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NUMERICAL SOLUTION OF SINGULAR INTEGRAL EQUATIONS USING CUBIC SPLINE INTERPOLATIONS

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Product integration methods using cubic spline interpolations have been obtained for real one dimensional singular integral equations of the second kind with Cauchy type kernels and index equal to 1. Here, we use the representation $p_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$, $x_j \le x \le x_{j+1}, j = O(1)\ n - 1$, for approximating the unknown function for obtaining the numerical method.

Key Words: Singular Integral Equation; Cauchy Type Kernels; Index; Cubic Spline Interpolations; Product Integration

1. INTRODUCTION

Singular integral equations (with Cauchy type kernels) appear frequently in physical and engineering applications. Their classical field of appearance is the theory of elasticity for cracked media, where crack problems were solved by the method of singular integral equations. The reduction of a physical or engineering problem to a singular integral equation generally requires the numerical solution of this equation by some simple and sufficiently accurate algorithm.

Here, we consider real one dimensional singular integral equation of the second kind of the form:

$$aw(x)\, g(x) + (b/\pi) \;⨍_{-1}^{1} w(t)\,(g(t)/(t-x))\, dt$$

$$+ \int_{-1}^{1} w(t)\, k(x,t)\, g(t)\, dt = f(x), \qquad -1 < x < 1, \qquad \text{... (1)}$$

where a and b are constants, $k(t, x)$ and $f(x)$ are known Hölder-continuous functions, $g(x)$ is the unknown function and $w(x)$ is a weight function, having the form

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}, \qquad \text{... (2)}$$

where

$$-\cot \pi\alpha = \cot \pi\beta = a/b, \qquad \alpha, \beta > -1, \qquad \text{... (3)}$$

with

$$\kappa = -(\alpha+\beta). \qquad \text{... (4)}$$

In this paper, we consider the case where

$$\kappa = 1, \quad -1 < \alpha, \beta < 0. \qquad \text{... (5)}$$

This is the case most frequently arising problem in practice, particularly in crack problems.

Under these conditions, eq. (1) does not possess a unique solution unless supplemented by a condition of the form

$$\int_{-1}^{1} w(t)\, g(t)\, dt = C, \qquad \text{... (6)}$$

where C is a constant.

In past, a global approximation to the solution of a general linear Fredholm integral equation of the second kind is constructed by means of cubic spline quadrature by Netravali and Figueiredo[2]. The numerical solution of singular integral equations (with Cauchy type kernels) were described by Jen and Srivastav[8] by expressing the unknown function as the product of an appropriate weight function and a cubic spline. Here, the method has been discussed for only one weight function $(1-t^2)^{-1/2}$. A modification of the direct collocation and quadrature methods for the numerical solution of singular integral equations with Cauchy type kernels and index equal to 1 has been proposed in Ioakimidis[7]. Numerical methods based on "practical" abscissas $x_k = \cos(k\pi/n)$, $k = 0(1)n$ and "classical" abscissas $x_k = \cos((2k+1)\pi/(2n+2))$, $k = O(1)n$ for (1) have been described in Kumar[10]

In the present method, the unknown function is replaced by cubic spline interpolation and the weight function considered is $(1-x)^{\alpha}(1+x)^{\beta}$, where $-\cot\pi\alpha = \cot\pi\beta = a/b$, $\alpha, \beta > -1$; with $\kappa = -(\alpha+\beta)$. In this paper, we consider the case where $\kappa = 1, -1 < \alpha, \beta < 0$, but this method can be applied in other cases also with a suitable supplemented condition. The problem is reduced to a system of linear algebraic equations using quadrature rules for evaluation of Cauchy principal value integrals described in Chawla and Kumar[1], Kumar[9] and Kumar and Sangal[11] to desired degree of accuracy. The non-singular integrals can be evaluated to desired degree of accuracy using quadrature formulas already existing in the literature.

It may be mentioned here, that the convergence of the quadrature formulas using cubic spline interpolations for a suitable class of functions for Cauchy principal value integrals (with weight function $(1-x)^{\alpha}(1+x)^{\beta}$; $\alpha, \beta > -1$ has been established in Dagnino and Santi[3].

2. THE NUMERICAL METHOD

Let the points $\{t_i\}$ be given on a mesh defined by $-1 = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = 1$ not necessarily to be equally spaced, then the cubic spline interpolations $g_k(t)$ for $g(t)$ on the interval $t_k \le x \le t_{k+1}$, $k = O(1)\ n-1$ may be written as :

$$g_k(t) = \sum_{r=1}^{4} d_{k,r}(t-t_k)^{r-1}, \qquad k = 0(1)\ n-1. \qquad \dots (7)$$

Natural splines are frequently used, as they have minimum total curvature among all sufficiently smooth interpolating curves. For natural splines, we are given below the determining equations for $d_{k,r}$ (see Davis and Rabinowitz[4] (pp. 67).

Determining equations for $d_{k,r}$:

$$h_k = t_{k+1} - t_k, \qquad k = O(1)\ n-1,$$

$$d_{k,1} = g(t_k), \qquad k = O(1)\ n, \qquad \dots (8a)$$

$$d_{k,2} = \{(d_{k+1,1} - d_{k,1})/h_k\} - \{(2d_{k,3} + d_{k+1,3})\, h_k/3\}, \quad k = 0(1)\ n-1, \qquad \dots (8b)$$

$$d_{k,4} = (d_{k+1,3} - d_{k,3})/(3h_k), \qquad k = O(1)\ n-1, \qquad \dots (8c)$$

$$h_{k-1}\, d_{k-1,3} + 2(h_{k-1} + h_k)\, d_{k-3} + h_k\, d_{k+1,3}$$

$$= 3[\{d_{k+1,1} - d_{k,1}/h_k\} - \{(d_{k,1} - d_{k-1,1})/h_{k-1}\}],$$

$$k = O(1)\ n-1, \qquad d_{0,3} = d_{n,3} = 0. \qquad \dots (8d)$$

For other specific conditions see Gerald and Wheatley[5] (pp. 233-240). For calculations of cubic smoothing splines for equally spaced data, see Cuplin[2].

We derive our method by approximating the integral in (1) in the following discrete form

$$aw(x)\, g(x) + \sum_{k=0}^{n-1} \left\{ (b/\pi) ⨍_{t_k}^{t_{k+1}} w(t)\,(g(t)/(t-x))\, dt + \int_{t_k}^{t_{k+1}} w(t)\, k(t,x)\, g(t)\, dt \right\}$$

$$= f(x), -1 < x < 1. \qquad \dots (9)$$

Approximating function $g(t)$ in each subinterval $[t_k, t_{k+1}]$, $k = O(1)\ n-1$, $g_k(t)$, a natural cubic spline interpolation polynomial, we obtain from (9)

$$aw(x)\, g(x) + \sum_{k=0}^{n-1}$$

$$\left\{ (b/\pi) \fint_{t_k}^{t_{k+1}} w(t)\,(g_k(t)/(t-x))\,dt + \int_{t_k}^{t_{k+1}} w(t)\,k(t,x)\,g_k(t)\,dt \right\}$$

$$= f(x), -1 < x < 1. \qquad \dots (10)$$

Using (7) and choosing the collocation points s_j, so that $t_j < s_j < t_{j+1}, j = O(1)\ n-1$, the equation (10) is reduced to the following discrete analogue

$$\sum_{k=0}^{j-1} \left(\sum_{r=1}^{4} d_{k,r} \left\{ A_{k,r}(s_j) + B_{k,r}(s_j) \right\} \right)$$

$$\sum_{r=1}^{4} d_{j,r} \left\{ aw(s_j)(x-t_j)^{r-1} + A_{j,r}(s_j) + B_{j,r}(s_j) \right\}$$

$$\sum_{k=j+1}^{n-1} \left(\sum_{r=1}^{4} d_{k,r} \left\{ A_{k,r}(s_j) + B_{k,r}(s_j) \right\} \right) = f(s_j), \qquad j = O(1)\ n-1, \qquad \dots (11)$$

where

$$A_{k,r}(s_j) = (b/\pi) \int_{t_k}^{t_{k+1}} w(t)\,((t-t_k)^{r-1}/(t-s_j))\,O\,dt, \qquad k = O(1)\ n-1, \quad r = 1(1)4,$$

$$B_{k,r}(s_j) = \int_{t_k}^{t_{k+1}} w(t)\,K(t,s_j)\,(t-t_k)^{r-1}\,dt, \qquad k = O(1)\ n-1, \quad r = 1(1)4.$$

It may be noted here that only $A_{j,r}(s_j)$ is a Cauchy principal value integral and others are non-singular integrals. Evaluaton of $A_{k,r}(s_j)$ for $r = 2(1)4$ may be done as:

$$A_{k,2}(s_j) = (b/\pi) \int_{t_k}^{t_{k+1}} w(t)\,dt + (s_j - t_k)\,A_{k,1}(s_j),$$

$$A_{k,3}(s_j) = (b/\pi) \int^{t_{k+1}} w(t)\,(t-t_k)\,dt + (s_j - t_k)\,A_{k,2}(s_j),$$

$$A_{k,4}(s_j) = (b/\pi) \int_{t_k}^{t_{k+1}} w(t)\,(t-t_k)^2\,dt + (s_j - t_k)\,A_{k,3}(s_j).$$

Proceeding similar to the above, we can approximate[6] as

$$\sum_{k=0}^{n-1} \sum_{r=1}^{4} d_{k,r} \int_{t_k}^{t_{k+1}} w(t)\,(t-t_k)^{r-1}\,dt = C. \qquad \text{... (12)}$$

Now, all the quantities in (11) can be evaluated to yield n linear equations from each (11), (8b), (8c), $(n + 1)$ linear equations from (8d) and one linear equaton from (12) forming a total of $(4n + 2)$ linear equatons in $(4n + 2)$ unknowns, taking into account $d_{n,1}$ and $d_{n,3}$ arising in eqs. (8b), (8c) and (8d). Thus, $(4n + 2)$ equations in $(4n + 2)$ unknowns can be solved using partition method or any other suitable methods for obtaining the approximate solution of the unknown function.

3. Numerical Examples

To illustrate the above method computationally, we consider the following two examples of singular integral equations.

Example 1 — Consider the singular integral equation

$$(1-x)^{-3/4}(1-x)^{-1/4}g(x) - (1/\pi) \fint_{-1}^{1} (1-t)^{-3/4}(1-t)^{-1/4}(g(t)/(t-x))\,dt$$

$$\int_{-1}^{1} (1-t)^{-3/4}(1-t)^{-1/4}(g(t)/(t+x+6))\,dt = f(x) \qquad -1 < x < 1, \qquad \text{... (13)}$$

where

$$f(x) = (\sqrt{2}/5)\,(3/5)^{-1/4}\ \{(x+4)^{-1} + \pi(x+2)^{-1}\}$$

$$-\pi\sqrt{2}\,(x+2)^{-1}\,((x+5)/(x+7))^{-1/4}$$

and the exact solution is given by

$$g(x) = (x+4)^{-1}.$$

We solved the singular integral eq. (3.1) by the present method for $n = 2(2)4$ for equally spaced mesh and choosing the collocation points $s_k = (t_k + t_{k+1})/2$, $k = 0(1)n$, under the condition

$$\int_{-1}^{1} (1-t)^{-3/4}(1+t)^{-1/4}g(t)\,dt = (\pi\sqrt{2}/5)\,(3/5)^{-1/4}.$$

We have solved linear algebraic system of equations using partition method for matrix inversion. The corresponding percentage errors $e_k = (\,|\text{ exact value }(x_k) - \text{approximate value }(x_k)\,| \times 100) / |\text{ Exact value }(x_k)\,|$, for $k = O(1)\ n$ are listed in Table I.

Example 2 — Consider the singular integral equation

$$(1-x)^{-1/4}(1+x)^{-3/4}(g(x)+(1/\pi)\,\frac{}{}\!\!\!\!\!\!\int_{-1}^{1}(1-t)^{-1/4}(1-t)^{-3/4}(g(t)/(t-x)\,dt$$

$$\int_{-1}^{1}(1-t)^{-1/4}(1-t)^{-3/4}(g(t)/(t+x+8))\,dt=f(x) \qquad -1<x<1, \qquad \ldots (14)$$

where

$$f(x)=(\sqrt{2}/11)\,(9/11)^{-3/4}\,\{\pi(x-2)^{-1}-(x+10)^{-1}\}$$
$$-\pi\sqrt{2}\,(x-2)^{-1}\,(x+9)^{-1}\,((x+7)/(x+9))^{-3/4}$$

and the exact solution is given by

$$g(x) = (x+10)^{-1}.$$

We solved the singular integral eq. (14) by the present method for $n = 2(2)4$ for equally spaced mesh and choosing the collocation points $s_k=(t_k+t_{k+1})/2$, $k = 0(1)n$; under the condition

$$\int_{-1}^{1}(1-t)^{-1/4}(1+t)^{-3/4}\,g(t)\,dt=(\pi\sqrt{2}/11)\,(9/11)^{-3/4}.$$

We have solved linear algebraic system of equations using partition method for matrix inversion. The corresponding percentage errors e_k = (| exact value (x_k) − approximate value (x_k) | × 100)/ | Exact value (x_k) |, for $k = 0(1)n$ are listed in Table I.

For exact evaluation of integrals (13) and (14), see Gradshteyn and Ryzhik[6] (pp. 338-339).

TABLE I

x_k	eq. (13) % error e_k $n = 2$	eq. (14) % error e_k $n = 2$	eq. (13) % error e_k $n = 4$	eq. (14) % error e_k $n = 4$
−1.00	0.187	0.266	0.003	0.003
−0.50			0.008	0.004
0.00	0.317	0.401	0.017	0.006
0.50			0.023	0.009
1.00	0.545	0.659	0.041	0.011

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