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NEIGHBOURLY IRREGULAR GRAPHS

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A connected graph G is said to be neigbourly irregular if no two adjacent vertices of G have the same degree. Given a positive inter n and a partition of n with distinct parts, this paper suggests a method to construct a neighbourly irregular graph of order n. This paper also includes a few properties possessed by these neighbourly irregular graphs.

Key Words : Highly Irregular Graphs; Neighbourly Irregular Graphs; Partitions of an Integer; Graphoidal Covering Number

INTRODUCTION

By a graph we mean a finite undirected, connected graph without loops or multiple edges. For graph theoretic terminology we refer Parthasarathy[4].

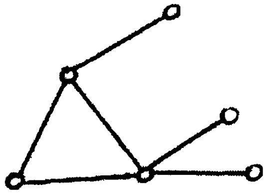
Regular graphs are those graphs for which each vertex has the same degree. There are plenty of regular graphs, for example, complete graphs. The problem arises when a graph is not regular. If it is irregular how much of irregularity is thrust upon its vertices? In this connection two new concepts called Highly Irregular graphs and k-neighbourhood graphs ([5] & [2]) have evolved.

A connected graph G is said to be highly irregular if each neighbour of any vertex has different degree. A connected graph G is said to be a k-neighbourhood regular graph if each of its vertices is adjacent to exactly k vertices of the same degree (If $k = 1$, it becomes a highly irregular graph)

Inspired by these two definitions, we define the concept of neighbourly irregular graphs abbreviated as NI graphs.

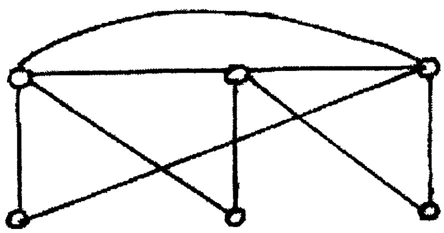
NEIGHBOURLY IRREGULAR GRAPHS

A connected graph G is said to be neighbourly irregular (NI) if no two adjacent vertices of G have the same degree. The following graph G_1 is a NI graph whereas it is not a k-neighbourhood regular graph.



G_1

The graph G_2 as shown below is a 2-neighbourhood regular graph but not a NI graph.



G_2

Fact 1 — If v is a vertex of maximum degree in a NI graph, then at least two of the adjacent vertices of v have the same degree.

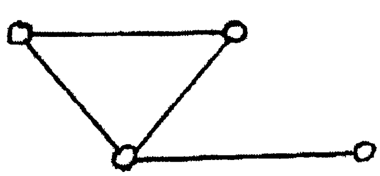
PROOF : Let v be a vertex of maximum degr Δ. Let $v_1, v_2 \ldots v_\Delta$ be the vertices adjacent to v. If their degrees are distinct, then there is one vertex v_i such that deg $(v_i) = \Delta =$ deg (v) which contradicts the neighbourly irregularness of the graph.

Fact 2 — $K_{m,n}$ is NI if and only if $m \neq n$.

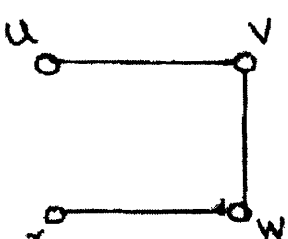
Fact 3 — Let G be a NI graph of order n. Then for any positive integer $m < n$, there exist atmost m vertices of degree $(n-m)$. For, if G has $(m + 1)$ vertices of degree $(n-m)$, then due to their nonadjacency nature, there must be at least $m + 1 + n - m$ vertices that is $(n + 1)$ vertices contradicting the order of G.

Fact 4 — If a graph G is NI, then no P_4 contains internal vertices of same degree in G. For, if a P_4 is in G, then the internal vertices of P_4 being adjacent, these must be of different degrees.

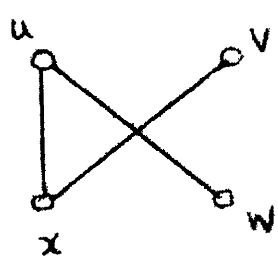
The converse of Fact 4 need not be true. For example, the graph G_3 is not NI but no P_4 contains internal vertices of same degree.



G_3



P_4



$P_4{}^c$

Fact 5 — If a graph G is NI, then G^c is not NI.

PROOF : By Fact 1, there are two nonadjacent vertices of same degree i in G. Those vertices are then adjacent vertices and also of same degree $n-1-i$ in G^c.

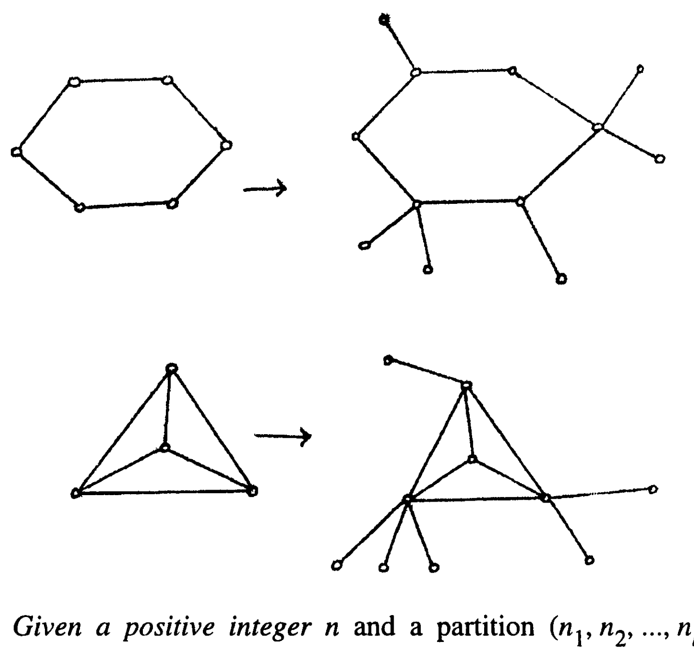
Converse of Fact 5 is also not true. For example, neither the graph P_4 nor its complement is NI.

Theorem 1 — *Any graph of order n can be made to be an induced subgraph of a NI graph of order atmost* $n+1\ C_2$.

PROOF : Choose any two adjacent vertices of G. If they are of same degree, introduce a new vertex and join this to exactly one of these adjacent vertices. This process is repeated pairwise inductively till no two adjacent vertices are of same degree.

As it involves atmost ${}^nC_2$ steps only, the order of the induced NI graph is $n+{}^nC_2={}^{n+1}C_2$.

The induced NI graph of C_6 and K_4 are given below.



Theorem 2 — *Given a positive integer n* and a partition $(n_1, n_2, ..., n_k)$ of n such that all n_is are distinct, there exists a NI graph of order n and size

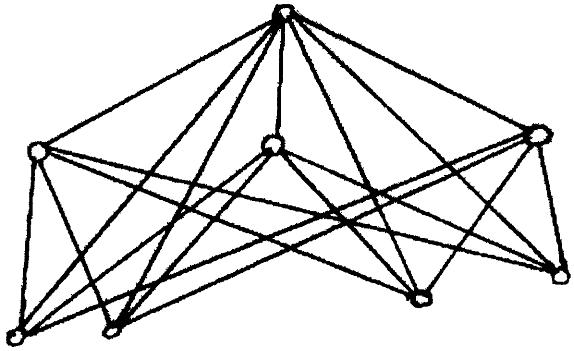
$$\frac{1}{2}\left\{n^2-\left(n_1^2+n_2^2+\ldots n_k^2\right)\right\}.$$

PROOF : The required NI graph is constructed as folows. The n vertices are partitioned into k sets. The first set consists of n_1 vertices $u_1, u_2, ..., u_{n_1}$; the second consists of n_2 vertices $v_1, v_2, ..., v_{n_2}$ and so on and finally the kth set consists of n_k vertices $z_1, z_2, ..., z_{n_k}$. Then every vertex in the first set is joined to all the other vertices in the remaining $(k-1)$ sets. Similarly each vertex in the remaining sets are joined to all those vertices in the other remaining sets. The vertices in the same set are non-adjacent. Therefore, degree of each vertex in the ith set is $n - n_i$. As all the $n_i s$ are distinct, the connected graph so constructed is NI and it is denoted by $NI_{(n_1, n_2, ..., n_k)}$.

$$\text{The size of this graph} = \frac{1}{2} \sum \deg V$$

$$= \frac{1}{2} \sum_{i=1}^{k} n_i (n - n_i) = \frac{1}{2} \left[n^2 - \left(n_1^2 + n_2^2 + \ldots + n_k^2 \right) \right].$$

Example — For $n = 8$ and the partition (1, 3, 4) of 8, the graph $NI_{(1, 3, 4)}$ is as shown below:



Corollary 2.1. — The maximum size of such a NI graph of order n is

$$\frac{1}{2} \left[n^2 - \left(n_1^2 + n_2^2 + \ldots + n_k^2 \right) \right]$$

where

$$n_1^2 + n_2^2 + \ldots + n_k^2 \text{ is minimum.}$$

The following lemma is helpful to find that maximum size.

Lemma 2.2 — If $(m_1, m_2, ..., m_r)$ and $(n_1, n_2, ..., n_r)$ are two partitions of a positive integer n with $r < k$, then $m_1^2 + m_2^2 + \ldots + m_r^2 > n_1^2 + n_2^2 + \ldots + n_k^2$.

PROOF : This is proved by induction on n. Assuming up to n, take two partitions $(m_1, m_2, ..., m_r)$ and $(n_1, n_2, ..., n_k)$ of the positive integer $(n + 1)$.

$$\text{i.e. } m_1 + m_2 + \ldots + m_r = n + 1 = n_1 + n_2 + \ldots + n_k$$

Let i and j be the first indices of m_i's and n_j's respectively such that $m_i \geq n_j$. Now, $(m_1, m_2, ..., m_i - 1, ..., m_r)$ and $(n_1, n_2, ..., n_j - 1, ..., n_k)$ are two partitions of n with $r < k$. Applying the induction, we have

$$m_1^2 + m_2^2 + \ldots + (m_i - 1)^2 + \ldots + m_r^2 > n_1^2 + n_2^2 + \ldots + (n_j - 1)^2 + \ldots + n_k^2$$

$$\text{(i.e.)}\quad m_1^2 + m_2^2 + \ldots + m_i^2 + \ldots + m_r^2 - 2m_i > n_1^2 + n_2^2 + \ldots + n_j^2 + \ldots + n_k^2 - 2n_j$$

$$\text{(i.e)}\quad m_1^2 + m_2^2 + \ldots + m_i^2 > n_1^2 + n_2^2 + \ldots + n_k^2 + 2(m_i - n_j)$$

$$> n_1^2 + n_2^2 + \ldots + n_k^2 \text{ as } m_i > n_j.$$

This proves the lemma.

Corollary 2.3 —

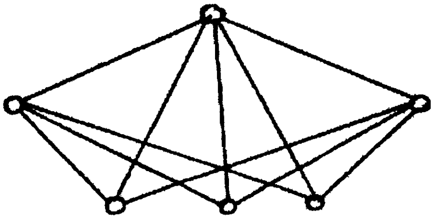
$$n_1^2 + n_2^2 + \ldots + n_k^2 \text{ is minimum iff } n_1 = 1 \text{ or } 2.$$

PROOF : If $n_1 \geq 3$, then $(1, n_1 - 1, n_2, \ldots, n_k)$ is also a partition of n with more number of parts of n than that of $(n_1, n_2, \ldots, n_k)$. Thus, by Lemma 2.2, $n_1^2 + n_2^2 + \ldots + n_k^2$ cannot be minimum.

Corollary 2. 4 — Actual partition of n for which $NI_{(n_1, n_2, \ldots, n_k)}$ is of maximum size is given by $\{1, 2, 3, \ldots, r-1, r+1, \ldots, k\}$ where k is the least positive integer such that $\frac{k(k+1)}{2} > n$ and $r = \frac{k(k+1)}{2} - n$.

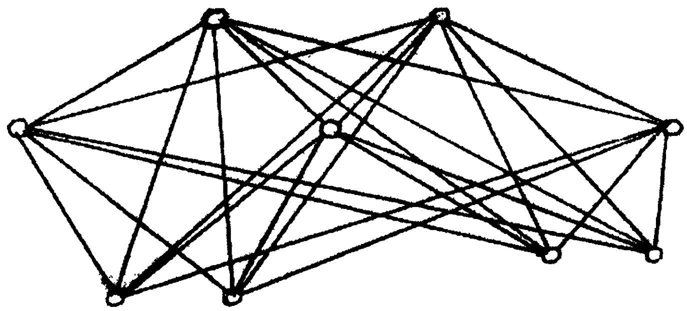
PROOF : If there exists a least positive integer k such that $n = \frac{k(k+1)}{2}$ then $(1, 2, 3, \ldots, k)$ is the partition of n of maximum parts. Therefore, by Lemma 2.2, $1^2 + 2^2 + \ldots + k^2$ is the minimum. On the other hand, if k is the least positive integer such that $\frac{k(k+1)}{2} > n$, then $(1, 2, \ldots, r-1, r+1, \ldots, n)$ {where $r = \frac{k(k+1)}{2} - n$} is the partition of n of maximum number of parts and so the result is true by Lemma 2.2.

Illustrations — For $n = 6$, the partition (1, 2, 3) gives the maximum size and the $NI_{(1,2,3)}$ graph is



$$\text{Size} = \frac{1}{2}[6^2 - (1^2 + 2^2 + 3^2)]$$

$$= \frac{1}{2} \times 22 = 11$$



For $n = 9$, the partition (2, 3, 4) gives the maximum size and the $\mathrm{NI}_{(2,3,4)}$ graph is

$$\text{Size} = \frac{1}{2}[9^2 - (2^2 + 3^2 + 4^2)]$$

$$= \frac{1}{2} \times 52 = 26$$

SOME PROPERTIES OF $\mathrm{NI}_{(n_1, n_2, ..., n_k)}$. GRAPHS

1. *Graphoidal covering number*

Let $G = (V, E)$ be a graph. A graphoidal cover of G [3], is a set ψ of (not necessarily open) paths in G satisfying the following conditions:1. Every path in ψ has at least two vertices.

2. Every vertex of G is an internal vertex of atmost one path in ψ.

3. Every edge of G is in some path in ψ.

Let $\xi(G)$ denote the set of all graphoidal covers of G. Then $\gamma(G) = \min_{\psi \Sigma \xi(G)} |\psi|$ is called the graphoidal covering number of G.

The following results established by Arumugam *et al.* in[1] are used to find $\gamma(G)$ of $\mathrm{NI}_{(n_1, n_2, ..., n_k)}$.

Result 1 — Let ψ be a graphoidal cover of G such that every vertex v with $d(v) > 1$ is an internal vertex of a path in ψ. Then $|\psi| = \gamma$.

Result 2 — Let G be a complete bipartite graph $K_{m,n}$ with $m > 2$ and $n > 2$ or $m = 2$ and $n > 3$. Then $\gamma(G) = q - p$. (q is the number of edges and p the number of vertices).

Theorem* 3 — *Let n be a positive integer and $(n_1, n_2, ..., n_k)$ be a partition of n with distinct*

*It is knwon that $\gamma(G)$ = size-order for all graphs for which the minimum degree $\delta \geq 3$[3]. However, in the proof of Theorem 3, we actually construct a graphoidal cover of minimum cardinality γ and so the proof is independent by itself.

parts and $n_k > 3$. *Then* $\gamma\,(\mathrm{NI}_{(n_1, n_2, \ldots, n_k)}) = e - n$ *wjere* e *is its size and* n *its order.*

PROOF : Consider the $\mathrm{NI}_{(n_1, n_2, \ldots, n_k)}$ labelled as follows:

u_1 u_2 . . . u_{n_1}

o o o

v_1 v_2 . . . v_{n_2}

o o o

: : . . . :

x_1 x_2 . . . $x_{n_{k-2}}$

o o o

y_1 y_2 . . . $y_{n_{k-1}}$

o o o

z_1 z_2 . . . z_{n_k}

o o o

Let ψ_K be a graphoidal cover of the complete bipartite graph K_{n_{k-1}, n_k} with vertices $y_1, y_2, \ldots, y_{n_{k-1}}$ and $z_1, z_2, \ldots, z_{n_k}$ such that

$$|\,\psi_K\,| = \gamma(K_{n_{k-1}, n_k}) = n_k\, n_{k-1} - (n_k + n_{k-1}) \text{ by result 2)}$$

Consider now the complete bipartite graph $K_{n_{k-2}, n_{k-1}}$ with vertices $x_1, x_2, \ldots, x_{n_{k-2}}$ and $y_1, y_2, \ldots, y_{n_{k-1}}$. As the vertices $y_1, y_2, \ldots, y_{n_{k-1}}$ are already internal vertices of paths in ψ_k we construct a graphoidal cover ψ_{k-1} for $K_{n_{k-1}, n_{k-2}}$ in such a way that $x_1, x_2, \ldots, x_{n_{k-2}}$ are the only internal vertices of paths in ψ_{k-1}. The collection $\psi_{k-1} = \{y_1\, x_1\, y_2, y_1\, x_2\, y_2, \ldots, y_1\, x_{n_{k-2}}\, y_2\}$: U $\{x_i\, y_j$: $i = 1, 2, \ldots, n_{k-2}$ and $j = 3, 4, \ldots, n_{k-1}\}$ satisfies our need. Clearly $|\,\psi_{k-1}\,| = n_{k-2} + n_{k-1}\, n_{k-2} - 2n_{k-2} = n_{k-1}\, n_{k-2} - n_{k-2}$. By similar methods, all other such covers $\psi_{k-2}, \ldots, \psi_2$ are found for the remaining complete bipartite graphs

k_{n_{i-1}, n_i}; $i = 2, 3, \ldots$, k − 2. Then $\left|\, \psi_1\, \psi_2 + \ldots + \psi_k \,\right|$

$$= (n_1\, n_2 - n_1) + (n_2\, n_3 - n_2) + \ldots + (n_{k-1}\, n_{k-2} - n_{k-2})$$

$$+ n_k\, n_{k-1} - (n_k + n_{k-1})$$

$$= (n_1\, n_2 + n_2\, n_3 + \ldots + n_{k-1}\, n_k) - (n_1 + n_2 + \ldots + n_k)$$

$$= (n_1 n_2 + n_2 n_3 + \dots + n_{k-1} n_k) - n.$$

The edges in other bipartite graphs $K_{n_1, n_3}, K_{n_2, n_4}, \dots K_{n_{k-2}, n_k}$ are not included in $\psi_2 \cup \psi_3 \cup \dots \cup \psi_k$. If we let ψ_1 as the collection of all these edges, each edge being considered as a path, then $\psi = \psi_1 \cup \psi_2 \cup \dots \cup \psi_k$ is a graphoidal cover of $NI_{(n_1, n_2, n_k)}$ where each vertex is an internal vertex of path in ψ. Thus, by result 1 of[1], $|\psi| = \gamma$.

$$\text{Now } |\psi| = (n_1 n_3 + n_2 n_4 + \dots + n_{k-2} n_k$$

$$+ n_1 n_{2+} n_2 n_3 + \dots + n_{k-1} n_k) - n$$

$$= e - n.$$

2. Gracefulness

Let $G = (V, E)$ be a graph with order n and size m and let $f: V \rightarrow \{0, 1, 2, \dots, m\}$ be an injection. Define $g: E \rightarrow \{1, 2, \dots, m\}$ by $g(e) = |f(u) - f(v)|$ where $e = uv$. If g is also an injection, then (f, g) is called a graceful labelling of the graph G. Graphs with graceful labellings are called graceful graphs[4] (p. 433).

Theorem 4 — *Let n be any positive integer and (n_1, n_2, n_3) be a partition of n with distinct parts and $n_1 = 1$. Then $NI_{(n_1, n_2, n_3)}$ is graceful.*

Proof : Let v_1, v_2, v_n be the vertices of $NI_{(n_1, n_2, n_3)}$. Label v_1 by 0; $v_2, v_3, \dots, v_{n_2+1}$ by 1, 2, 3, ..., n_2 respectively and label the remaining n_3 vertices by $e, e - (n_2 + 1), e - 2(n_2 + 1), \dots, 2n_2 + 1$.

Note that $2n_2 + 1 = e - (n_3 - 1)(n_2 + 1)$ as $e = n_2 + n_3 + n_2 n_3$. This labelling clearly shows $NI_{(n_1, n_2, n_3)}$ is graceful.

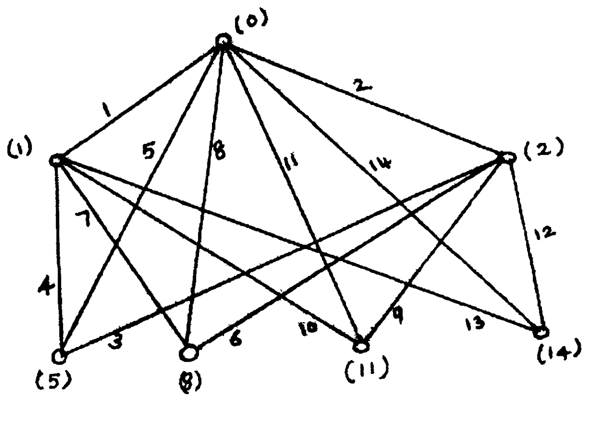


Illustration — Consider $NI_{(1, 2, 4)}$. The labellings are as shown in the figure.

Graceful labelling of $NI_{(1, 2, 4)}$

Open Problem

It is not known whether $\mathrm{NI}_{(n_1, n_2, \ldots, n_k)}$ is graceful for all partitions (with distinct parts) of a positive integer n.

3. Ply Number and Lace Number

A thread between vertices s and t of a graph G is a set of paths between s and t which have pairwise no vertices in common except s and t. The number of paths in the thread is the ply number of the thread. A thread with ply number q is called a q-thread. The maximum ply number of a thread between s and t is called the $s-t$ ply number and is denoted by $p(s, t)$. The minimum value of $p(s, t)$ for all pairs $s, t \varepsilon V$ such that $st \notin E$ is denoted by $p(G)$ and is called the ply number of G.[4]. (Def. 5.14, p. 116).

Theorem 5 — *Let n be any positive integer and $(n_1, n_2, \ldots, n_k)$ be any partition of n with distinct parts. Then the ply number of* $\mathrm{NI}_{(n_1, n_2, \ldots, n_k)}$ *is* $n - n_k$.

Proof : Let u and v be any two nonadjacent vertices of $\mathrm{NI}_{(n_1, n_2, \ldots, n_k)}$. Then u and v are in the ith independent set of vertices for some $i \varepsilon \{1, 2, \ldots, k\}$. Clearly the thread consisting of 2-paths $uv_j v$ for all v_j's in the other independent set of vertices will yield the maximum plynumber $p(u, v)$. The number of paths in the thread is $n_1 + n_2 + \ldots + n_{i-1} + n_{i+1} + \ldots + n_k = n - n_i$. This implies that the ply number of $\mathrm{NI}_{(n_1, n_2, \ldots, n_k)} = n - n_k$.

A lace between two vertices s and t is a set of paths which are pairwise edge disjoint. Analogously lace number of a graph is defined and can be found to be $n - n_k$ for the graph $\mathrm{NI}_{(n_1, n_2, \ldots, n_k)}$.

4. Clique Graphs

A clique of a graph is a maximal complete subgraph of G. The clique graph of G is the intersection graph of all cliques of G.

Consider the NI graph $\mathrm{NI}_{(n_1, n_2, \ldots, n_k)}$ represented by k sets of independent vertices $u_1, u_2, \ldots, u_{n_1}$; $v_1, v_2, \ldots, v_{n_2}$; $w_1, w_2, \ldots, w_{n_3}$; ... $z_1, z_2, \ldots, z_{n_k}$. By the very construction of this NI graph it follows that its clique is the complete graph K_k. For example, the complete graph represented by $u_1, v_1, w_1, \ldots, z_1$ is a clique.

Varying these vertices, we get $n_1 n_2 \ldots n_k$ cliques. Some of them are disjoint. For example, the clique $u_1, v_1, w_1, \ldots, z_1$ is disjoint from the clique $u_2, v_2, w_2, \ldots, z_2$. In fact, there are $(n_2 - 1)(n_3 - 1) \ldots (n_k - 1)$ cliques disjoint from the given clique $u_1 v_1 w_1 \ldots z_1$. Thus the degree of the vertex in the clique graph representing the clique $u_1 v_1 w_1 \ldots z_1$ is $n_1 n_2 \ldots n_k - (n_2 - 1)(n_3 - 1) \ldots (n_k - 1) - 1$. This shows that the clique graph is a m regular graph of order

$n_1 n_2 \ldots n_k$ where

$$m = n_1 n_2 \ldots n_k - (n_2 - 1)(n_3 - 1) \ldots (n_k - 1) - 1$$

and its size is of course $\frac{m}{2} \times n_1 n_2 \ldots n_k$.

If $n_1 = 1$, all the cliques of $\mathrm{NI}_{(n_1, \ldots, n_k)}$ have the common point u_1 and so the clique graph is the complete graph K_l where $l = n_1 n_2 \ldots n_k$.

Summerizing, therefore, we get

Theorem* 6 — *The clique graph of $\mathrm{NI}_{(n_1, n_2, \ldots, n_k)}$ *is a* m-regular graph of order $n_1, n_2, \ldots, n_k$. and of size $\frac{m\, n_1 n_2 \ldots n_k}{2}$ where $m = n_1 n_2 \ldots n_k - (n_2 - 1)(n_3 - 1) \ldots (n_k - 1) - 1$. If $n_1 = 1$, then its clique graph is the complete graph K_l where $l = n_1 n_2 \ldots n_k$.

5. Minimal Edge Covering

A family of edges of a graph is called a covering edge family when it has at least one edge at each vertex[4] (p. 322).

***Theorem* 7** — *A minimal covering edge family of the* $\mathrm{NI}_{(n_1, n_2, \ldots, n_k)}$ *graph has the cardinality* $n_1 + n_3 + \ldots + n_k$ *if* k is odd or $n_2 + n_4 + \ldots + n_k$ if k is even.

PROOF : Let $u_1, u_2, \ldots, u_{n_1}$; $v_1, v_2, \ldots, v_{n_2}$; $w_1, w_2, \ldots, w_{n_3}$ ... $z_1, z_2, \ldots, z_{n_k}$ be k sets of independent vertices of $\mathrm{NI}_{(n_1, n_2, \ldots, n_k)}$.

Let E_1 be the set of edges $u_1 v_1, u_2 v_2, \ldots u_{n_1} v_{n_1}$. This set E_1 covers all the vertices of u_1's and the first n_1 vertices $v_1 v_2 \ldots v_{n1}$ of v_i's. The remaining vertices $v_{n_1+1}, v_{n_1+2}, v_{n_2}$ are joined to the next $n_2 - n_1$ vertices $w_1, w_2, \ldots, w_{n_2 - n_1}$ of w_i's to form the set E_2. The number of vertices uncovered in w_i's is $n_3 - (n_2 - n_1)$. Proceeding like this, we get the set $E_1, E_2, \ldots, E_{k-1}$ of edges covering all vertices except the remaining vertices $n_k - [\, n_{k-1} - \{ n_{k-2} - \ldots - (n_2 - n_1) \} \,]$ on the kth set of independent vertices. Joining these vertices to any of the vertices in the other sets of independent vertices we get a covering set $E = E_1 \cup E_2 \cup \ldots \cup E_k$. As they are pairwise disjoint

$$
\begin{aligned}
|E| &= |E_1| + |E_2| + \ldots + |E_k| \\
&= n_1 + (n_2 - n_1) + n_3 - (n_2 - n_1) + \ldots + n_k \\
&\quad - [\, n_{k-1} - \{ n_{k-2} - n_{k-3} - \ldots - (n_2 - n_1) \} \,]
\end{aligned}
$$

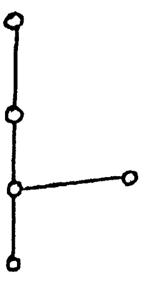
$$= \begin{cases} n_1 + n_3 + \ldots + n_k \text{ if } k \text{ is odd} \\ n_2 + n_4 + \ldots + n_k \text{ if } k \text{ is even} \end{cases} \qquad \ldots (*)$$

As $|E| = \left[\frac{n}{2}\right]$, the integral part of $\frac{n}{2}$, this (*) is minimal also.

Open Problem

Given a positive integer n, can we characterize all those NI graphs of order n ?

The class of all $NI_{(n_1, n_2, \ldots, n_k)}$ graphs described in this paper is only a proper subclass of the class of all NI graphs. For example, the following graph is a NI graph of order 5; but it does not come under the purview of any $NI_{(n_1, n_2, \ldots, n_k)}$ graph of order 5.



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