{co}\left\{\left\{u_0\right\} \cup A(B_1, B_2)\right\},$$

and therefore, $\overline{B}_1 \subset \overline{co}\left\{\left\{u_0\right\} \cup A(B_1, B_2)\right\}$. Similarly, we can show $\overline{B}_2 \subset \overline{co}\left\{\left\{v_0\right\}, A(B_2, B_1)\right\}$. It follows from (*ii*) that B_1, B_2 both are relatively compact, i.e., $\{u_n \mid n = 0, 1, 2, ...\}$ and $B_2 = \{v_n \mid n = 0, 1, 2, ...\}$ are relatively compact, and therefore, there exists a subsequence $\left\{u_{n_i}\right\} \subset \left\{u_n\right\}$ such that $u_{n_i} \to x^* \in E$. Suppose that $\{u_n\}$ is not convergent, then there exists another subsequence $\left\{u_{n_j}\right\} \subset \left\{u_n\right\}$ such that $u_{n_j} \to w^*$ and $w^* \neq x^*$. From (2), we know $\{u_n\}$ is a monotone increasing sequence in D. For any fixed n_{i_0}, we have $u_{n_{i_0}} \leq u_{n_j}$ when n_j is sufficiently large. Since

P is a cone and $u_{n_j} \to w^*, u_{n_{i0}} \le w^*$; and for any n_i, we have $u_{n_i} \le w^*$. It follows from $u_{n_i} \to x^*$ that $x^* \le w^*$. Similarly, we get $w^* \le x^*$. Hence $x^* = w^*$. This brings a contradiction. Therefore, $u_n \to x^*$ $(n \to \infty)$. In the same way, we have $v_n \to y^*$ $(n \to \infty)$. From (2), we have $u_n \le x^* \le y^* \le v_n$.

We now show that (x^*, y^*) is a minimax coupled fixed point of A. By the mixed monotony of A, we have

$$u_{n+1} = A(u_n, v_n) \le A(x^*, y^*)$$

$$\le A(y^*, x^*) \le (Av_n, u_n) = v_{n+1}, \quad n = 0, 1, 2, \dots .$$

Taking limit in above inequality as $n \to \infty$, we get

$$x^* \le A(x^*, y^*) \le A(y^*, x^*) \le y^*. \qquad \dots (3)$$

For any n and m, from

$$A(u_n, v_{n+m}) \le A(u_{n+m}, v_{n+m}) = u_{n+m+1},$$

$$u_{n+m+1} = A(v_{n+m}, u_{n+m}) \le A(v_n, u_{n+m})$$

and the demicontinuity of $A(x, \cdot)$, letting $m \to \infty$, we have

$$A(u_n, y^*) \le x^*, \quad y^* \le A(v_n, x^*), \qquad n = 0, 1, 2, \dots$$

By the demicontinuity of $A(\cdot, y)$, letting $n \to \infty$, we find

$$A(x^*, y^*) \le x^*, \quad y^* \le A(y^*, x^*). \qquad \dots (4)$$

Hence, from (3) and (4), we have (x^*, y^*) is a coupled fixed point of A. By standard arguments in[4], we can easily know (x^*, y^*) is a minimax coupled fixed point of A. The proof is completed.

Remark : In Lemma 1, condition (*ii*) is a weaker compactness condition, because the condensing condition "for any countable and bounded subsets $D_1, D_2 \subset [u_0, v_0]$, when $\alpha(D_1) + \alpha(D_2) \neq 0$, $\alpha(A(D_1, D_2)) < \max\{\alpha(D_1), \alpha(D_2)\}$" implies condition (*ii*).

For $B \subset PC^1[J, E]$, we denote $B' = \{u' \mid u \in B\} \subset PC[J, E]$ $B_{rj} = \{u|_{J_{rj}} : u \in B\}$ $B'_{rj} = \{u|_{J_{rj}} : u \in B\}$ $(j = 0, 1, 2, \dots)$. For $t \in J, B(t) = \{u(t) \mid u \in B\} \subset E, B'(t) = \{u'(t) \mid u \in B\} \subset E$. Let α and α_{PC^1} denote the Kuratowski measure of noncompactness of bounded sets in E and $PC^1[J, E]$, respectively. For details on definitions and properties of measure of noncompactness, one can refer to [5].

Lemma 2 — Let $B_1, B_2 \subset PC^1[J_r, E]$ be two countable subsets satisfying $\overline{B}_1 = \overline{co}\left\{\{x_0\} \cup B_2\right\}$ for some $x_0 \in PC^1[J_r, E]$, then

$$\overline{B_1(t)} = \overline{co}\left\{\left\{x_0(t)\right\} \cup B_2(t)\right\}, \qquad \overline{B_1(t)} = \overline{co}\left\{\left\{x_0'(t)\right\} \cup B_2'(t)\right\}, \ \forall t \in J_r.$$

Using the similar method as Lemma 4 in[6], we can easily obtain the conclusion of Lemma 2.

Lemma 3[5] — If $B \subset PC^1[J_r, E]$ is bounded and the elements of B are equicontinuous on each J_{rj} ($j = 0, 1, 2, ..., k$), then $\alpha(\{u(t) \mid u \in B_{rj}\})$ is continuous on $t \in J_{rj}$.

Lemma 4[5] — If $B \subset PC^1[J_r, E]$ is bounded and the elements of B' are equicontinuous on each J_{rj} ($j = 0, 1, 2, ..., k$), then

$$\alpha_{PC^1}(B) = \max\left\{\sup_{t \in J_r} \alpha(B(t)), \quad \sup_{t \in J_r} \alpha(B'(t))\right\}.$$

Lemma 5[5] — Let $m \in C[J_{rj}, R^+]$ ($j = 0, 1, ..., k$) and $k \in C[J_r, R^+]$, β_i ($i = 1, 2, ..., k$) be nonnegative constants. Suppose that

$$m(t) \leq \int_0^t k(s)\, m(s)\, ds \;=\; \sum_{0 < t_i < t} \beta_i\, m(t_i), \quad \forall t \in J_r.$$

Then $m(t) \equiv 0$, $\forall t \in J_r$.

2. Main Theorem

Theorem* 1 — *Suppose that

(H_1) *$I_i(u, v), \overline{I}_i(u, v)$ ($i = 1, 2, ...$) are nondecreasing in u and nonincreasing in v.*

(H_2) *$f(t, u, v, w)$ is nondecreasing in u and nonincreasing in v.*

(H_3) *For any $r > 0$ and $s \in J_r, y_j, v_j \in PC^1[J_r, E]$ ($i = 1, 2$), there exist a $a \in C[J_r, R^+]$ and constants $b_i \geq 0, c_i \geq 0$ ($i = 1, 2, ..., k$) such that*

$$\| f(s, u_1(s), v_1(s), u_1'(s)) - f(s, u_2(s), v_2(s), u_2'(s)) \|$$

$$\leq a(s)\, \phi(\max\{\| u_1(s) - u_2(s) \|, \| v_1(s) - v_2(s) \|, \| u_1'(s) - u_2'(s) \|\}),$$

$$\| I_i(u_1(s), v_1(s)) - I_i(u_2(s), v_2(s)) \|$$

$$\leq b_i\, \phi(\max\{\| u_1(s) - u_2(s) \|, \| v_1(s) - v_2(s) \|\}),$$

$$\|\overline{I}_i(u_1(s), v_1(s)) - \overline{I}_i(u_2(s), v_2(s))\|$$

$$\leq c_i\,\phi(\max\{\|u_1(s) - u_2(s)\|, \|v_1(s) - v_2(s)\|\})$$

$$(s \in J_r,\ i = 1, 2, \ldots, k),$$

where

$$\phi : [0, +\infty) \to [0, +\infty) \text{ is nondecreasing and } \phi(R(s)) \leq R(s),\ \forall R(s) > 0\ (s \in J_r).$$

(H_4) There exist $u_0, v_0 \in PC^1[J, E] \cap C^2[J', E]$, $u_0 \leq v_0$, and $[u_0, v_0] = \{u \in PC^1[J, E] \cap C^2[J', E] \mid u_0(t) \leq u(t) \leq v_0(t)$ for $t \in J\}$ is bounded with regard to the norm in $PC^1[J, E]$, such that

$$\begin{cases} u_0''(t) \leq f(t, u_0(t), v_0(t), u_0'(t)), & t \in J,\ t \neq t_i, \\ \Delta u_0|_{t=t_i} \leq I_i(u_0(t_i), v_0(t_i)), & \\ \Delta u_0'|_{t=t_i} \leq \overline{I}_i(u_0(t_i), v_0(t_i)) & (i = 1, 2, \ldots), \\ u_0(0) \leq w_0,\ u_0'(0) \leq w_1, & \end{cases}$$

$$\begin{cases} v_0''(t) \geq f(t, v_0(t), u_0(t), v_0'(t)), & t \in J,\ t \neq t_i, \\ \Delta v_0|_{t=t_i} \geq I_i(v_0(t_i), u_0(t_i)), & \\ \Delta v_0'|_{t=t_i} \geq \overline{I}_i(v_0(t_i), u_0(t_i)) & (i = 1, 2, \ldots), \\ v_0(0) \geq w_0,\ v_0'(0) \geq w_1, & \end{cases}$$

Then IVP(1) has an unique solution $u^* \in PC^1[J, E] \cap C^2[J', E]$, and $\|u_n(t) - u^*(t)\| \to 0$, $\|v_n(t) - u^*(t)\| \to 0\ (n \to \infty)$, where

$$u_n(t) = w_0 + w_1 t + \int_0^t (t-s) f(s, u_{n-1}(s), v_{n-1}(s), u_{n-1}'(s))\, ds$$

$$+ \sum_{0 < t_i < t} \left[I_i(u_{n-1}(t_i), v_{n-1}(t_i)) + (t - t_i)\overline{I}_i(u_{n-1}(t_i), v_{n-1}(t_i)) \right],$$

$$v_n(t) = w_0 + w_1 t + \int_0^t (t-s) f(s, v_{n-1}(s), u_{n-1}(s), v'_{n-1}(s))\, ds$$

$$+ \sum_{0<t_i<t} \left[I_i(v_{n-1}(t_i), u_{n-1}(t_i)) + (t-t_i) \bar{I}_i(v_{n-1}(t_i), u_{n-1}(t_i)) \right]$$

$(t \in J, \quad n = 1, 2, \ldots)$.

PROOF : It is clear that $u \in PC^1[J, E] \cap C^2[J', E]$ is a solution of IVP(1) if and only if u is a solution of the following impulsive integral equation

$$u(t) = w_0 + w_1 t + \int_0^t (t-s) f(s, u(s), v(s), u'(s))\, ds$$

$$+ \sum_{0<t_i<t} \left[I_i(u(t_i), v(t_i)) + (t-t_i) \bar{I}_i(u(t_i), v(t_i)) \right], \qquad t \in J.$$

Define operator A by

$$A(u, v)(t) = w_0 + w_1 t + \int_0^t (t-s) f(s, u(s), v(s), u'(s))\, ds$$

$$+ \sum_{0<t_i<t} \left[I_i(u(t_i), v(t_i)) + (t-t_i) \bar{I}_i(u(t_i), v(t_i)) \right], \qquad t \in J. \qquad \ldots (5)$$

For any $u, v \in PC^1[J, E], t \in J, t \neq t_i$ $(i = 1, 2, \ldots)$, we have by (5)

$$A'(u, v)(t) = w_1 + \int_0^t f(s, u(s), v(s), u'(s))\, ds$$

$$+ \sum_{0<t_i<t} \bar{I}_i(u(t_i), v(t_i)). \qquad \ldots (6)$$

By virtue of condition (H_1) and (H_2), we know A is mixed monotone. Let $r > 0$ be arbitrary given. For any $(u_1, v_1), (u_2, v_2) \in PC^1[J_r, E] \times PC^1[J_r, E]$, from condition (H_3), we have

$$\| A(u_1, v_1)(t) - A(u_2, v_2)(t) \|$$

$$\leq \int_0^t (t-s) \| f(s, u_1(s), v_1(s), u_1'(s)) - f(s, u_2(s), v_2(s), u_2'(s)) \|\, ds$$

$$+ \sum_{0<t_i<t} [\, \| I_i(u_1(t_i), v_1(t_i)) - I_i(u_2(t_i), v_2(t_i)) \|$$

$$+ (t - t_i) \| \overline{I}_i(u_1(t_i), v_1(t_i)) - \overline{I}_i(u_2(t_i), v_2(t_i)) \| \,]$$

$$\le \int_0^t r \cdot a(s)\, \phi\Big(\max\Big\{ \| u_1(s) - u_2(s) \|, \| v_1(s) - v_2(s) \|, \| u_1'(s) - u_2'(s) \| \Big\} \Big) ds$$

$$+ \sum_{0<t_i<t} (b_i + rc_i)\, \phi\Big(\max\Big\{ \| u_1(t_i) - u_2(t_i \|, \| v_1(t_i) - v_2(t_i) \| \Big\} \Big), \quad t \in J_r.$$

Similarly, we have

$$\| A'(u_1, v_1)(t) - A'(u_2, v_2)(t) \|$$

$$\le \int_0^t \| f(s, u_1(s), v_1(s), u_1'(s)) - f(s, u_2(s), v_2(s), u_2'(s)) \| ds$$

$$+ \sum_{0<t_i<t} \| \overline{I}_i(u_1(t_i), v_1(t_i)) - \overline{I}_i(u_2(t_i), v_2(t_i)) \|$$

$$\le \int_0^t a(s)\, \phi\Big(\max\Big\{ \| u_1(s) - u_2(s) \|, \| v_1(s) - v_2(s) \|, \| u_1'(s) - u_2'(s) \| \Big\} \Big) ds$$

$$+ \sum_{0<t_i<t} c_i\, \phi\Big(\max\Big\{ \| u_1(t_i) - u_2(t_i \|, \| v_1(t_i) - v_2(t_i) \| \Big\} \Big), \quad t \in J_r.$$

Therefore A is a continuous operator from $PC^1[J_r, E] \times PC^1[J_r, E]$ into $PC^1[J_r, E]$. For any countable subsets $D_1, D_2 \subset [u_0, v_0]$, we have

$$\operatorname{diam} A(D_1, D_2)(t) \le \int_{t0}^t r \cdot a(s)\, \phi(\max\{\operatorname{diam} D_1(s), \operatorname{diam} D_2(s), \operatorname{diam} D_1'(s)\})\, ds$$

$$+ \sum_{0<t_i<t} (b_i + rc_i)\, \phi(\max\{\operatorname{diam} D_1(t_i), \operatorname{diam} D_2(t_i)\}), \quad t \in J_r.$$

For any given $\varepsilon > 0$, there exist a finite number of subsets $D_1^{(1)}, D_2^{(1)}, \ldots, D_1^{(m)}$ and $D_2^{(1)}, D_2^{(2)}, \ldots, D_2^{(l)}$ of $[u_0, v_0]$, wuch that

$$D_1(t) \subset \bigcup_{i_1=1}^m D_1^{(i_1)}(t), \operatorname{diam} D_1^{(i_1)}(t) < \alpha(D_1(t)) + \varepsilon, \ t \in J_r,$$

$$D_2(t) \subset \bigcup_{j_1=1}^{l} D_1^{(j_1)}(t), \operatorname{diam} D_2^{(j_1)}(t) < \alpha(D_2(t)) + \varepsilon, \ t \in J_r,$$

$$D_1'(t) \subset \bigcup_{i_1=1}^{m} D_1^{'(i_1)}(t), \operatorname{diam} D_1^{(i_1)}(t) < \alpha(D_1'(t)) + \varepsilon, \ t \in J_r.$$

Since

$$A(D_1, D_2(t) \subset \bigcup_{j_1, j_1} A(D_1^{(i_1)}, D_2^{(j_1)})(t), \ t \in J_r$$

and

$$\operatorname{diam} A(D_1^{(i_1)}, D_2^{(j_1)})(t)$$

$$\leq \int_0^t r \cdot a(s)\, \phi\Big(\max\Big\{\operatorname{diam} D_1^{(i_1)}(s), \operatorname{diam} D_2^{(j_1)}(s), \operatorname{diam} D_1^{'(i_1)}(s)\Big\}\Big) ds$$

$$+ \sum_{0<t_i<t} (b_i + rc_i)\, \phi\Big(\max\Big\{\operatorname{diam} D_1^{(i_1)}(t_i), \operatorname{diam} D_2^{(j_1)}(t_i)\Big\}\Big)$$

$$\leq \int_0^t r \cdot a(s)\, \phi\Big(\max\Big\{\alpha D_1(s)), \alpha D_2(s)), \alpha D_1'(s))\Big\} + \varepsilon\Big) ds$$

$$+ \sum_{0<t_i<t} (b_i + rc_i)\, \phi\Big(\max\Big\{\alpha(D_1(t_i)), \alpha D_2'(t_i))\Big\} + \varepsilon\Big)$$

$$(t \in J_r, i_1 = 1, 2, \dots, m, j_1 = 1, 2, \dots, l),$$

we have

$$\alpha(A(D_1, D_2)(t)) \leq \int_0^t r \cdot a(s)\, \phi\Big(\max\Big\{\alpha D_1(s)), \alpha D_2(s)), \alpha D_1'(s))\Big\} + \varepsilon\Big) ds$$

$$+ \sum_{0<t_i<t} (b_i + rc_i)\, \phi\Big(\max\Big\{\alpha(D_1(t_i)), \alpha D_2'(t_i))\Big\} + \varepsilon\Big), \quad t \in J_r.$$

Let $R_1(s) = \max\{\alpha D_1(s)), \alpha D_2(s)), \alpha D_1'(s))\} + \varepsilon > 0$, $R_3(t_i) = \max\{\alpha D_1(s)), \alpha D_2(s)), \alpha D_1'(s))\} + \varepsilon > 0$, then by condition (H_3), we have $\phi(R_1(s)) \leq R_1(s)$, $\phi(R_2(t_i)) \leq R_2(t_i)$. Since ε is arbitrary, this implies

$$\alpha(A(D_1, D_2)(t)) \le \int_0^t r \cdot a(s)\, \phi\Big(\max\{\alpha D_1(s)), \alpha D_2(s)), \alpha D_1'(s))\}\Big) ds$$

$$+ \sum_{0<t_i<t} (b_i + rc_i)\, \phi\Big(\max\{\alpha(D_1(t_i)), \alpha D_2'(t_i))\}\Big), \quad t \in J_r. \qquad \ldots (7)$$

Similarly, we get

$$\alpha(A'(D_1, D_2)(t)) \le \int_0^t a(s)\, \phi \max\{\alpha D_1(s)), \alpha D_2(s)), \alpha D_1'(s))\} ds$$

$$+ \sum_{0<t_i<t} c_i\, \phi \max\{\alpha(D_1(t_i)), \alpha D_2'(t_i))\}, \quad t \in J_r. \qquad \ldots (8)$$

Using the same method as above, we also get

$$\alpha(A(D_1, D_2)(t)) \le \int_0^t r \cdot a(s)\, \phi \max\{\alpha D_1(s)), \alpha D_2(s)), \alpha D_1'(s))\} ds$$

$$+ \sum_{0<t_i<t} (b_i + rc_i)\, \phi \max\{\alpha(D_1(t_i)), \alpha D_2'(t_i))\}, \quad t \in J_r. \qquad \ldots (9)$$

and

$$\alpha(A'(D_1, D_2)(t)) \le \int_0^t a(s)\, \phi \max\{\alpha D_1(s)), \alpha D_2(s)), \alpha D_1'(s))\} ds$$

$$+ \sum_{0<t_i<t} c_i\, \phi \max\{\alpha(D_1(t_i)), \alpha D_2'(t_i))\}, \quad t \in J_r. \qquad \ldots (10)$$

For some $u, v \in [u_0, v_0]$, $D_1, D_2 \subset [u_0, v_0]$ countable, and $\overline{D}_1 \subset \overline{co}\,\{\{u\} \cup A(D_1, D_2)\}$, $\overline{D}_2 \subset \overline{co}\,\{\{v\} \cup A(D_2, D_1)\}$, then by Lemma 2 and the properties of measure of noncompactness, for any $t \in J_r$, we have

$$\overline{D_1(t)} \subset \overline{co}\,\{\{u(t)\} \cup A(D_1, D_2)\}, \qquad \overline{D_2(t)} \subset \overline{co}\,\{\{v(t)\} \cup A(D_2, D_1)\},$$

$$\overline{D_1'(t)} \subset \overline{co}\,\{\{u'(t)\} \cup A'(D_1, D_2)\}, \qquad \overline{D_2'(t)} \subset \overline{co}\,\{\{v'(t)\} \cup A'(D_2, D_1)\}$$

and

$$\alpha_{PC^1}(D_1) = \alpha_{PC^1}(\overline{D_1}) \le \alpha_{PC^1} A(D_1, D_2)),$$

$$\alpha_{PC^1}(D_2) = \alpha_{PC^1}(\overline{D_2}) \le \alpha_{PC^1} A(D_2, D_1)), \qquad \ldots (11)$$

$$\alpha(D_1(t)) = \alpha(\overline{D_1(t)}) \le \alpha(A(D_1, D_2)(t)),$$

$$\alpha(D_2(t)) = \alpha(\overline{D_2(t)}) \le \alpha(A(D_2, D_1)(t)), \quad \dots (12)$$

$$\alpha(D_1'(t)) = \alpha(\overline{D_1'(t)}) \le \alpha(A'(D_1, D_2)(t)),$$

$$\alpha(D_2'(t)) = \alpha(\overline{D_2'(t)}) \le \alpha(A'(D_2, D_1)(t)). \quad \dots (13)$$

Condition (H_3) implies A is a bounded operator, and from (5) and (6), that the elements of $A(D_1, D_2), A(D_2, D_1), A'(D_1, D_2), A'(D_2, D_1)$ are all equicontinuous on each J_{rj} ($j = 0, 1, 2, \dots, k$). By (12) and (13), we have

$$\max\{\alpha(D_1(t)), \alpha(D_2(t))\} \le \max\{\alpha(A(D_1, D_2)(t)), \alpha(A(D_2, D_1)(t))\}, \quad t \in J_r,$$

$$\max\{\alpha(D_1'(t)), \alpha(D_2'(t))\} \le \max\{\alpha(A'(D_1, D_2)(t)), \alpha(A'(D_2, D_1)(t))\}, \quad t \in J_r.$$

Let $m(t) = \max\{\alpha(A(D_1, D_2)(t),)\alpha(A(D_2, D_1)(t)), \ \alpha(A'(D_1, D_2)(t),)\alpha(A'(D_2, D_1)(t))\}$, $t \in J_r$, then by Lemma 3, we have $m \Rightarrow C[J_{rj}, R^{+}]$ ($j = 0, 1, \dots, k$), and by (7)-(10) and (12), (13), we get

$$m(t) \le \int_0^t (r+1)\, a(s)\, m(s)\, ds + \sum_{0 < t_i < t} [b_i + (r+1)\, c_i]\, m(t_i), \quad t \in J_r,$$

which implies by virtue of Lemma 5 that $m(t) \equiv 0$, $\forall t \in J_r$, $\forall r > 0$, i.e.,

$$\alpha(A(D_1, D_2)(t)) = \alpha(A(D_2, D_1)(t))$$

$$= \alpha(A'(D_1, D_2)(t)), \alpha(A'(D_2, D_1)(t)) = 0, \quad \forall t \in J_r.$$

Consequently, by Lemma 4, we have

$$\alpha_{PC^1}(D_1) \le \alpha_{PC^1} A(D_1, D_2))$$

$$= \max\left\{\sup_{j \in J_r} (A(D_1, D_2)(t)), \ \sup_{j \in J_r} (A'(D_1, D_2)(t))\right\} = 0,$$

$$\alpha_{PC^1}(D_2) \le \alpha_{PC^1} A(D_2, D_1))$$

$$= \max\left\{\sup_{j \in J_r} (A(D_2, D_1)(t)), \ \sup_{j \in J_r} (A'(D_2, D_1)(t))\right\} = 0,$$

that is D_1, D_2 are relatively compact in $PC^1[J_r, E]$.

From condition (H_4), by performing direct integration of the inequality $u_0'' \le f(t, u_0, v_0, u_0')$ and $v_0'' \le f(t, v_0, u_0, v_0')$ $(t \in J)$ twice, we can easily prove that $u_0 \le A(u_0, v_0)$ and $A(v_0, u_0) \le v_0$.

Summing up, A satisfies all the conditions of Lemma 1, then A has a minimax coupled fixed point $(u^*, v^*) \in [u_0, v_0] \times [u_0, v_0]$. In addition,

$$\| u^*(t) - v^*(t) \| = \| A(u^*(t), v^*(t)) - A(v^*(t), u^*(t)) \|$$

$$\le \int_0^t r \cdot a(s)\, \phi(\| u^*(s) - v^*(s) \|)\, ds$$

$$+ \sum_{0 < t_i < t} (b_i + rc_i)\, \phi(\| u^*(t_i) - v^*(t_i)) \|), \quad t \in J_r.$$

Let $m_1(t) = \| u^*(t) - v^*(t)) \|$, $t \in J_r$, then by virtue of the characters of ϕ, we have

$$m_1(t) \le \int_0^t r \cdot a(s)\, m_1(s)\, ds + \sum_{0 < t_i < t} (b_i + rc_i)\, m_1(t), \quad t \in J_r,$$

which also implies that $m_1(t) \equiv 0$, $\forall t \in J_r$, $\forall r > 0$, i.e., $u^*(t) = v^*(t)$, $t \in J_r$. If there exists a fixed point $w^* \in PC^1[J_r, E] \setminus [u_0, v_0]$ of A, by using the above method, we have $w^*(t) = u^*(t)$ $(t \in J_r)$. Hence, A has an unique fixed point $u^* \in PC^1[J_r, E]$, i.e., IVP(1) has an unique solution in $PC^1[J_r, E] \cap C^2[J_r', E]$. Since $r > 0$ is arbitrary, we see that $u^* \in PC^1[J. E] \cap C^2[J', E]$, and $u_n = A(u_{n-1}, v_{n-1})$, $v_n = A(v_{n-1}, u_{n-1})$ $(n = 1, 2, \ldots)$, and $\| u_n(t) - u^*(t) \| \to 0$, $\| v_n(t) - u^*(t) \| \to 0$, $(t \in J, n \to \infty)$.

Remark 2 : In this paper, we discuss the IVP(1) on infinite interval in case f possess u' and have no any demands on $a(t)$ $(t \in J_r)$, b_i and c_i $(i = 1, 2, \ldots, k)$. When we consider the IVP(1) on a finite interval $J = [0, 1]$ with the finite number p of impulsive times, we cross out the restriction condition $a + \sum_{i=1}^{p} (b_i + c_i) \le 1$ which is given in[2]. One can also discuss the IVP of the first order impulsive differential equations in a similar way.

3. An Example

Consider the IVP of the infinite system for nonlinear second order impulsive integro-differential equations

$$\begin{cases} u_0'' = \dfrac{1}{(n+2)(t+1)} + \dfrac{1}{6(t+1)^3} u_n(t) - \dfrac{1}{2^{n+3}(t+1)^3} u_{n+2}(t) \\ \quad + \dfrac{1}{8(t+1)^2} u_n'(t), \qquad t \geq 0, t \neq k \ (k = 1, 2, \ldots), \\ \Delta u_n |_{t=k} = \dfrac{1}{2^{k+2}} u_n(k), \\ \Delta u_n' |_{t=k} = \dfrac{1}{2^{n+k+1}} u_{n+1}(k), \qquad (k = 1, 2, \ldots), \\ u_n(0) = 0, \ u_n'(0) = 0, \qquad (n = 1, 2, \ldots). \end{cases} \quad \ldots (14)$$

Conclusion — System (14) has an unique solution which is continuously differential on $[0, k) \cup (k, \infty)$ $(k = 1, 2, \ldots)$.

PROOF : Let $E = l^1 = \{ u = (u_1, \ldots, u_n, \ldots) \mid \sum_{n=1}^{\infty} | u_n | < \infty \}$ with norm $\| u \| = \sum_{n=1}^{\infty} | u_n |$ and $P = \{u = (u_1, \ldots, u_n, \ldots) \in l^1 \mid u_n \geq 0, \ n = 1, 2, \ldots\}$. Then P is a normal cone in E. System (14) can be regarded as an IVP of form (1) in E, where $J = [0, \infty), f = (f_1, \ldots, f_n, \ldots)$, $t_k = k (k = 1, 2, \ldots)$, $I_k = (I_{k1}, \ldots, I_{kn}, \ldots)$, $\bar{I}_k = (\bar{I}_{k1}, \ldots, \bar{I}_{kn}, \ldots)$, $u = (u_1, \ldots, u_n, \ldots)$, $v = (v_1, \ldots, v_n, \ldots)$, $w = (w_{01}, \ldots, w_{0n}, \ldots)$, $w_1 = (w_{11}, \ldots, w_{1n}, \ldots)$, in which $w_{0n} = 0, w_{1n} = 0$ and

$$f_n(t, u(t), v(t), u'(t))$$

$$= \frac{1}{(n+2)(t+1)} + \frac{1}{6(t+1)^3} u_n(t) - \frac{1}{2^{n+3}(t+1)^3} u_{n+2}(t) + \frac{1}{8(t+1)^2} u_n'(t)$$

$$I_{kn}(u(k), v(k)) = \frac{1}{2^{k+2}} u_n(k), \quad \bar{I}_{kn}(u(k), v(k)) = \frac{1}{2^{n+k+1}} u_{n+1}(k).$$

Obviously $f \in C[J \times E \times E \times E, E]$, $I_k, \bar{I}_k \in C[E \times E, E]$ $(k = 1, 2, \ldots)$, $f(t, u, v, w)$ is non-decreasing in u and nonincreasing in v, $I_k(u, v) \bar{I}_k(u, v)$ are nondecreasing in u. Let $a(t) = \frac{4}{(t+1)^2}$, $b_k = \frac{6}{2^k}$, $c_k = \frac{12}{2^k}$, then for any $s \in J$, $u_j, v_j \in PC^1[J, E]$ $(j = 1, 2)$, we have

$$\| f(s, u_1(s), v_1(s), u_1'(s)) - f(s, u_2(s), v_2(s), u_2'(s)) \|$$

$$= \sum_{n=1}^{\infty} | f_n(s, u_1(s), v_1(s), u_1'(s)) - f_n(s, u_2(s), v_2(s), u_2'(s)) |$$

$$\le \frac{1}{6(s+1)^2}$$

$$\left[\sum_{n=1}^{\infty} |u_{1n}(s) - u_{2n}(s)| + \sum_{n=1}^{\infty} |v_{1(n+2)}(s) - v_2 n + 2)(s)| + \sum_{n=1}^{\infty} |u'_{1n}(s) - u'_{2n}(s)| \right]$$

$$\le a(s)\, \phi\, (\max \{\| u_1(s) - u_2(s) \|, \| v_1(s) - v_2(s) \|, \| u'_1(s) - u'_2(s) \|\}),$$

$$\| I_k(u_1(s), v_1(s)) - I_k(u_2(s), v_2(s)) \|$$

$$\le \frac{1}{2^{k+1}} \quad \| u_1(s) - u_2(s) \| \le b_k\, \phi\, (\| u_1(s) - u_2(s) \|),$$

$$\| \bar{I}_k (u_1(s), v_1(s)) - I_k (u_2(s), v_2(s) \|$$

$$\le \frac{1}{2^{k+1}} \sum_{n=1}^{\infty} | u_{1(n+1)}(s) - u_{2(n+1)}(s) |$$

$$\le c_k\, \phi\, (\| u_1(s) - u_2(s) \|),$$

in which, $\phi(x(s)) = \frac{1}{24} x(s)$, and for any $x(s) > 0$, we have $\phi(x(s)) < x(s)$ $(\forall s \in J)$. Hence condition (H_3) of Theorem 1 is satisfied for $s \in J_r$ $(\forall r > 0)$.

Finally, let $u_0(t) = (0, 0, \ldots)$, $t \in J$ and

$$v_0(t) = \begin{cases} \left(t^2, \ldots, \frac{t^2}{n}, \ldots \right), & 0 \le t \le k, \\ \left(t, (t+1), \ldots, \frac{t(t+1)}{n}, \ldots \right), & t > k, \end{cases}$$

then we have $u_0 \in C^2[J, E]$, $v_0 \in PC^1[J, E] \cap C^2[J', E]$ and $u_0(t) \le v_0(t)$, $u'_0(t) \le v'_0(t)$. Since p is a normal cone in l_1, it is easy to know $P_c = \{u \in PC^1[J, e] \mid u(t) \ge \theta, u'(t) \ge \theta,\ t \in J\}$ is a normal cone in $PC^1[J, E]$; hence $[u_0, v_0]$ is bounded in $PC^1[J, E]$. For any $n = 1, 2, \ldots$, we have

$$\Delta u_{0n} |_{t=k} = 0 \le I_{kn}(u_{0n}(k), v_{0n}(k)),$$

$$\Delta v_{0n} |_{t=k} = \frac{k}{n} > \frac{k}{2^{k+2}\, n} = 0 \le I_{kn}(v_{0n}(k), u_{0n}(k)),$$

$$\Delta u'_{0n} |_{t=k} = 0 \le \bar{I}_{kn}(u_{0n}(k), v_{0n}(k)),$$

$$\Delta v'_{0n} |_{t=k} = \frac{1}{n} > \frac{k^2}{2^{n+k+1}} \frac{1}{n+1} = 0 \le \bar{I}_{kn}(v_{0n}(k), u_{0n}(k)),$$

$$f_n(t, u_0(t), v_0(t), u_0'(t)) \geq \frac{1}{(n+2)(t+1)} - \frac{1}{2^{n+3}(t+1)^3}\frac{t(t+1)}{n+2}$$

$$\geq \frac{16(t+1)-t}{2^{n+3}(t+1)^2(n+2)} > 0 = u_{0n}''(t), \quad t \in J,$$

$$f_n(t, u_0(t), v_0(t), u_0'(t)) \geq \frac{1}{(n+2)(t+1)} + \frac{1}{6(t+1)^3}\frac{t(t+1)}{n} + \frac{1}{8(t+1)^2}\frac{2t+1}{n}$$

$$< \frac{1}{n}\frac{1}{t+1} + \frac{1}{6n}\frac{t}{t+1}\frac{1}{t+1} + \frac{1}{8n}\frac{2}{t+1}$$

$$< \frac{17}{12n} < \frac{2}{n} = v_{0n}''(t), \quad t \in J, \ t \neq k.$$

Consequently, u_0 and v_0 satisfy (H_4) of Theorem 1. Therefore, our conclusion follows from Theorem 1. The proof is completed.

Remark : In this example, when $t \in J = [0, 1]$, $a(t) + \sum_{i=1}^{p} (b_i + c_i) = + \frac{4}{(t+1)^2}$ $\sum_{i=1}^{p} \left(\frac{6}{2^i} + \frac{12}{2^i}\right) > 1$. So even if we consider the system (14) on the finite interval $J = [0, 1]$, our conclusioncan't follow from the main theorem in[2]. This shows that this paper improves and generalizes the main result in[2].

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