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EFFECT OF OBLATENESS ON THE LOCATION AND STABILITY OF EQUILIBRIUM POINTS IN ROBE'S CIRCULAR PROBLEM

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The effect on the location and linear stability of the equilibrium points in the Robe's (*Celest. Mech.* **16** (1977) 343) circular problem, when the primary other than the spherical shell is an oblate spheroid, has been studied. It is proved that the center of the first primary (spherical shell) is always an equilibrium point, for all values of the density parameter K and the mass parameter μ. For $K > 1$, there is another equilibrium point lying on the line joining the centers of the primaries. For $-\mu < K < 0$, there are two equilibrium points in the $x - z$ plane, equidistant and forming triangles with the line joining the centers of the primaries. For $K = \left(1 + \frac{3}{2}\alpha\right)(1 - \mu)$, where α is the oblateness factor, there are infinite number of equilibrium points in the $x - y$ plane, lying on a circle of radius one and having center at the center of the second primary provided the points lie inside the shell. Further, it is proved that the circular and the triangular points are always unstable. And the equilibrium points lying on the line joining the centers of the shell and the second primary are stable when K and μ satisfy certain inequalities.

Key Words: Restricted Three-Body Problem; Equilibrium Points; Oblate Spheroid; Stability

1. INTRODUCTION

A new kind of restricted three body problem has been considered by Robe (1977), in which one of the primaries is a rigid spherical shell m_1^* filled with a homogeneous incompressible fluid of density ρ_1. The second primary is a mass point m_2 outside the shell and the third body m_3 is a small solid sphere of density ρ_3, inside the shell, with the assumption that the mass and radius of m_3 are infinitesimal. He has shown the existence of an equilibrium point with m_3 at the center of the shell, while m_2 describes a Keplerian orbit around it. Further he has discussed the linear stability of the equilibrium point.

Shrivastava and Garain (1991) have studied the effect of small perturbations in the Coriolis and Centrifugal forces on the location of the equilibrium point in the Robe's problem by taking the orbit of m_2 around m_1 as a circle and assuming densities ρ_1 and ρ_3 to be equal. Plastino and Plastino (1995) have considered the Robe's problem by taking the shape of the fluid body as Roche's ellipsoid (Chandrasekhar 1987). Giordano, Plastino and Plastino (1997) have studied the effect of drag force on the stability of equilibrium point, both in Robe's problem (1977) and the problem studied by Plastino and Plastino (1995). Hallan and Rana (2001a) have studied the existence of all the equilibrium points, their locations and linear stability in the Robe's problem.

The results of stability of the equilibrium point, the center of the first primary, are same as those given by Robe (1977). Again Hallan and Rana (2001b) studied the effect of small perturbations ε and ε' in the coriolis and centrifugal forces respectively on the location and linear stability of the equilibrium point in the Robe's circular problem with densities ρ_1 and ρ_3 equal that is K has the value zero. They have shown that there is only one equilibrium point which is stable for $\mu_c < \mu < 1$ and unstable for $0 < \mu < \mu_c$, where $\mu_c = \frac{8}{9} + \frac{2}{3}\left(\frac{43}{25}\varepsilon' - \frac{10}{3}\varepsilon\right)$. When the second primary m_2 is an oblate spheroid, then it not only perturbs the coriolis and centrifugal forces but also perturbs the potentials between the bodies. So, in the present paper we propose to study the existence and stability in the linear sense, of equilibrium points in the Robe's circular problem when the second primary is an oblate spheroid and the resulting perturbations in the potentials between the bodies are small, since oblateness factor $\alpha << 1$. Most of the equilibrium points are found as perturbations of the equilibrium points obtained in the case when the second primary is a mass point that is $\alpha = 0$ (Hallan and Rana, 2001a).

The model studies in the paper could be applied to the restricted three body problem consisting of Earth, its core and Moon where Earth is first primary and Moon, the second primary.

2. The Location of the Equilibrium Points

In the uniformly rotating dimensionless coordinate system Oxyz, the origin O at the center of mass of the two primaries, Ox pointing towards m_2 and Oxy being the orbital plane of the finite bodies, the equations of motion of m_3 in Robe's circular problem, when the primary m_2 is an oblate spheroid are :

$$\ddot{x} - 2n\dot{y} = V_x,$$

$$\ddot{y} + 2n\dot{x} = V_y,$$

$$\ddot{z} = V_z, \qquad \dots (1)$$

where

$$V = \frac{n^2}{2}(x^2 + y^2) - \frac{K}{2}\left[(x+\mu)^2 + y^2 + z^2\right] + \frac{\mu}{r_2} + \frac{\mu\alpha}{2r_2^3}\left(\frac{1 - 3z^2}{r_2^2}\right), \qquad \dots (2)$$

$$r_2^2 = (1 - \mu - x)^2 + y^2 + z^2, \quad \mu = \frac{m_2}{m_1^* + m_2}, \quad 0 < \mu < 1,$$

$$K = \frac{4}{3}\frac{\pi \rho_1 a^3}{m_1^* + m_2}\left(1 - \frac{\rho_1}{\rho_3}\right), \qquad \alpha = \frac{a_0^2 - b_0^2}{5}. \quad 0 < \alpha << 1.$$

Here, a is the radius of the circular orbit which m_2 describes around m_1^*.

a_0 and b_0 are the lengths of the semi axes of m_2, whose equatorial plane coincides with the $x - y$ plane. Coordinates of the center of the shell and the second primary are $(-\mu, 0, 0)$ and $(1-\mu, 0, 0)$ respectively. The terms $\frac{\mu}{r_2} + \frac{\mu\alpha}{2r_2^3}\left(1 - \frac{3z^2}{r_2^2}\right)$ occurring in the potential V given by (2) are due to second primary m_2 being oblate and have been obtained by using the formula for the potential of a triaxial rigid body given in Meirovitch (1970). The other term in V are similar to those occurring in Robe's problem (1977) with eccentricity $e = 0$.

The perturbed mean motion, n, of the primaries, is given by

$$n^2 = 1 + \frac{3}{2}\alpha.$$

Equilibrium points are given by the equations

$$V_x = n^2 x - K(x+\mu) + \mu\left[\frac{1}{r_2^3} + \frac{3\alpha}{2r_2^5} - \frac{15\,\alpha}{2r_2^7}z^2\right](1-\mu-x) = 0,$$

$$V_y = n^2 y - Ky - \mu\left[\frac{1}{r_2^3} + \frac{3\alpha}{2r_2^5} - \frac{15\,\alpha}{2r_2^7}z^2\right]y = 0,$$

$$V_z = -Kz - \mu\left[\frac{1}{r_2^3} + \frac{3\alpha}{2r_2^5} - \frac{15\,\alpha}{2r_2^7}z^2\right]z - \frac{3\mu\,\alpha z}{r_2^5} = 0. \qquad \text{... (3)}$$

These equations have no general solution. But there are specific cases, which have solutions. They are given by the following system of equations:

(*i*) when $y = 0$, $z = 0$

$$n^2 x + \mu\left(\frac{1}{r_2^3} + \frac{3\alpha}{2r_2^5} - \frac{15\,\alpha}{2r_2^7}z^2\right)(1-\mu-x) - K(x+\mu) = 0. \qquad \text{... (4)}$$

(*ii*) when $y = 0$

$$n^2 x + \mu\left(\frac{1}{r_2^3} + \frac{3\alpha}{2r_2^5} - \frac{15\,\alpha}{2r_2^7}z^2\right)(1-\mu-x) - K(x+\mu) = 0,$$

and

$$\mu\left(\frac{1}{r_2^3} + \frac{9\alpha}{2r_2^5} - \frac{15\,\alpha}{2r_2^7}z^2\right) + K = 0. \qquad \text{... (5)}$$

(*iii*) when $z = 0$,

$$n^2 - \mu\left(\frac{1}{r_2^3} + \frac{3\alpha}{2r_2^5}\right) - K = 0,$$

$$n^2 x + \mu\left(\frac{1}{r_2^3} + \frac{3\alpha}{2r_2^5}\right)(1 - \mu - x) - K(x + \mu) = 0. \qquad \text{... (6)}$$

Consider eq. (4). The solutions of this equation lie on the line joining the centers of the shell and the second primary and the x-coordinates of the equilibrium points are the roots of the equation

$$n^2 x + \mu\left(\frac{1}{(1 - \mu - x)^2} + \frac{3\alpha}{(1 - \mu - x)^2}\right) - K\,(x + \mu) = 0. \qquad \text{... (6a)}$$

$x = -\mu$ is a root of the eq. (6*a*), for all values of α.

When $\alpha = 0$, the eq. (6*a*) has two more roots (Hallan and Rana, 2001a). Only one of them lies within the spherical shell, it is

$$x_{20} = \frac{-2 + \mu + 2K - 2K\mu - \sqrt{\mu\,(-4 + 4K + \mu)}}{2(K - 1)}, \text{ for } K > 1,$$

provided

$$\mu + x_{20} < R \text{ where } R = \left[\frac{3(m_1^* - m_3)}{4\pi\rho_1}\right]^{1/3}$$

When $\alpha \neq 0$, let the root be

$$x_2 = x_{20} + p, \qquad |\,p\,| << 1.$$

Since oblateness factor α is very small, we reject second and higher order terms of α.

Substituting the value of x_2 in eq. (6*a*) and rejecting second and higher powers of p and α, we get

$$p = p'\,\alpha,$$

where

$$p' = \frac{3[(-2 + \mu + 2K - 2K\mu - \sqrt{\mu\,(-4 + 4K + \mu)})\,(\mu + \sqrt{\mu - (-4 + \mu K + \mu)^4} + 32\mu(K - 1)^5}{4(K - 1)^2\,(\mu + \sqrt{\mu\,(-4 + 4K + \mu)})\,[(\mu + \sqrt{\mu\,(-4 + 4K + \mu)})^3 - 16\mu(K - 1)^2]}$$

Consider the equilibrium point $(x_2, 0, 0)$. When m_2 is a mass point, the equilibrium point is $(x_{20}, 0, 0)$. When m_2 is an oblate spheroid, the equilibrium point shifts to $(x_{20} + p'\,\alpha, 0, 0)$ which means that the point shifts to the right or left of the equilibrium point $(x_{20}, 0, 0)$ according as p' is positive or negative.

Consider the other equilibrium point $(-\mu, 0, 0)$ that is the center of the first primary. When the shape of m_2 changes from a mass point to an oblate spheroid, the position of the equilibrium point remains same that is the center of the shell. Thus, center of the first primary is always an equilibrium point independent of the shape of the second primary.

Consider the system of Eqs. (5). When $\alpha = 0$, system has two solutions $(x_{30}, 0, \pm z_{30})$, where

$$x_{30} = K, \quad z_{30} = \sqrt{b_1^2 - a_1^2},$$

$$a_1 = 1 - \mu - K, \quad b_1 = \left(-\frac{\mu}{K}\right)^{1/3},$$

provided $K < 0$ and $K + \mu > 0$ (Hallan and Rana (2001a)) and $K^2 + z_{30}^2 < R^2$.

When $\alpha \neq 0$, let system[5] has the solution

$$x_3 = x_{30} + q_1, y_3 = 0, \quad z_3 = z_{30} + q_2, \qquad |q_1|, |q_2| << 1.$$

Substituting the values of x_3 and z_3 in the system of eqs. (5) and rejecting second and higher powers of q_1, q_2 and α, we get the solution as

$$x_3 = x_{30} + q_1' \alpha, \qquad y_3 = 0, \qquad z_3 = z_{30} + q_2' \alpha,$$

where

$$q_1' = \frac{-3K}{2b_1^2}(2a_1 + b_1^2), \qquad q_2' = \frac{1}{2z_{30} b_1^2}[a_1^2(5 - 6K) - b_1^2(2 + 3a_1 K)].$$

Thus, for $K < 0$ and $K + \mu > 0$, there are two equilibrium points $(x_3, 0, \pm z_3)$ making triangles with the line joining the centers of the shell and the second primary. And also the points are equidistant from the line joining the centers of the shell and the second primary. When the shape of m_2 becomes an oblate spheroid, the location of the equilibrium points shifts from $(x_{30}, 0, \pm z_{30})$ to $\left(x_{30} + q_1' \alpha, 0, \pm(z_{30} + q_2' \alpha)\right)$.

Consider the system of Eqs. (6). The system has solution only for $K = \left(1 + \frac{3}{2}\alpha\right)(1 - \mu)$. And then there are infinite number of equilibrium points lying on the circle $(1 - \mu - x)^2 + y^2 = 1$, $z = 0$, provided they lie inside the spherical shell. The circle has center at $(1 - \mu, 0, 0)$ which is the center of the second primary and radius one which is the distance between the centers of the primaries.

When m_2 is a mass points, circular points exist only for $K = 1 - \mu$, but when m_2 becomes an oblate spheroid, circular points exist only for $K = \left(1 + \frac{3}{2}\alpha\right)(1 - \mu)$. So, change in shape of the second primary only affects the value of K and the radius and the center of the circle remain same.

3. Stability

To discuss the linear stability of an equilibrium point (x_0, y_0, z_0), we substitute

$$\xi = x - x_0, \quad \eta = y - y_0, \quad \zeta = z - z_0,$$

in the equations of motion (1). Then, in the linearized form, the variational equations are

$$\ddot{\xi} - 2n\dot{\eta} = V^0_{xx}\,\xi + V^0_{xy}\,\eta + V^0_{xz}\,\zeta,$$

$$\ddot{\eta} - 2n\dot{\xi} = V^0_{yx}\,\xi + V^0_{yy}\,\eta + V^0_{yz}\,\zeta,$$

$$\ddot{\zeta} = V^0_{zx}\,\xi + V^0_{zy}\,\eta + V^0_{zz}\,\zeta. \quad \text{... (7)}$$

Here, the superscript 0 denotes that the second derivatives are to be evaluated at the point (x_0, y_0, z_0).

(*i*) Stability of the equilibrium points lying on the line joining the centers of the shell and second primary.

(*ii*) Equilibrium point $(-\mu, 0, 0)$. At this point

$$V^0_{xx} = 1 - K + 2\mu + \frac{3}{2}\alpha(1 + 6\mu),$$

$$V^0_{yy} = 1 - K - \mu + \frac{3\alpha}{2}(1 - \mu),$$

$$V^0_{zz} = 1 - K - \mu - \frac{3}{2}\alpha(3\mu - 1),$$

$$V^0_{xy} = V^0_{yz} = V^0_{zx} = 0.$$

Then, variational eq. (7) become

$$\ddot{\xi} - 2n\dot{\eta} = \left[1 - K + 2\mu + \frac{3\alpha}{2}(1 + 6\mu)\right]\xi, \quad \text{... (8)}$$

$$\ddot{\eta} + 2n\dot{\xi} = \left[1 - K - \mu + \frac{3\alpha}{2}(1 - \mu)\right]\eta, \quad \text{... (9)}$$

$$\ddot{\zeta} = -\left[K + \mu + \frac{9\mu\alpha}{2}\right]\zeta. \quad \text{... (10)}$$

The eq. (10) shows that the motion parallel to z-axis is always stable for $K + \mu + \frac{9\mu\,\alpha}{2} > 0$. This inequality is true when m_3 is denser than the fluid $(\rho_3 > \rho_1)$. The characteristic equation corresponding to the eqs. (8) and (9) is

$$\lambda^4 + \left\{ 2K - \mu\left(1 + \frac{15}{2}\alpha\right) + 2(1+\alpha) \right\} \lambda^2 + \left\{ K - (2+9\alpha)\mu\left(1 + \frac{3}{2}\alpha\right)\right\}$$

$$\left\{ K + \left(1 + \frac{3}{2}\alpha\right)\mu - \left(1 + \frac{3}{2}\alpha\right)\right\} = 0.$$

Its roots are

$$\lambda^2_{1,2} = \frac{-\left\{ 2K - \mu\left(1 + \frac{15}{2}\alpha\right) + 2(1+\alpha)\right\} \pm \sqrt{\Delta}}{2},$$

where

$$\Delta = 9(1+7\alpha)\,\mu^2 - 8(1+9\alpha)\mu + 8(2+3\alpha)\,K.$$

Thus,

$$\lambda_1^2 + \lambda_2^2 = -2K + \mu\left(1 + \frac{15}{2}\alpha\right) - 2(1+\alpha),$$

$$\lambda_1^2\,\lambda_2^2 = \left[K - (2+9\alpha)\mu - \left(1 + \frac{3}{2}\alpha\right)\right]\left[K + \left(1 + \frac{3}{2}\alpha\right)\mu - \left(1 + \frac{3}{2}\alpha\right)\right].$$

The equilibrium point is stable if the roots are real and negative. This implies $\Delta > 0$, $\lambda_1^2 + \lambda_2^2 < 0$ and $\lambda_1^2\lambda_2^2 > 0$. Thus, in the $\mu - K$ place, stability regions are I and II (Fig. 1) where region I is the shaded area which is, above the line $P'Q'$ with equation $K - (2+9\alpha)\mu - \left(1 + \frac{3}{2}\alpha\right) = 0$ and between $\mu = 0$ and $\mu = 1$, the co-ordinates of points P', Q' being $\left(0, 1 + \frac{3}{2}\alpha\right)$, $\left(1, 3 + \frac{21}{2}\alpha\right)$ respectively. And region II corresponds to the points within the shaded area bounded by the curves: $K + \left(1 + \frac{3}{2}\alpha\right)\mu - \left(1 + \frac{3}{2}\alpha\right) = 0$ (Line $P'H$), $9(1+7\alpha)\mu^2 - 8(1+9\alpha)\mu + 8(2+3\alpha)K = 0$ (Parabola $OI'J'$), the lines $\mu = 0$ and $\mu = 1$, the co-ordinates of points H, I', J' being $(1, 0)$, $\left[\frac{8}{9}(1+2\alpha), 0\right]$, $\left[1, -\frac{1}{16}\left(1 - \frac{21}{2}\alpha\right)\right]$ respectively. When m_2 is a mass point, stability region I is above the line $K - 2\mu = 1$ (line PQ) and stability region II is in between the curves : $\mu + K = 1$ (line PH) and $9\mu^2 - 8\mu + 16K = 0$ (Parabola OIJ) (Fig. 1), the co-ordinates of points P, Q, I, J being $(0, 1)$, $(1, 3)$, $\left(\frac{8}{9}, 0\right)$, $\left(1, -\frac{1}{16}\right)$ respectively.

(b) Equilibrium point $(x_2, 0, 0)$. At this point

$$V^0_{xx} = 1 - K + 4A_1 + \alpha S_1,$$

$$V^0_{yy} = 1 - K - 2A_1 + \alpha S_2,$$

$$V^0_{zz} = n^2 - K - 2A_1 - \alpha S_3,$$

$$V^0_{xy} = V^0_{yz} = V^0_{zx} = 0.$$

where

$$A_1 = \frac{\mu A_0^3}{2}, \quad A_0 = \frac{2(K-1)}{[\mu + \sqrt{\mu(-4+4K+\mu)}]},$$

$$S_1 = \frac{3}{2} + 6\mu A_0^4 (p' + A_0),$$

$$S_2 = \frac{3}{2} - 3\mu A_0^4 \left(\frac{p' + A_0}{2}\right),$$

$$S_3 = 3\mu A_0^4 \left(p' + \frac{3A_0}{2}\right).$$

Then, variational eq. (7) become

$$\ddot{\xi} - 2n\dot{\eta} = (1 - K + 4A_1 + \alpha S_1)\,\xi, \qquad \text{... (11)}$$

$$\ddot{\eta} + 2n\dot{\xi} = (1 - K - 2A_1 + \alpha S_2)\,\eta, \qquad \text{... (12)}$$

$$\ddot{\zeta} = (K + 2A_1 + \alpha S_3)\,\zeta. \qquad \text{... (13)}$$

Eq. (13) shows that the motion parallel to z-axis is stable if $K + 2A_1 + \alpha S_3 > 0$. As $\alpha << 1$, therefore the motion is stable if $\rho_3 > \rho_1$. The characteristic equation corresponding to the system of eqs. (11) and (12) is

$$\lambda^4 + (2 + 2K - 2A_1 - \alpha A_2)\lambda^2 + (-1 + K - 4A_1 - \alpha S_1)(-1 + K + 2A_1 - \alpha S_2) = 0, \qquad \text{... (14)}$$

where

$$A_2 = -3 + 3\mu p' A_0^4 + 9\mu A_0^5.$$

Roots of eq. (14) are

$$\lambda^2_{1,2} = \frac{-(2 + 2K - 2A_1 - \alpha A_2) \pm \sqrt{D}}{2},$$

where

$$D = 16(K - A_1) + 36A_1^2 + \alpha S_4,$$

$$S_4 = 24K - 6\mu A_0^3 \left\{ 3 + 4p' A_0 + 3A_0^2 (K+3) - 10\mu p' A_0^4 - 12\mu A_0^5 \right\}.$$

The equilibrium point is stable if the roots are real and negative. That is $(x_2, 0, 0)$ is stable if

$$16\mu (K-1)^3 < (K - 1 - \alpha S_1) \left(\mu + \sqrt{\mu (-4 + 4K + \mu)} \right)^3 \qquad \text{... (15)}$$

Hence the equilibrium point $(x_2, 0, 0)$ is stable if $K + 2A_1 + \alpha S_3 > 0$ and condition (15) is satisfied. In other words, stability region of the equilibrium point $(x_2, 0, 0)$ in $\mu - K$ plane is the set of points common to the regions determined by the inequalities $K + 2A_1 + \alpha S_3 > 0$ and (15).

(*ii*) Stability of triangular points.

Triangular points are $(x_3, 0, \pm z_3)$. At these points

$$V_{xx}^0 = 1 - \frac{3a_1^2 K}{b_1^2} + \alpha R_1,$$

$$V_{yy}^0 = 1 + \alpha R_2,$$

$$V_{zz}^0 = 1 + \frac{3K}{b_1^2} \left(a_1^2 - b_1^2 \right) + \alpha R_3,$$

$$V_{xz}^0 = \pm \left[3Ka_1 \frac{\sqrt{b_1^2 - a_1^2}}{b_1^2} + \alpha R_4 \right],$$

$$V_{xy}^0 = V_{yz}^0 = 0,$$

where

$$R_1 = \frac{3}{2} \left\{ 1 - \frac{2K}{b_1^2} \left(1 + 3a_1 K - \frac{10a_1^2}{b_1^2} + \frac{5a_1^4}{b_1^4} + \frac{6a_1^2 K}{b_1^2} \right) \right\},$$

$$R_2 = \frac{3}{2} \left(1 + \frac{2K}{b_1^2} \right),$$

$$R_3 = \frac{3}{2} \left\{ 1 + \frac{2K}{b_1^2} \left(7 + 3a_1 K + \frac{6a_1^2 K}{b_1^2} - \frac{15a_1^2}{b_1^2} + \frac{5a_1^4}{b_1^4} \right) \right\},$$

$$R_4 = \frac{3K}{2b_1^4} \left[\frac{a_1}{\sqrt{b_1^2 - a_1^2}} (5a_1^2 - 2b_1^2 - 6a_1^2 K - 3a_1 b_1^2 K) \right.$$

$$+\sqrt{b_1^2-a_1^2}\left(-10a_1+6a_1K+3Kb_1^2+\frac{10a_1^2}{b_1^3}\right)\Bigg],$$

Thus, characteristic equation becomes

$$f(\lambda)=0,$$

where

$$f(\lambda)=\lambda^6+\left(5n^2-V_{xx}^0-V_{yy}^0-V_{zz}^0\right)\lambda^4+(n^2\left(4-V_{xx}^0-V_{yy}^0-V_{zz}^0\right)+V_{yy}^0V_{zz}^0$$
$$+V_{xx}^0V_{zz}^0+V_{xx}^0V_{yy}^0+V_{yy}^0V_{xz}^{02}-V_{xz}^{02})\lambda^2$$
$$\left(V_{xx}^0V_{yy}^0n^2-V_{xx}^0V_{yy}^0V_{zz}^0+V_{xz}^{02}V_{yy}^0\right).$$

Now

$$f(\lambda)\to\infty \quad \text{as} \quad \lambda\to\infty.$$

and

$$f(0)=\frac{3K}{b_1^2}\left(b_1^2-a_1^2\right)+\alpha R_0$$

$$\simeq-\frac{3K}{b_1^2}\left(b_1^2-a_1^2\right)<0 \qquad \text{as } \alpha<<1,$$

where

$$R_0=\frac{3}{2}+9K-\frac{21K}{b_1^2}+\frac{9a_1^2K}{2b_1^2}+\frac{45a_1^2K}{b_1^4}-\frac{9a_1K^2}{b_1^2}-\frac{15a_1^4K}{b_1^6}+\frac{45a_1^2K^2}{b_1^4}-\frac{45K^2a_1^2}{b_1^6}$$
$$+\frac{54a_1K^3}{b_1^2}+\frac{225a_1^4K^2}{b_1^6}-\frac{45a_1^4K^2}{b_1^8}-\frac{27a_1^3K^2}{b_1^4}-\frac{54a_1^4K^3}{b_1^6}-\frac{54a_1^3K^3}{b_1^4}$$
$$-\frac{90a_1^2K^2}{b_1^4}+\frac{54a_1^2K^3}{b_1^4}-\frac{90a_1^6K^2}{b_1^8}.$$

Thus, there is at least one positive root of the characteristic equation and consequently, both the equilibrium points are unstable.

(*iii*) Stability of circular points.

These points exist only when $K=\left(1+\frac{3}{2}\alpha\right)(1-\mu)$ and lie on the circle $(1-\mu-x)^2+y^2=1$, $z=0$. Coordinates of any point on the circle are of the form $(1-\mu+\cos\phi, \sin\phi, 0)$. At this point

$$V^0_{xx} = C_1 \cos^2 \phi,$$

$$V^0_{yy} = C_1 \sin^2 \phi,$$

$$V^0_{zz} = -3\mu\,\alpha,$$

$$V^0_{xy} = C_1 \cos\phi \sin\phi,$$

$$V^0_{yz} = V^0_{xz} = 0,$$

where $$C_1 = 3\mu\left(1+\frac{5}{2}\alpha\right).$$

Thus, variational eq. (7) become

$$\ddot{\xi} - 2n\dot{\eta} = (C_1\cos^2\phi)\,\xi + (C_1\cos\phi\sin\phi)\,\eta,$$

$$\ddot{\eta} + 2n\dot{\xi} = (C_1\cos\phi\,\sin\phi)\,\xi + (C_1\sin^2\phi)\,\eta,$$

$$\ddot{\zeta} = -\left[1+\frac{3}{2}\alpha(1+2\mu)\right]\zeta.$$

Last equation shows that the motion parallel to z-axis is always stable. The characteristic equation corresponding to the other two equations is

$$\lambda^2\left[\lambda^2 - \left\{3\mu - 4 + \frac{3}{2}\alpha(5\mu-4)\right\}\right] = 0. \qquad \dots (16)$$

Here, two of the root of the eq. (16) are equal to zero, so the solution of the equations of motion will contain secular terms. Hence, the equilibrium points are unstable.

4. CONCLUSION

In the Robe's problem, when the second primary (m_2) is an oblate spheroid, with its equatorial plane coinciding with the plane of motion of the primaries, center of the shell is an equilibrium point for all values of K and μ, as in the case when m_2 is a mass point. For $K > 1$, there is another equilibrium point lying on the line joining the centers of the primaries, namely $(x_{20} + p'\alpha, 0, 0)$. This point lies to the right or left of the point $(x_{20}, 0, 0)$, its position when m_2 is a mass point, according as p' is positive or negative. For $-\mu < K < 0$, there are two equilibrium points in the $x-z$ plane, having coordinates $\left(x_{30} + q_1'\alpha, 0, \pm\left(z_{30} + q_2'\alpha\right)\right)$, equidistant and forming triangles with the line joining the centers of the primaries. For $K = \left(1+\frac{3}{2}\alpha\right)(1-\mu)$, there are infinite number of equilibrium points in the $x-y$ plane, lying on a circle of radius one and having center at the center of the second primary, provided the points lie inside the spherical shell. This result is same as that when m_2 is a mass point except that these equilibrium points exist when K and μ

satisfy the equation $K = \left(1 + \frac{3}{2}\alpha\right)(1-\mu)$ instead of $K = 1 - \mu$. The circular and triangular points are always unstable. And

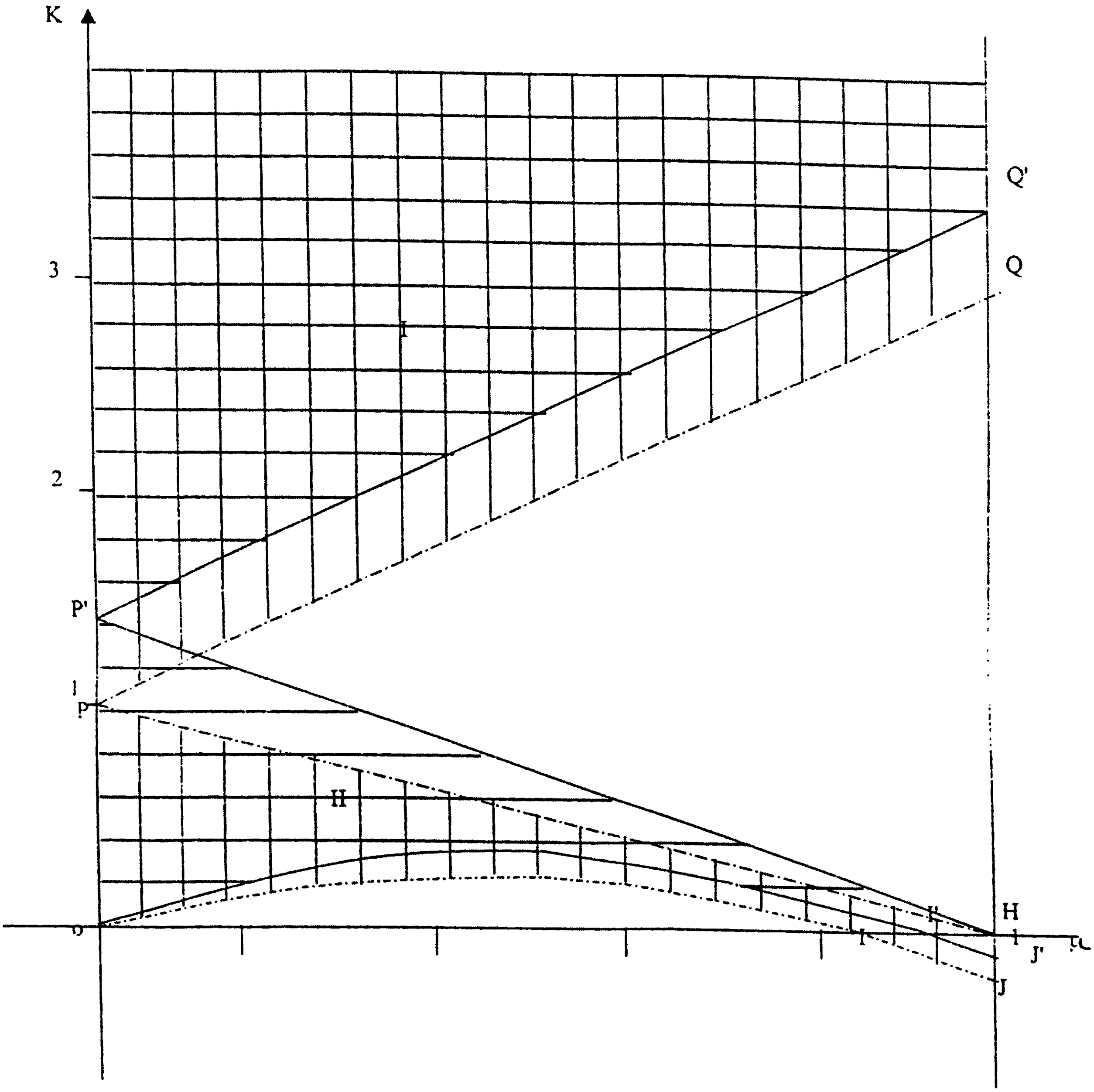

FIG. 1. Stability region of the equilibrium point $(-\mu, 0, 0)$. Stable region, when m_2 is a mass point. Stable region, when m_2 is an oblate spheroid.

(*a*) The equilibrium point $(x_1, 0, 0)$ is stable if the point (μ, K) lies in the region (Fig. 1) shaded by the horizontal lines.

(*b*) The equilibrium point $(x_2, 0, 0)$ is stable if

$$16\mu(K-1)^3 < (K - 1 - \alpha S_1)\left(\mu + \sqrt{\mu(-4+4K+\mu)}\right)^3$$

and

$$K + 2A_1 + \alpha S_3 > 0 \text{ hold.}$$

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