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# A NEW ITERATIVE APPROXIMATION OF FIXED POINTS FOR ASYMPTOTICALLY CONTRACTIVE TYPE MAPPINGS IN BANACH SPACES*

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In this paper, we introduce the concept of $p$-asymptotically contractive type mappings, and study the modified Mann and Ishikawa iteration processes with errors for approximating fixed points of the $p$-asymptotically contractive type mappings in uniformly convex Banach spaces. The results presented in this paper improve and extend the corresponding results of[1-5, 10, 15, 17, 23]

**Key Words : $p$-Asymptotically contractive Type Mappings; $p$-Uniformly Convex; Fixed Point; Mann Iteration Process with Errors; Ishikawa Iteration Process with Errors**

## 1. INTRODUCTION

Some convergence and stability results for certain classes of nonlinear single-valued and set-valued mappings have been studied by many authors[1-5, 7-16, 21] and the references therein. By using the modified Mann iteration method introduced by Schu[18], Liu[10] proved convergence theorem for the iterative approximation of fixed points of $k$-strictly asymptotically pseudocontractive mappings and asymptotically demicontractive mappings in Hilbert spaces. In 1994, Rhoades[17] presented some generalizations of Schu[18] on the convergence of Mann and Ishikawa iterations of asymptotically nonexpansive mappings in uniformly convex Banach spaces. Recently, Osilike[15], extended the corresponding results of Liu[10] from Hilbert spaces to much more general real $q$-uniformly smooth Banach spaces, $1 < q < \infty$, and to the much more general modified Ishikawa iteration method.

On the other hand, Ghosh and Debnath gave a necessary and sufficient condition for Ishikawa iterative sequence to converge to a fixed point of quasi-nonexpansive mappings and Liu[11, 12] extended the above result and obtained some necessary and sufficient conditions for Ishikawa iterative sequence or the Ishikawa iterative sequences with errors to converge to a fixed point of asymptotically quasi-nonexpansive mappings. In 1998, Xu[22] introduced the more geneal Ishikawa and Mann iteration procedures with errors, and proved some convergence theorems for nonlinear mappings in Banach spaces. Since then, the more general Ishikawa and Mann iteration procedures with errors have been studied by many authors[1, 2, 13, 23].

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Motivated and inspired by the above works, in this paper, we introduce the concept of $p$-asymptotically contractive type mappings and study of modified Mann and Ishikawa iteration processes with errors for approximating fixed points of the $p$-asymptotically contractive type mappings in Banach spaces much more general than Hilbert spaces. Our results improve and extend the corresponding result of[1-5, 10, 15, 17, 23]. In particular, our theorems hold in $L_p$, $l_p$, $W^{1,p}$ spaces, for $1 < p < \infty$.

## 2. PRELIMINARIES

Throughout this paper, let $X$ be a real Banach space with dual space, $\langle \cdot , \cdot \rangle$ denote the dual pair between $X$ and $X^*$, and $J : X \to 2^{X^*}$ be the normalized duality mapping defined by

$$J(x) = \left\{ f \in X^*, \langle x, f \rangle = \| x \| \| f \|, \| x \| = \| f \| \right\}, \qquad x \in X.$$

*Definition* 2.1 — Let $X$ be a real Banach space.

(1) $X$ is said to be uniformly convex if the modulus of convexity of $X$:

$$\delta_X \varepsilon = \inf \left\{ 1 - \frac{1}{2} \| x + y \| : \| x \| = \| y \| = 1, \| x - y \| \geq \varepsilon \right\} \geq 0 \quad \forall 0 < \varepsilon \leq 2.$$

(2) $X$ is said to be $p$-uniformly convex, $p > 1$, if there exists a constant $d > 0$ such that $\delta_X \varepsilon \geq d\varepsilon^p$.

*Remark* 2.1. : (1) If $X$ is a real uniformly convex Banach space, then $X$ is reflexive and strictly convex, and so the mapping $J$ is single-valued.

(2) It is known[20] that

$$L_p \text{ is } \begin{cases} 2 - \text{uniformly convex, if } 1 < p \leq 2 \\ p - \text{uniformly convex, if } p \geq 2. \end{cases}$$

*Definition* 2.2 — Let $D$ be a nonempty closed subset of $X, T : D \to D$ be a mapping and $k \in [0, 1]$, $p > 0$ be two constants.

(1) $T$ is said to be $p$-asymptotically contractive type mapping with respect to the constant $k$, if

$$\limsup_{n \to \infty} \sup_{x \in D} \left\{ \| T^n x - T^n y \|^p - \| x - y \|^p - k \| (x - T^n x) - y - T^n y) \|^p \right\} \leq 0, \quad \forall y \in D; \quad \text{... (2.1)}$$

(2) $T$ is said to be $p$-asymptotically hemi-contractive type mapping with respect to the constant $k$, if $F(T) = \{x \in D : Tx = x\} \neq \phi$ and for each $q \in F(T)$

$$\limsup_{n \to \infty} \sup_{x \in D} \left\{ \| T^n - q \|^p - \| x - q \|^p - k \| x - T^n x \|^p \right\} \leq 0; \quad \text{... (2.2)}$$

(3) $T$ is said to be uniformly $L$-Lipschitzian, where $L$ is a positive constant, if

$$\| T^n x - T^n y \| \leq L \| x - y \| \quad \forall x, y \in D \text{ and } n \in N = \{1, 2, ...\};$$

(4) $T$ is said to be semi-compact, if for any bounded sequence $\{x_n\} \subset D$, if $\| x_n - Tx_n \| \to 0$ $(n \to \infty)$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x^* \in D$ $(n_i \to \infty)$.

*Remark* 2.2 : It is easy to see that the mapping of $p$-asymptotically contractive type satisfying the condition (2.1) is a kind of more general and more important nonlinear mappings which contains the following nonlinear mappings as its special cases.

(1) If $T$ is a nonexpansive mapping, then it is obvious that $T$ is of 1-asmptotically contractive type mapping with respect to 0.

(2) If $\{a_n\}$ is a non-negative real sequence such that $a_n \to 1$ and $T$ is a mapping satisfying the following condition:

$$\| T^n x - T^n y \| \le a_n \| x - y \|, \quad \forall x, y \in D \quad \text{and} \quad n \in N,$$

then $T$ is called an asymptotically nonexpansive mapping, which ws first introduced and studied by Goebel and Kirk[4]. It is obvious that $T$ is 1-asymptotically contractive type mapping with respect to 0 and is also uniformly $L$-Lipschitzian[5].

(3) If $p = 1$, and $k = 0$, then the mapping $T$ satisfying condition (2.1) is called the asymptotically nonexpansive type mapping, which was introduced and studied in Goebel-Kirk[4], Kirk[7] and Xu[21].

(4) If $X$ is a real Hilbert space, $\{a_n\}$ is a non-negative real sequence such that $a_n \to 1$ and $T$ is a mapping satisfying the following condition:

$$\langle x - y, T^n x - T^n y \rangle \le a_n \| x - y \|^2, \quad \forall x, y \in D \quad \text{and} \quad n \in N, \qquad \dots (2.3)$$

then the mapping $T$ is said to be an asymptotically pseudo-contractive, which was first introduced and studied in Schu[18]. It is easy to prove that (2.3) is equivalent to the following inequality:

$$\| T^n x - T^n y \|^2 \le (2a_n - 1) \| x - y \|^2$$

$$+ \| (x - T^n x) - (y - T^n y) \|^2, \quad \forall x, y \in D \quad \text{and} \quad n \in N.$$

This implies that $T$ is a 2-asymptotically contractive type mapping with respect to 1.

(5) If $X$ is a real Hilbert space, $\{a_n\}$ is a non-negative real sequence such that $a_n \to 1$ and $T$ is a mapping with $F(T) \ne \phi$ and satisfying the following condition:

$$\| T^n x - q \|^2 \le a_n \| x - q \|^2 + \| x - T^n x \|^2, \qquad \forall x, \in D, \ q \in F(T),$$

then $T$ is called an asymptotically hemi-contractive mapping, which was first introduced and studied in Liu[10]. It is obvious that $T$ is 2-asymptotically contractive type mapping with respect to 1.

*Definition* 2.3 — Let $D$ be a nonempty bounded closed convex subset of $X$ and $T: D \to D$ be a mapping. Suppose that $\{\alpha_n\}$, $\{\gamma_n\}$, $\{\beta_n\}$, $\{\delta_n\}$ are real sequences in [0, 1] and $\{u_n\}$, $\{v_n\}$ are arbitrary sequences in $D$.

(1) The sequence $\{x_n\} \subset D$ defined by

$$\begin{cases} x_0 \in D \\ x_{n+1} = (1 - \alpha_n - \gamma_n) x_n + \alpha_n T^n y_n + \gamma_n u_n, & n \geq 0, \\ y_n = (1 - \beta_n - \delta_n) x_n + \beta_n T^n x_n + \delta_n v_n, & n \geq 0, \end{cases} \quad \text{... (2.4)}$$

is called the modified Ishikawa iterative with errors in the sense of Xu[22].

(II) The sequence $\{x_n\}$ define by

$$\begin{cases} x_0 \in D \\ x_{n+1} = (1 - \alpha_n - \gamma_n) x_n + \alpha_n T^n x_n + \gamma_n u_n, & n \geq 0, \end{cases} \quad \text{... (2.5)}$$

is called the modified Mann iterative sequence with errors in the sense of Xu[22].

*Remark* 2.3 : (1) If $\gamma_n = \delta_n = 0$ for all $n \geq 0$, then the sequence $x_n\} \subset D$ defined by (2.4) is generally referred to as the modified Ishikawa iterative in the light of[6].

(2) If $\gamma_n = 0$ for all $n \geq 0$, then the sequence $\{x_n\} \subset D$ defined by (2.5) is called the modified Mann iterative in the light of[14].

For the sake of convenience, we need the following lemmas.

*Lemma* 2.1 — Huang[5], Lemma 4. For the above $p > 1$ and for all $a \geq 0, b \geq 0$, there exists a non-negative real number $c$ between $a$ and $a + b$ such that

$$(a + b)^p = a^p + pc^{p-1} b.$$

*Lemma* 2.2 — Xu[20] Theorem 1. Let $X$ be a real Banach space. Then $X$ is $p$-uniformly convex, $p > 1$, if and only if there exists a constant $c > 0$ such that for any $\lambda \in [0, 1]$ and $x, y \in X$

$$\| \lambda x + (1 - \lambda) y \|^p \leq \lambda \| x \|^p + (1 - \lambda) \| y \|^p - c \cdot \omega_p (\lambda) \| x - y \|^p,$$

where

$$\omega_p (\lambda) = \lambda^p (1 - \lambda) + \lambda (1 - \lambda)^p.$$

*Lemma* 2.3[19, 10] — Lemma 1. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two non-negative real sequences and for all $n \geq N_0$ (for some fixed $N_0$), $a_{n+1} \leq a_n + b_n$.

(1) If $\sum_{n=1}^{\infty} b_n < \infty$ then $\lim_{n \to \infty} a_n$ exists;

(2) If $\sum_{n=1}^{\infty} b_n < \infty$, and $\left\{a_n\right\}_{n=1}^{\infty}$ has a subsequence $\left\{a_{n_k}\right\}_{k=1}^{\infty}$ converging to 0, then $\lim_{n \to \infty} a_n = 0$.

*Lemma* 2.4 — Let $X$ be a real normed space, $D$ a nonempty bounded convex subset of $X$, and $T : D \to D$ be a uniformly $L$-Lipschitzian mapping with $L \geq 0$. Let $\{x_n\}$ be the Ishikawa type iterative sequence defined by (2.4), and $\lim_{n \to \infty} \gamma_u = \lim_{n \to \infty} \delta_n = 0$. If $\| x_n - T^n x_n \| \to 0$ as $n \to \infty$, then $\| x_n - Tx_n \| \to 0$.

PROOF : Set $\theta_n = \| x_n - T^n x_n \|$. Then, for all $n \in N$, we have

$$\begin{aligned}
\| y_{n-1} - x_{n-1} \| &= \| \beta_{n-1} (T^{n-1} x_{n-1} - x_{n-1}) + \delta_{n-1} (v_{n-1} - x_{n-1}) \| \\
&\leq \beta_{n-1} \| T^{n-1} x_{n-1} - x_{n-1} \| + \delta_{n-1} \| v_{n-1} - x_{n-1} \| \\
&\leq \theta_{n-1} + \delta_{n-1} \| v_{n-1} - x_{n-1} \|,
\end{aligned}$$

$$\begin{aligned}
\| x_{n-1} - T^{n-1} y_{n-1} \| &\leq \| x_{n-1} - T^{n-1} x_{n-1} \| + \| T^{n-1} x_{n-1} - T^{n-1} y_{n-1} \| \\
&\leq \theta_{n-1} + L \| x_{n-1} - y_{n-1} \| \leq (1+L)\theta_{n-1} + L\delta_{n-1} \| v_{n-1} - x_{n-1} \|,
\end{aligned}$$

$$\begin{aligned}
\| x_n - x_{n-1} \| &= \| \alpha_{n-1} (T^{n-1} y_{n-1} - x_{n-1}) + \gamma_{n-1} (u_{n-1} - x_{n-1}) \| \\
&\leq \| T^{n-1} y_{n-1} - x_{n-1} \| + \gamma_{n-1} \| u_{n-1} - x_{n-1} \| \\
&\leq (1+L)\theta_{n-1} + \gamma_{n-1} \| u_{n-1} - x_{n-1} \| + L\delta_{n-1} \| v_{n-1} - x_{n-1} \|
\end{aligned}$$

and so

$$\begin{aligned}
\| x_{n-1} - T^{n-1} x_n \| &\leq \| x_{n-1} - T^{n-1} x_{n-1} \| + \| T^{n-1} x_{n-1} - T^{n-1} x_n \| \\
&\leq \theta_{n-1} + L \cdot \| x_{n-1} - x_n \|
\end{aligned}$$

$$\leq (1 + L + L^2)\, \theta_{n-1} + L\gamma_{n-1} \| v_{n-1} - x_{n-1} \| + L^2\, \delta_{n-1} \| v_{n-1} - x_{n-1} \|,$$

$$\| x_n - T^{n-1} x_n \| \leq \| x_n - x_{n-1} \| + \| x_{n-1} - T^{n-1} x_n \|$$

$$\leq (2 + 2L + L^2)\, \theta_{n-1} + (1+L)\, \gamma_{n-1} \| u_{n-1} - x_{n-1} \|$$

$$+ L\,(1+L)\, \delta_{n-1} \| v_{n-1} - x_{n-1} \|,$$

$$\| x_n - Tx_n \| \le \| x_n - T^n x_n \| + \| T^n x_n - Tx_n \| \le \theta_n + L \cdot \| x_n - T^{n-1} x_n \|$$

$$\le \theta_n + L(2 + 2L + L^2)\theta_{n-1}$$

$$+ L(1+L)\gamma_{n-1} \| v_{n-1} - x_{n-1} \| + L^2(1+L)\delta_{n-1} \| v_{n-1} - x_{n-1} \|. \quad \text{... (2.6)}$$

Since $\lim_{n\to\infty} \gamma_u = \lim_{n\to\infty} \delta_n = 0$, it follows from (2.6) that $\| x_n - Tx_n \| \to 0$. This completes the proof.

## 3. Main Results

***Theorem*** 3.1 — *Let X be a real p-uniformly convex Banach space, $1 < p < \infty$, D be a nonempty bounded closed convex subset of X, and $T : D \to D$ be a semi-compact uniformly L-Lipschitzian p-asymptotically hemi-contractive type mapping with respect to be constant $k \in [0, 1]$. Suppose that $\{x_n\}$ is the modified Mann iterative sequence with errors defined by (2.5). If*

$$(i)\ k < ca(1-a_2)^{-1}, \quad a = a_1 a_2 (a_1^{p-1} + a_2^{p-1});$$

$$(ii)\ a_1 \le \lambda_n = a_n + \gamma_n \le 1 - a_2;$$

$$(iii)\ \sum_{n=0}^{\infty} \gamma_n < \infty$$

*for all $n \ge 0$ and some $a_1, a_2 \in (0, 1)$, where c is the positive constant appeared in Lemma 2.2, then $\{x_n\}$ converges strongly to some fixed points $x^*$ of T in D.*

Proof : Since $X$ is $p$-uniformly convex, $T : D \to D$ is a $p$-asymptotically hemi-contractive type mapping, $F(T) \ne \phi$ and $T$ satisfies the condition (2.2). For any given $q \in F(T)$, it follows from Lemmas 2.1 and 2.2 that

$$\| x_{n+1} - q \|^p = \| (1-\lambda_n)(x_n - q) + \lambda_n (T^n x_n - q) - \gamma_n (T^n x_n - u_n) \|^p$$

$$\le \left\{ \| (1-\lambda_n)(x_n - q) + \lambda_n (T^n x_n - q) \| + \gamma_n (T^n x_n - u_n) \| \right\}^p$$

$$= \| (1-\lambda_n)(x_n - q) + \lambda_n (T^n x_n - q) \|^p + p e_n^{p-1} \gamma_n (T^n x_n - u_n) \|$$

$$\le (1-\lambda_n) \| x_n - q \|^p + \lambda_n \| T^n x_n - q \|^p$$

$$- c\omega_p(\lambda_n) \| x_n - T^n x_n \|^p + p e_n^{p-1} \gamma_n \| T^n x_n - u_n \|, \quad \text{... (3.1)}$$

where

$$\| (1-\lambda_n)(x_n-q)+\lambda_n (T^n x_n - q) \|$$

$$\leq e_n \leq \| (1-\lambda_n)(x_n-q)+\lambda_n (T^n x_n - q) \| + \gamma_n \| T^n x_n - u_n \|.$$

Setting $d = \sup_{x, y \in D} \| x-y \|$, we know that $d < \infty$. By the uniformly $L$-Lipschitzian property of $T$ and boundedness of $D$, for all $n \geq 0$, we get

$$\| T^n x_n - q \| = \| T^n x_n - T^n q \| \leq L \| x_n - q \| \leq Ld,$$

$$\| (1-\lambda_n)(x_n-q)+\lambda_n (T^n x_n - q) \|$$

$$\leq (1-\lambda_n) \| x_n - q \| + \lambda_n \| T^n x_n - q \| \leq (1+L)d,$$

$$\| T^n x_n - u_n \| \leq \| T^n x_n - q \| + \| u_n - q \| \leq (1+L)d.$$

It follows from condition (*iii*) that there exists a number $\overline{N} \in N$ such that $\gamma_n < t$ for all $n > \overline{N}$, and hence $pe_n^{p-1} < p[d(1+L)(1+t)]^{p-1} < \infty$ for all $n > \overline{N}$. Set

$$M = \max\{pd^p (1+L)^p (1+\gamma_{1)}^{p-1}, \ldots,$$

$$pd^p (1+L)^p (1+\gamma_{\overline{N}-1})^{p-1}, pd^p (1+L)^p (1+t)^{p-1}\}.$$

Then

$$pe_n^{p-1} \| T^n x_n - u_n \| \leq M < \infty, \quad \forall n \geq 0. \qquad \ldots (3.2)$$

It follows from (2.2) that there exists a positive integer $n_0$ such that

$$\| T^n x_n - q \|^p - \| x_n - q \|^p - k \| x_n - T^n x_n \|^p \leq 0, \quad \forall n \geq n_0. \qquad \ldots (3.3)$$

By condition (*ii*), we get

$$\omega^p (\lambda_n) \geq a > 0. \qquad \ldots (3.4)$$

Substituting (3.2)-(3.4) into (3.1) and simplifying, we have

$$\| x_{x+1} - q \|^p \leq \| x_n - q \|^p + \lambda_n$$

$$\{\| T^n x_n - q \|^p - \| x_n - q \|^p - k \| T^n x_n - x_n \|^P\}$$

$$+ \{k\lambda_n - c\omega_p(\alpha_n)\} \| T^n x_n - x_n \|^p + M\gamma_n$$

$$\leq \| x_n - q \|^p + \{k(1-a_2) - ca\} \| T^n x_n - x_n \|^p + M\gamma_n, \qquad \ldots (3.5)$$

i.e.,

$$0 < \{ca - k(1 - a_2)\} \ \| T^n x_n - x_n \|^p$$

$$\leq \| x_n - q \|^p - \| x_{x+1} - q \|^p + M\gamma_n, \quad \forall n \geq n_0. \qquad \text{... (3.6)}$$

Furthermore, (3.5) implies that

$$\| x_{x+1} - q \|^p \leq \| x_n - q \|^p + M\gamma_n. \qquad \text{... (3.7)}$$

Since

$$\sum_{n=0}^{\infty} M\gamma_n < \infty, \quad \| x_n - q \|^p \geq 0, \quad \forall n \geq 0,$$

it follows from Lemma 2.3, part (*i*), that $\lim_{n \to \infty} \| x_n - q \|^p$ exists. Let $\lim_{n \to \infty} \| x_n - q \|^p = s$. We now have, by (3.6),

$$\lim_{n \to \infty} \| T^n x_n - x_n \| = 0.$$

Letting $\beta_n = \delta_n = 0$ for all $n \geq 0$ in Lemma 2.4, we conclude

$$\lim_{n \to \infty} \| {}^n x_n - T x_n \| = 0. \qquad \text{... (3.8)}$$

Since $D$ is bounded, it follows from (3.8) and the semi-compactness of $T$ that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x^* \in D(k \to \infty)$. By the virtue of the continuity of $T$ and (3.8), we have

$$\lim_{k \to \infty} \| x_{n_k} - T x_{n_k} \| = \| x^* - T x^* \| = 0.$$

This implies that $x^*$ is a fixed point of $T$ in $D$. Therefore, $\{x_n\}_{n=1}^{\infty}$ has a subsequence which converges to the fixed point $x^*$ of $T$. Replacing the $q$ in (3.7) by $x^*$, from Lemma 2.3, we have

$$\lim_{n \to \infty} \| x_n - x^* \|^p = 0,$$

and so $\lim_{n \to \infty} x_n - x^*$. This completes the proof of Theorem 3.1.

***Theorem*** 3.2 — *Let $X$ be a real p-uniformly convex Banach space, $1 < p < \infty$, $D$ be a nonempty bounded closed convex subset of $X$, and $T : D \to D$ be a semi-compact uniformly L-Lipschitzian p-asymptotically hemi-contractive type mapping with respect to a constant $k \in [0, 1]$. Suppose that $\{x_n\}$ is the modified Ishikawa iterative sequence with errors defined by (2.4). If*

(i) $[L(1-a_2)]^p < k < ca(1-a_2)^{-1}, \quad a = a_1 a_2 (a_1^{p-1} + a_2^{p-1});$

(ii) $a_1 \le \lambda_n, \mu_n \le 1 - a_2, \quad \lambda_n = \alpha_n + \gamma_n, \quad \mu_n = \beta_n + \delta_n;$

(iii) $\sum_{n=0}^{\infty} \gamma_n < \infty, \quad \sum_{n=0}^{\infty} \delta_n < \infty$

*for all* $n \ge 0$ *and some* $a_1, a_2 \in (0, 1)$, *where* $c$ *is the positive constant appeared in Lemma 2.2, then* $\{x_n\}$ *converges strongly to some fixed points* $x^*$ *of* $T$ *in* $D$.

PROOF : Since $X$ is $p$-uniformly convex, $T: D \to D$ is a $p$-asymptotically hemi-contractive type mapping, $F(T) \ne \phi$ and $T$ satisfies the condition (2.2), for any given $q \in F(T)$, from Lemmas 2.1 and 2.2 and the boundedness of $D$, we have, for some constant $M_1 > 0$,

$$\begin{aligned}
\| x_{n+1} - q \|^p &= \| (1-\lambda_n)(x_n - q) + \lambda_n (T^n y_n - q) - \gamma_n (T^n y_n - u_n) \|^p \\
&\le \| (1-\lambda_n)(x_n - q) + \lambda_n (T^n y_n - q) \|^p + p \cdot \gamma_n (T^n y_n - u_n) \| \cdot \| x_{n+1} - q \|^{p-1} \\
&\le (1-\lambda_n) \| x_n - q \|^p + \lambda_n \| (T^n y_n - q) \|^p - c\omega_p(\lambda_n) \| x_n - T^n y_n \|^p + M_1 \gamma_n \\
&\le \| x_n - q \|^p + \lambda_n \{ \| T^n y_n - q \|^p - \| y_n - q \|^p - k \| y_n - T^n y_n \|^p \} \\
&+ \lambda_n \{ \| y_n - q \|^p + k \| y_n - T^n y_n \|^p - \| x_n - q \|^p \} \\
&- c\omega_p(\lambda_n) \| x_n - T^n y_n \|^p + M_1 \gamma_n. \qquad \text{... (3.9)}
\end{aligned}$$

Moreover, for some constants $M_2 > 0$ and $M_3 > 0$, we have

$$\begin{aligned}
\| y_n - q \|^p &= \| (1-\mu_n)(x_n - q) + \mu_n (T^n x_n - q) - \delta_n (T^n x_n - v_n) \|^p \\
&\le (1-\mu_n) \| x_n - q \|^p + \mu_n \| T^n x_n - q \|^p - c\omega_p(\mu_n) \| x_n - T^n x_n \|^p + M_2 \delta_n. \qquad \text{... (3.10)}
\end{aligned}$$

$$\begin{aligned}
\| y_n - T^n y_n \|^p &= \| (1-\mu_n)(x_n - T^n y_n) + \mu_n (T^n x_n - T^n y_n) - \delta_n (T^n x_n - v_n) \|^p \\
&\le (1-\mu_n) \| x_n - T^n y_n \|^p + \mu_n \| T^n x_n - T^n y_n \|^p \\
&- c\omega(\mu_n) \| x_n - T^n x_n \|^p + M_3 \delta_n \\
&\le (1-\mu_n) \| x_n - T^n y_n \|^p + L^p \mu_n \| x_n - y_n \|^p \\
&- c\omega(\mu_n) \| x_n - T^n x_n \|^p + M_3 \delta_n. \qquad \text{... (3.11)}
\end{aligned}$$

It follows from (2.4) and Lemma 2.1 that

$$\| x_n - y_n \|^p = \| \mu_2(x_n - T^n x_n) + \delta_n (T^n x_n - v_n) \|$$

$$\leq \{\mu_n \| x_n - T^n x_n \| + \delta_n \| T^n x_n - v_n \|\}^p$$

$$= \mu_n^p \|x_n - T^n x_n \|^p + ph_n^{p-1} \| T^n x_n - v_n \|, \quad \dots (3.12)$$

where

$$0 \leq \mu_n \| (x_n - T^n x_n) \| \leq h_n \leq \mu_n \| x_n - T^n x_n \| + \delta_n \| T^n x_n - v_n \|.$$

By using the same method as in the proof of Theorem 3.1, there exists a constant $M_4 > 0$ such that

$$ph_n^{p-1} \| T^n x_n - v_n \| \leq M_4 < \infty, \quad \forall n \geq 0. \quad \dots (3.13)$$

Substituting (3.10)-(3.13) into (3.9) and simplifying, we conclude

$$\| x_{n+1} - q \|^p \leq \| x_n - q \|^p + \lambda_n \{\| T^n y_n - q \|^p - \| y_n - q \|^p - k \| y_n - T^n y_n \|^p\}$$

$$+ \lambda_n \mu_n \{\| T^n x_{n-q} \| \| x_{n-q} \|^p - k \| x_n - T^n x_n \|^p\}$$

$$- \lambda_n \{c\omega_p (\mu_n) - k\mu_n - k [L^p \mu_n^{p+1} - c\omega_p ((\mu_n)]\} \| x_n - T^n x_n \|^p$$

$$+ \{k\lambda_n (1 - \mu_n) - c\omega_p (\lambda_n)\} \| x_n - T^n y_n \|^p$$

$$+ M_1 \gamma_n + \lambda_n (M_2 + kM_3 + kM_4 L^p \mu_n) \delta_n. \quad \dots (3.14)$$

Since $x_n, y_n \in D$, it follows from (2.2) that there exists a positive integer $n_0$ such that

$$\| T^n y_n - q \|^p - \| y_n - q \|^p - k \| y_n - T^n y_n \|^p \leq 0, \quad \forall n \geq n_0, \quad \dots (3.15)$$

$$\| T^n x_n - q \|^p - \| x_n - q \|^p - k \| x_n - T^n x_n \|^p \leq 0, \quad \forall n \geq n_0, \quad \dots (3.16)$$

By condition (*ii*), we have

$$\omega_p (\lambda_n) \geq a > 0, \quad \omega_p (\delta_n) \geq a > 0,$$

and so

$$c\omega_p (\mu_n) - k\mu_n - k[L^p \mu_n^{p+1} - c\omega_p (\mu_n)]$$

$$= c (1 + k) \omega_p (\mu_n) - k\mu_n (1 + L^p \mu_n^p)$$

$$\geq (k + 1) ca - k(1 - a_2) [1 + (1 - a_2)^p L^p ]$$

$$\geq (k+1)\, ca - ca\,[1 + (1-a_2)^p L^p\,]$$

$$= ca\,[k - (1-a_2)^p L^p > 0, \qquad \text{... (3.17)}$$

$$k\lambda_n (1-\mu_n) - c\omega_p(\lambda_n) \leq k\lambda_n - c\omega_p(\lambda_n) \leq k(1-a_2) - ca < 0. \qquad \text{... (3.18)}$$

Substituting (3.15)-(3.18) into (3.14) and simplifying, we get

$$\leq \| x_{n+1} - q \|^p + \leq \| x_n - q \|^b - ca(1-a_2)\,[k - (1-a_2)^p L^p]\, \| x_n - T^n x_n \|^p$$

$$+ M_1 \gamma_n + (M_2 + kM_3 + kM_4 L^p)\, \delta_n, \quad \forall n \geq n_0,$$

which implies that

$$ca(1-a_2)\,[k - (1-a_2)^p L^p]\, \| x_n - T^n x_n \|^p$$

$$\leq \| x_n - q \,|^p - \| x_{n+1} - q \|^p + M_1 \gamma_n + (M_2 + kM_3 + kM_4 L^p)\, \delta_n, \quad \forall n \geq n_0,$$

Let $m \geq n_0$ be any positive integer. It follows that

$$ca(1-a_2)\,[k - (1-a_2)^p L^p] \sum_{n=n_0}^{m} \| x_n - T^n x_n \|^p$$

$$\leq \| x_{n_0} - q \,|^p - \| x_{m+1} - q \|^p$$

$$+ M_1 \sum_{n=n_0}^{m} \gamma_n + (M_2 + kM_3 + k L^p M_4) \sum_{n=n_0}^{m} \delta_n$$

$$\leq \| x_{n_0} - q \|^p + M_1 \sum_{n=n_0}^{m} \gamma_n + (M_2 + kM_3 + k L^p M_4) \sum_{n=n_0}^{m} \delta_n. \qquad \text{... (3.19)}$$

Letting $m \to \infty$ in (3.19), we now have

$$ca(1-a_2)\,[k - (1-a_2)^p L^p] \sum_{n=n_0}^{\infty} \| x_n - T^n x_n \|^p$$

$$\leq \| x_{n_0} - q \|^p + M_1 \sum_{n=n_0}^{\infty} \gamma_n + (M_2 + kM_3 + k L^p M_4) \sum_{n=n_0}^{\infty} \delta_n < \infty,$$

and so

$$\sum_{n=1}^{\infty} \| x_n - T^n x_n \|^p < \infty.$$

This implies that

$$\lim_{n \to \infty} \| x_n - T^n x_n \| = 0.$$

It follows from Lemma 2.4 that $\lim_{n \to \infty} \| x_n - T x_n \| = 0$. The rest of proof follows the same arguments in the proof of Theorem 3.1. This completes the proof.

From Theorem 3.2 and Schauder fixed point theorem, we can obtain the following result:

***Theorem* 3.3 —** *Let $X$ be a real p-uniformly convex Banach space, $1 < p < \infty$, $D$ be a nonempty bounded closed convex subset of $X$, and $T : D \to D$ be a compact uniformly L-Lipschitzian p-asymptotically hemi-contractive type mapping with respect to a constant $k \in [0, 1]$. Suppose that $\{x_n\}$ is the modified Ishikawa iterative sequence with errors defined by (2.4) and conditions (i)-(iii) of Theorem 3.2 hold. Then $\{x_n\}$ converges strongly to some fixed points $x^*$ of $T$ in $D$.*

*Remark* 3.1 : Let $X$ be a real $p$-uniformly convex Banach space, $1 < p < \infty$, $D$ be a nonempty bounded closed convex subset of $X$, and $T : D \to D$ be a compact uniformly $L$-Lipschitzian $p$-asymptotically contractive type mapping with respect to a constant $k \in [0, 1]$. Suppose that $\{x_n\}$ is the modified Mann iterative sequence with errors defined by (2.5) and the conditions (*i*)-(*iii*) in Theorem 3.2 hold. Then $\{x_n\}$ converges strongly to some fixed points $x^*$ of $T$ in $D$.

From Theorem 3.3 and the proof method in Theorem 3.1, we have the following result:

***Theorem* 3.4 —** *Let $X$ be a real p-uniformly convex Banach space, $1 < p < \infty$, $D$ be a nonempty bounded closed convex subset of $X$, and $T : D \to D$ be a compact uniformly L-Lipschitzian p-asymptotically contractive type mapping with respect to a constant $k \in [0, 1]$. Let $\{x_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$ satisfying $a_1 \le \alpha_n, \beta_n \le 1 - a_2$ and condition (i) of Theorem 3.2 hold. Then the modified Ishikawa iterative sequence $\{x_n\}$ defined by*

$$\begin{cases} x_0 \in D \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n y_n, & n \ge 0, \\ y_n = (1 - \beta_n) x_n + \beta_n T^n x_n, & n \ge 0, \end{cases}$$

converges strongly to some fixed points $x^*$ of $T$ in $D$.

*Remark* 3.2 : If particularly $a_1 = a_2 = \varepsilon$ for some constant $\varepsilon > 0$, then Theorem 3.4 of this paper reduces to Theorem 2 of[17].

From Theorem 3.1 and Schauder fixed point theorem, we can obtain the following result:

***Theorem* 3.5 —** *Let $X$ be a real p-uniformly convex Banach space, $1 < p < \infty$, $D$ be a nonempty bounded closed convex subset of $X$, and $T : D \to D$ be a compact uniformly L-Lipschitzian* *p-asymptotically contractive type mapping with respect to a constant* $k \in [0, 1]$. *Let* $\{x_n\}$ *be two sequences in* $[0, 1]$ *satisfying* $a_1 \le \alpha_n \le 1 - a_2$ *and condition (i) of Theorem 3.1 hold. Then the modified Mann iterative sequence* $\{x_n\}$ *defined by*

$$\begin{cases} x_0 \in D \\ x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n T^n x_n, \quad n \ge 0, \end{cases}$$

*converges strongly to some fixed points* $x^*$ *of T in D.*

*Remark* 3.4 : (1) Theorems 3.1 and 3.5 improve and extend Theorem 1.5 of Schu[8], Theorems 1 and 2 of Liu[10].

(2) Theorem 2.3 in Schu[8] and Theorem 3 in Liu[10] are special cases of Theorems 3.2-3.4.

(3) Our Theorems are significant generalization from Hilbert spaces to the much general Banach spaces that include the $L_p$, $l_p$ and $W^{1 \cdot p}$ spaces for $1 < p < \infty$.

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