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OSCILLATORY SOLUTIONS OF LINEAR ITERATIVE FUNCTIONAL EQUATIONS

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This paper contains sufficient conditions for the oscillation of all solutions of linear iterative functional equation.

Key Words: Iterative Equations; Oscillatory Solutions

In this paper we continue our investigation of oscillation criteria for solutions of iterative functional equations of higher order[4, 5].

Let $\mathfrak{R}$ be the set of real numbers and let I denotes an unbounded subset of $\mathfrak{R}_+ = [0, \infty)$. By g^m we mean the mth iterate of the function g, i.e.

$$g^0(t) = t, \quad g^{m+1}(t) = g(g^m(t)), \qquad t \in I, \qquad m = 0, 1, \ldots .$$

By g^{-1} we mean the inverse function to g and $g^{-m-1}(t) = g^{-1}(g^{-m}(t))$. In the whole of this paper upper indices at the sign of a function will denote iterations. In each instance we have the relation $g^1(t) = g(t)$.

We consider the oscillatory solutions of linear function equations of the form

$$Q_0(t)\, x(t) + Q_1(t)\, x(g(t)) + Q_2(t)\, x(g^2(t))$$

$$+ \ldots + Q_{m+1}(t)\, x(g^{m+1}(t)) = 0, \qquad m \geq 1, \qquad \ldots (E)$$

where $Q_k : I \to \mathfrak{R}$ for $k = 0, 1, \ldots, m + 1$ and $g : I \to I$ are given functions, x is an unknown real valued fucntion. We also assume that

$$g(t) \neq t \quad \text{and} \quad \lim_{t \to \infty} g(t) = \infty, \quad t \in I. \qquad \ldots (1)$$

Moreover we assume that g has an inverse function.

By a solution of eq. (E) we mean a function $x : I \to \mathfrak{R}$ such that $sup\{|x(s)| : s \in I_{t_0} = [t_0, \infty) \cap I\} > 0$ for any $t_0 \in \mathfrak{R}_+$ and x satisfies (E) in I.

A solution x of eq. (E) is called oscillatory if there exists a sequence of points $\{t_n\}_{n=1}^{\infty}$, $t_n \in I$, such that $\lim_{n \to \infty} t_n = \infty$ and $x(t_n)\, x(t_{n+1}) \le 0$ for $n = 1, 2, \ldots$. Otherwise it is called nonoscillatory.

Problem of oscillation of solutions of differential equations has been investigated by many authors. Since, in the literature there are many oscillation criteria for these equations[3, 7]. For the functional equations the situation is different however some papers concerned oscillation of functional equations appear[2,9,10,11,12,14]. We are of the opinion that it is worth considering iterative functional equations because, in particular, they are recurrence equations which have a lot of applications. The recurrence equations can be used to describe processes in many areas such biology, meteorology, economics and so on[8].

Our aim is to give some new oscillation criteria for eq. (E). The analogous problem has been considered in[9,10,11].

One can observe that existence of oscillatory solutions of Eq. (E) is connected with the sign of the functions $Q_i (i = 0, 1, \ldots, m + 1)$ on I. For example, it is easy to prove that $Q_i(t) > 0$ or $Q_i(t) < 0$ for $i = 0, 1, \ldots, m + 1$, $t \in I$, implies that eq. (E) possesses only oscillatory solutions. If one of the coefficients Q_i, $(i = 1, \ldots, m)$ has an opposite sign to that of others, i.e. if there exists $s \in \{1, \ldots, m\}$ such that $Q_s(t) < 0$ and $Q_i(t) > 0$, $i \in \{0, 1, \ldots, m + 1\} - \{s\}$ then eq. (E) can possess both oscillatory and nonoscillatory solutions. For example, the functional equation

$$3x(t) + 2x(t+\pi) - 8x(t+2\pi) + x(t+3\pi) + 2x(t+4\pi) = 0, \qquad t \in [0,\infty)$$

has oscillatory solutions $x = \cos 2t$, $x = \sin 2t$ and a nonoscillatory solution $x = 1$. So, a question arises: if the last case holds, under what additional conditions on the coefficients Q_i every solution of (E) will be oscillatory. We present some answers to this question in case if for some $s \in \{1, 2, \ldots, m\}$

$$Q_s(t) < 0,\ Q_i(t) \ge 0 \ \ (i = 0, 1, \ldots, s-1, s+1, \ldots, m+1)$$

with

$$Q_0(t),\ Q_{s+1}(t) > 0 \ \text{ for } t \in I.$$

Without loss of generality we may assume that $Q_s(t) = -1$, $t \in I$. Then equation (E) takes the form

$$x(g^s(t)) = \sum_{k=0}^{s-1} Q_k(t)\, x(g^k(t)) + \sum_{k=s+1}^{m+1} Q_k(t)\, x(g^k(t)), \quad m \ge 1, \qquad \ldots (E_s)$$

where $s \in \{1, \ldots, m\}$ $Q_i(t) \ge 0$ $(i = 0, 1, \ldots, s-1, s+1, \ldots, m+1)$ and $Q_0(t)$, $Q_{s+1}(t) > 0$ for $t \in I$.

As usual we take

$$\sum_{j=k}^{r} a_j = 0 \quad \text{and} \quad \prod_{j=k}^{r} a_j = 1, \text{ where } r < k.$$

Further we will use the following lemma which is a slight modification of Lemmas 1 and 2 given in[9].

Lemma — Consider the functional inequalities

$$x(g^m(t)) \geq P(t)\, x(g^{m+1}(t)) + Q(t)\, x(t) \qquad \text{... (2)}$$

and

$$x(g^m(t)) \leq P(t)\, x(g^{m+1}(t)) + Q(t)\, x(t), \qquad \text{... (3)}$$

where $m \geq 1$, P, $Q : I \to \Re_+$, $g : I \to I$ are given functions and g satisfies condition (1). If

$$\liminf_{I \ni t \to \infty} \sum_{i=0}^{m-1} Q(g^{m-i}(t)) \prod_{j=1}^{m} P(g^{m-i-j}(t)) > \left(\frac{m}{m+1}\right)^{m+1},$$

then the functional inequality (2) [3] has not positive [negative] solutions for large $t \in I$.

Now we study oscillation criteria for equation (E_s).

***Theorem* 1** — *Let*

$$\liminf_{I \ni t \to \infty} \sum_{i=0}^{s-1} A(g^{s-i}(t)) \prod_{j=1}^{s} B(g^{s-i-j}(t)) > \left(\frac{s}{s+1}\right)^{s+1}, \qquad \text{... (4)}$$

where

$$A(t) = Q_0(t) + \sum_{k=1}^{s-1} Q_k(t)\, Q_{s-k}(g^{k-s}(t)) \qquad \text{... (5)}$$

and

$$B(t) = Q_{s+1}(t) + \sum_{k=s+2}^{m+1} Q_k(t) \prod_{j=2}^{k-s} Q_{s-1}(g^j(t)). \qquad \text{... (6)}$$

Then every solution of eq. (E_s) *is oscillatory.*

PROOF : Suppose that (E_s) has a nonoscillatory solution x and let $x(t) > 0$ for $t \in I_{t_1}$, $t_1 \geq 0$. Then also, in view of assumption (1) about function g, $x(g^i(t)) > 0$, $t \in \{1, 2, \ldots, m + 1\}$ and $t \in I_{t_2}$, $t_2 \geq t_1$. Thus from eq. (E_s) we have

$$x(g^s(t)) \geq Q_i(t)\, x(g^i(t)), \qquad \text{for} \qquad i = 0, 1, ..., s-1.$$

Hence we get

$$x(g^k(t)) \geq Q_{s-k}(g^{k-s}(t))\, x(t), \qquad (k = 1, 2, ..., s-1) \qquad ... (7)$$

and

$$x(g^k(t)) \geq x(g^{s+1}(t)) \prod_{j=2}^{k-s} Q_{s-1}(g^j(t)), \qquad (k = s+2, ..., m+1). \qquad ... (8)$$

Using now (7) and (8) in eq. (E_s) we obtain

$$x(g^s(t)) \geq A(t)\, x(t) + B(t)\, x(g^{s+1}(t)), \qquad ... (9)$$

where A and B are given by (5) and (6). Then by Lemma and condition (4) we obtain a contradiction with the fact that x is a positive solution of inequality (9).

We give now another oscillation criterion for eq. (E_s).

***Theorem* 2** — *Assume that*

$$\limsup_{I \supset t \to \infty} \sum_{i=0}^{s} A(g^{s-i}(t)) \prod_{j=1}^{s} B(g^{s-i-j}(t))$$

$$\times \left\{ 1 + \sum_{k=1}^{i} A(g^{-k}(t)) \prod_{l=1}^{s} B(g^{-k-l}(t)) \right\} > 1, \qquad ... (10)$$

where A and B are given by (5) and (6). Then eq. (E_s) has only oscillatory solutions.

PROOF : Let $x(t) > 0$ for $t \in I_{t_1}, t_1 \geq 0$ be a nonoscillatory solution of eq. (E_s). Then, as in the proof of Theorem 1, inequalities (7), (8) and (9) are satisfied. From inequality (9) we have for $i \in \{0, 1, ..., s\}$ and $t \in I_{t_2}, t_2 \geq t_1$

$$x(g^{s-i}(t)) \geq A(g^{-i}(t))\, x(g^{-i}(t)) + B(g^{-i}(t))\, x(g^{s-i+1}(t)). \qquad ... (11)$$

Multiplying both sides of above inequality by $\prod_{j=i+1}^{s} B(g^{-j}(t))$ and next summing from $i = 1$ to $i = s$ we obtain

$$x(t) \geq x(g^s(t)) \prod_{j=1}^{s} B(g^{-j}(t)) + \sum_{i=1}^{s} A(g^{-i}(t)) \prod_{j=i+1}^{s} B(g^{-j}(t))\, x(g^{-i}(t)). \qquad ... (12)$$

From inequality (9) we have for $t \in I_{t_3}, t_3 \geq t_2$

$$x(g^s(t)) \geq B(t)\, x(g^{s+1}(t)). \qquad \ldots (13)$$

Hence for $i \in \{0, 1, ..., s+1\}$ we get

$$x(g^{-i}(t)) \geq x(t) \prod_{j=s+1}^{s+i} B(g^{-j}(t)). \qquad \ldots (14)$$

From (11) we obtain for $i \in \{0, 1, ..., s\}$ and $l \in \{0, 1, ..., s\}$

$$x(g^{l-i}(t)) \geq A(g^{l-i-s}(t))\, x(g^{l-i-s}(t)) + B(g^{l-i-s}(t))\, x(g^{l-i+1}(t))$$

and in view of (14)

$$x(g^{l-i}(t)) \geq B(g^{l-i-s}(t))\, x(g^{l-i+1}(t)) + X(t)\, A(g^{l-i-s}(t)) \prod_{k=s+1}^{2s+i-l} B(g^{-k}(t)). \qquad \ldots (15)$$

Using now (9) and (15) for $l = 0$ in (12 we have

$$x(t) \geq \prod_{j=1}^{s} B(g^{-j}(t))\, \{A(t)\, x(t) + B(t)\, x(g^{s+1}(t))\} + \sum_{i=1}^{s} A(g^{-i}(t)) \prod_{j=i+1}^{s} B(g^{-j}(t)) \left\{ B(g^{-i-s}(t))\, x(g^{-i+1}(t)) + A(g^{-i-s}(t)) \prod_{k=s+1}^{2s+i} B(g^{-k}(t))\, x(t) \right\}. \qquad \ldots (16)$$

Hence we get

$$x(t) \geq x(t) \left\{ \sum_{i=0}^{1} A(g^{-i}(t)) \prod_{j=1}^{s} B(g^{-i-j}(t)) + \left\{ \sum_{i=1}^{s} A(g^{-i}(t))\, A(g^{-i-s}(t)) \prod_{j=1}^{2s} B(g^{-i-j}(t)) \right\} + \sum_{i=2}^{s} A(g^{-i}(t))\, B(g^{-i-s}(t)) \prod_{j=i+1}^{s} B(g^{-j}(t))\, x(g^{-i+1}(t)). \right.$$

From above inequality and from (15) for $l = 1$ we obtain

$$x(t) \geq x(t) \left\{ \sum_{i=0}^{2} A(g^{-i}(t)) \prod_{j=1}^{s} B(g^{-i-j}(t)) \right.$$

$$+ \sum_{i=1}^{s} A(g^{-i}(t)) A(g^{-i-s}(t)) \prod_{j=1}^{2s} B(g^{-i-j}(t))$$

$$+ \left\{ \sum_{i=2}^{s} A(g^{-i}(t)) B(g^{-1-s-i}(t)) \prod_{j=1}^{2s-1} B(g^{-i-j}(t)) B(g^{-i-s}(t)) \right\}$$

$$+ \sum_{i=3}^{s} A(g^{-i}(t)) \prod_{j=i+1}^{s} B(g^{-j}(t)) \prod_{k=0}^{1} B(g^{k-i-s}(t)) x(g^{2-i}(t)).$$

Similarly we have

$$x(g^{s}(t)) \geq x(g^{s}(t)) \left\{ \sum_{i=0}^{s} A(g^{s-i}(t)) \prod_{j=1}^{s} B(g^{s-i-j}(t)) \right.$$

$$+ \sum_{i=1}^{s} A(g^{s-i}(t)) A(g^{-i}(t)) \prod_{j=1}^{2s} B(g^{s-i-j}(t)) + \ldots +$$

$$+ \sum_{i=s-1}^{s} A(g^{s-i}(t)) A(g^{s-i-2}(t)) \prod_{j=1}^{s+2} B(g^{s-i-j}(t)) \prod_{k=0}^{s-3} B(g^{k-i}(t))$$

$$\left. + \; A(t) A(g^{-1}(t)) \prod_{j=1}^{s+1} B(g^{-j}(t)) \prod_{k=0}^{s-2} B(g^{k-s}(t)) \right\}.$$

Dividing now both sides of above inequality by $x(g^{s}(t))$ we get a contradiction with (10). Thus the proof is complete.

We present now another sufficient condition for the oscillation of all solutions of equation (E_s). It can be applied when conditions (4) and (10) are not satisfied.

***Theorem* 3** — *Assume that*

$$\sum_{i=0}^{s-1} A(g^{s-i}(t)) \prod_{j=1}^{s} B(g^{s-i-j}(t)) \geq \delta, \quad \delta < \left(\frac{s}{s+1}\right)^{s+1} \qquad \ldots (17)$$

for $t \in I_{t_1}, t_1 \geq t_0$, *and*

$$\limsup_{I \ni t \to \infty} \sum_{i=0}^{s} A(g^{s-i}(t)) \prod_{j=1}^{s} B(g^{s-i-j}(t)) \times \left\{ 1 + \sum_{k=1}^{i} A(g^{-k}(t)) \prod_{l=1}^{s} B(g^{-k-l}(t)) \right\} > 1 - \delta^{s+1}, \quad \dots (18)$$

where A and B are given by (5) and (6). Then eq. (E_s) possesses only oscillatory solutions.

PROOF : Let $x(t) > 0$ for $t \in I_{t_1}, t_1 \geq 0$ be a nonoscillatory solution of eq. (E_s). Then for $t \in I_{t_2}, t_2 \geq t_1$ inequalities (7) and (8) holds. So, inequality (9) is also true. Thus for sufficient large t inequalities (11) and (16) are also satisfied. Further, going the same way as in the proof of Theorem 2, from (16) we obtain

$$x(g^s(t)) \geq x(g^{2s+1}(t)) \prod_{j=0}^{s} B(g^{s-j}(t))$$
$$+ \; x(g^s(t)) \left\{ \sum_{i=0}^{s} A(g^{s-i}(t)) \prod_{j=1}^{s} B(g^{s-i-j}(t)) \right.$$
$$+ \sum_{i=1}^{s} A(g^{s-i}(t)) A(g^{-i}(t)) \prod_{j=1}^{2s} B(g^{s-i-j}(t)) + \dots +$$
$$+ \sum_{i=s-1}^{s} A(g^{s-i}(t)) A(g^{s-i-2}(t)) \prod_{j=1}^{s+2} B(g^{s-i-j}(t)) \prod_{k=0}^{s-3} B(g^{k-i}(t))$$
$$\left. + \; A(t) A(g^{-1}(t)) \prod_{j=1}^{s+1} B(g^{-j}(t)) \prod_{k=0}^{s-2} B(g^{k-s}(t)) \right\}. \quad \dots (19)$$

Multiplying both sides of inequality (11) by $\prod_{j=i+1}^{s-1} B(g^{-j}(t))$ and next summing from $i = 0$ to $i = s - 1$ we get

$$x(g(t)) \geq x(g^{s+1}(t)) \prod_{j=0}^{s-1} B(g^{-j}(t))$$
$$+ \sum_{i=0}^{s-1} A(g^{-i}(t)) \prod_{j=i+1}^{s-1} B(g^{-j}(t))\, x(g^{-i}(t)).$$

Multiplying both sides of above inequality by $B(g^{-s}(t))$ we obtain

$$B(g^{-s}(t))\,x(g(t)) \geq \sum_{i=0}^{s-1} A(g^{-i}(t)) \prod_{j=i+1}^{s} B(g^{-j}(t))\,x(g^{-i}(t)). \qquad \ldots (20)$$

From inequality (11) we have

$$x(g^{s-i}(t)) \geq B(g^{-i}(t))\,x(g^{s-i+1}(t)), \quad i = 0, 1, \ldots, s.$$

Hence

$$x(g^{-k}(t)) \geq x(t) \prod_{j=1}^{k} B(g^{-s-j}(t)).$$

Using above inequality in (20) and next from assumption (17) we obtain

$$B(g^{-s}(t))\,x(g(t)) \geq \sum_{i=0}^{s-1} A(g^{-i}(t)) \prod_{j=i+1}^{s} B(g^{-i-j}(t))\,x(t) \geq \delta x(t).$$

Thus nonoscillatory, positive solution $x(t)$ of eq. (E_s) satisfies the following inequalities

$$B(g^{-s}(t))\,x(g(t)) \geq \delta x(t)$$

and

$$x(g^{2s+1}(t)) \prod_{j=0}^{s} B(g^{s-j}(t)) \geq \delta^{s+1} x(g^{s}(t)).$$

Using above inequality in (19) and next dividing by $x(g^{s}(t))$ we have

$$1 - \delta^{s+1} \geq \sum_{i=0}^{s} A(g^{s-i}(t)) \prod_{j=1}^{s} B(g^{s-i-j}(t))$$

$$\times \left\{ 1 + \sum_{k=1}^{i} A(g^{-k}(t)) \prod_{l=1}^{s} B(g^{-k-l}(t)) \right\}.$$

The last inequality contradicts assumption (18). Thus the proof is complete.

Remark : As it was mentioned functional equations are some kind of generalization of the recurrence equations i.e. if we take $I = N$ and $g(n) = n + 1$ then from functional equations we obtain recurrence equations. So, results of this paper can be applied to recurrence equations, too. Now we compare our results with those known in the literature for recurrence equations. In order to let us consider very simple recurrence equation

$$\Delta x(n) = p(n)\,x(n+2), \qquad \ldots (R)$$

where $\Delta x(n) = x(n+1) - x(n)$ and $p : N \rightarrow (0, \infty), n \in N$. For this equation conditions for oscillation given in this paper have the form

$$\liminf_{n \to \infty} \ p(n) > \frac{1}{4}. \qquad \text{... } (9')$$

$$\limsup_{n \to \infty} \left\{ p(n) + p(n-1) + p(n-1)\, p(n-2) \right\} > 1 \qquad \text{... } (10')$$

and

$$\text{if } p(n) \geq \delta, \qquad \delta < \frac{1}{4}, \qquad \text{for } \ n \geq n_1 \geq 1 \qquad \text{... } (17')$$

and

$$\limsup_{n \to \infty} \left\{ p(n) + p(n-1) + p(n-1)\, p(n-2) \right\} > 1 - \delta^2. \qquad \text{... } (18')$$

Similar results have been obtained by Györi and Ladas in[3], Erbe and Zhang in [1] and Stavroulakis in [13]. They have forms

$$[3] \ \liminf_{n \to \infty} \ p(n-1) > \frac{1}{4} \qquad \text{... } (C_1)$$

$$[3] \ \limsup_{n \to \infty} \left\{ p(n) + p(n-1) \right\} > 1, \qquad \text{... } (C_2)$$

$$[1] \ \liminf_{n \to \infty} \ p(n) \equiv c > 0 \qquad \text{... } (C_3)$$

and

$$\limsup_{n \to \infty} \ p(n) > 1 - c,$$

at last

[13] if there exists $M > 0$ such that

$$\liminf_{n \to \infty} \ p(n-1) > M$$

and

$$\limsup_{n \to \infty} \ p(n-1) > 1 - \left(\frac{M}{2} \right)^2. \qquad \text{... } (C_4)$$

It is obvious that condition (10′) is better than (C_2). Consider now particular case of eq. (R), i.e., equation

$$\Delta x(n) = \left[\frac{4(2 + (-1)^n)}{19} + \frac{1}{n} \right] x(n+2). \qquad \text{... } (R_1)$$

Let us observe that conditions (9′) and (C_1) have the same form and for eq. (R_1) are not satisfied. Similarly conditions (10′) and (C_2) are not fulfilled because

$$\limsup_{n\to\infty}\ \{p(n)+p(n-1)+p(n-1)\,p(n-2)\} = \frac{352}{361} < 1$$

and

$$\limsup_{n\to\infty}\ \{p(n)+p(n-1)\} = \frac{16}{19} < 1.$$

On the other hand assumptions (17′) and (18′) for this equation holds since

$$p(n) = \frac{4(2+(-1)^n}{19} + \frac{1}{n} \geq \frac{4}{19} \quad \text{for} \quad n \geq 1$$

and

$$\limsup_{n\to\infty}\ \{p(n)+p(n-1)+p(n-1)\,p(n-2)\} = \frac{352}{361} > 1$$

$$1-\delta^2 = 1-\left(\frac{4}{19}\right)^2 = \frac{345}{361}.$$

Thus all solutions of (R_1) are oscillatory.

Now let us check that conditions (C_3) and (C_4) are not fulfilled, namely for considered eq. (R_1) condition (C_3) has the form

$$\liminf_{n\to\infty}\ p(n) = \frac{4}{19} \equiv c$$

and

$$\limsup_{n\to\infty}\ p(n) = \frac{12}{19} < 1-c = \frac{15}{19}.$$

Similarly, for considered $p(n)$ condition (C_4) is not true because there is no $M > 0$ to both inequalities hold.

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