Indian J. pure appl. Math., 35(4) : 455-461, April 2004

NONOSCILLATION THEOREMS FOR A CLASS OF THIRD ORDER DIFFERENTIAL EQUATIONS

PAKIZE TEMTEK AND M. TAMER SENEL

Department of Mathematics, Erciyes University 38039, *Kayseri, Turkey*
e-mail: temtek@erciyes.edu.tr; senel@erciyes.edu.tr

(*Received* 12 *September* 2002; *after final revision* 18 *June* 2003; *accepted* 18 *November* 2003)

Sufficient conditions for the solutions of

$$\frac{d}{ds}\left[\sigma(s)\frac{d}{ds}(r(s)\frac{dx}{ds})\right]+q(s)\,k\left(r(s)\frac{dx}{ds}\right)+p(s)\,x^{\alpha}=f(s)$$

to be nonoscillatory are presented, where σ, r, p, q and f are real valued continuous functions on $[0, \infty)$ such that $\sigma(s)>0, r(s)>0$ and $f(s)\geq 0$ and $k(rx')$ as continuous in $(-\infty,\infty)$ such that $k(rx')\,x' > 0$ for $x' \neq 0$. $\alpha > 0$ is ratio of odd integers.

Key Words : Oscillatory Solution; Nonoscillatory Solution; Behaviour of Solution; Non-Linear Differential Equation

1. INTRODUCTION

Finding sufficient conditions for nonoscillatory of solutions is a problem of general interest in the theory of ordinary and delay differential equations.

In this work we consider

$$\frac{d}{ds}\left[\sigma(s)\frac{d}{ds}(r(s)\frac{dx}{ds})\right]+q(s)\,k\left(r(s)\frac{dx}{ds}\right)+p(s)\,x^{\alpha}=f(s) \qquad \dots (1)$$

where σ, r, p, q and f are real valued continuous functions on $[0, \infty)$ such that $\sigma(s)>0, r(s)>0$ and $f(s)\geq 0$ and $k(rx')$ is continuous in $(-\infty,\infty)$ such that $k(rx')\,x' > 0$ for $x' \neq 0$. $\alpha > 0$ is the ratio of odd integers.

Several special cases of (1) have been studied by many authors[5-15]. Indeed, in the linear case eq. (1) with $\alpha = 1$, $\sigma(s)=r(s)=1, k(x')=x'$ and $f(s) = 0$ has been considered by Birkhoff[1], Gregus[4] and Hanan[6]. In[14], Philos considered eq. (1) with $\sigma(s)=r(s)=1, k(x')=x'$ and $f(s) = 0$ and $\alpha = 1$. Finally Parhi[13] studied the case $\alpha = 1$, $k(rx')=x'$ of (1). Recently when $q(s) = 0$ and $\alpha = 1$, this equation has been considered by Cechi[2].

In the nonlinear case, we should mention the papers of Erbe[3], Heidel[7] who have considered eq. (1) with $k(x')=x'$, $r(s) = 1$, $f(s) = 0$. Parhi[8-12] studied the case $k(x')=x'^{\beta}$, $\beta > 0$ is a ratio of odd integers, $r(s) = 1$ of (1). Later, when $k(rx')=x'^{\beta}$, eq. (1) has been considered by[15].

It seems that no work has been done on nonlinear third order differential equation of the form (1). In this paper, we give some sufficient conditions concerning the nonoscillation of solutions of eq. (1). The results obtained generalize some criteria stated in[8-12, 15]. It may be noted that finding sufficient conditions for nonoscillation of solutions of (1) directly seems to be a difficult task. If $\int_0^\infty \frac{ds}{r(s)} = \infty$, then we employ the classical change of variable $t = \int_0^s \frac{du}{r(u)}$ to transform (1) to an equation for which sufficient conditions are known.

We restrict our considerations to those real solutions of (1) which exist on the half line $[T, \infty)$, where $T \geq 0$ depends on the particular solution and are nontrivial in any neighbourhood of infinity. We may recall that a solution $x(s)$ of (1), $s \in [T, \infty)$, is said to be nonoscillatory if there exists a $s_1 \geq T$ such that $x(s) \neq 0$ for $s \geq s_1$; $x(s)$ is said to be oscillatory if for any $s_1 \geq T$ there exist s_2 and s_3 satisfying $s_1 < s_2 < s_3$ such that $x(s_2) > 0$ and $x(s_3) < 0$; and it is said to be a z-type solution if it has arbitrarily large zeros but is ultimately non-negative or non-positive.

2.

In this section we consider

$$(\lambda(t)\, y'')' + Q(t)\, k(y') + P(t)\, y^{\alpha} = F(t) \qquad \text{... (2)}$$

where $y = y(t)$, P, Q, F and λ are real valued continuous functions on $[0, \infty)$ such that $\lambda(t) > 0$, $F(t) \geq 0$ and $y'(t)\, k(y'(t)) > 0$ for $y'(t) \neq 0$.

We begin by mentioning the following known results.

***Theorem* 2.1** — *Let $P(t) \geq 0$ and $Q(t) \leq 0$. If $Q(t) + t^{\alpha} P(t) \leq 0$ for large t, then all solutions of (2) with $k(y') = y'^{\alpha}$ are nonoscillatory.*

***Theorem* 2.2** — *Let $P(t) \geq 0$ and $Q(t) \leq 0$. Let $Q(t)$ be once continuously differentiable. If $Q'(t) \geq 0$ such that $Q'(t) + P(t) > 0$ for large t and $\lim_{t \to \infty} \frac{Q'(t)}{P(t)} = \infty$, then all bounded solutions of (2) with $\alpha \geq 1$ and $k(y') = y'$ are nonoscillatory.*

The proof of Theorems 2.1 and 2.2 may be found in[11].

***Theorem* 2.3** — *Let $P(t) \geq 0$ and $Q(t) \leq 0$. If $F(t) \geq M^{\alpha} P(t)$ for large t, where $M > 0$, $\beta > 0$ is a ratio of odd integers, then all solutions $y(t)$ of (2) with $k(y') = y'^{\beta}$ which are bounded above by M in any interval where $y(t) > 0$ are nonoscillatory.*

The proof of this theorem may be found in[9].

Theorem 2.4 — *Let $P(t) \geq 0$ and $Q(t) \leq 0$. Let P and F once continuously differentiable functions. If $P'(t) \geq 0, F'(t) \geq 0$ and $(\alpha+1) F(t) + P(t) \geq 0$, $\beta > 0$ is a ratio of odd integers, then all solutions y(t) of (2) with $k(y') = y'^{\beta}$ which $|y(t)| \leq 1$ ultimately are nonoscillatory.*

For the proof of this theorem the reader is referred to[8].

In the case $Q(t) \leq 0$, the following theorem improves Theorem 3.5 in[13&15] and the assumption employed is weaker then in[10].

Theorem 2.5 — *Let $P(t) \geq 0$ and $Q(t) \leq 0$. Let P and F be once continuously differentiable functions. If $P'(t) \geq 0, F'(t) \leq 0$ and $(\alpha+1) F(t) + P(t)$, then all solutions y(t) of (2) for which $|y(t)| \leq 1$ ultimately are nonoscillatory.*

PROOF : Let $y(t)$ be a solution of (2) on $[T, \infty)$, $T \geq 0$ such that $|y(t)| \leq 1$ for $t_1 > T$. Suppose that $y(t)$ is of non-negative z-type with consecutive double zeros a and b $(T_1 \leq a < b)$. So there exists a $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a, c)$. Multiplying (2) through by $y'(t)$, we get

$$(\lambda(t)\, y'(t)\, y''(t))' = \lambda(t)\, y''^2(t) - Q(t)\, y'(t)\, k(y'(t))$$

$$- P(t)\, y'(t)\, y^{\alpha}(t) + F(t)\, y'(t). \qquad \dots (3)$$

Integrating (3) from a to c, we get

$$0 = \int_a^c [\, \lambda(t)\, y''^2(t) - Q(t)\, y'(t)\, k(y'(t))$$

$$- P(t)\, y'(t)\, y^{\alpha}(t) + F(t)\, y'(t)]\, dt. \qquad \dots (4)$$

But

$$\int_a^c P(t)\, y'(t)\, y^{\alpha}(t)\, dt = \frac{1}{\alpha+1}$$

$$\left[P(t)\, y^{\alpha+1}(t) \big|_a^c - \int_a^c P'(t)\, y^{\alpha+1}(t)\, dt \right] \leq \frac{1}{\alpha+1} P(c)\, y^{\alpha+1}(c)$$

and

$$\int_a^c F(t)\, y'(t) = F(t)\, y(t) \big|_a^c - \int_2^c F'(t)\, y(t)\, dt \geq F(c)\, y(c).$$

So

$$\int_a^c F(t)\, y'(t)\, dt - \int_a^c P(t)\, y'(t)\, y^{\alpha}(t)\, dt \geq F(c)\, y(c) - \frac{1}{\alpha+1} P(c)\, y^{\alpha+1}(c)$$

$$\geq \frac{P(c)}{\alpha+1} y(c) - \frac{1}{\alpha+1} P(c)\, y^{\alpha+1}(c)$$

$$= \frac{1}{\alpha+1} p(c)\, [y(c) - y^{\alpha+1}(c)] > 0$$

since $|y(t)| \leq 1$ for t_1. Hence (4) yields

$$0 = \int_a^c [\lambda(t)\, y''^2(t) - Q(t)\, y'(t)\, k(y'(t))$$

$$- P(t)\, y'(t)\, y^{\alpha}(t) + F(t)\, y'(t)]\, dt > 0$$

a contradiction.

Next suppose that $y(t)$ is of non-positive z-type with consecutive double zeros a and b $(T_1 \leq a < b)$. So there exists $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c, b)$.

Integrating (3) from c to b, we get

$$0 = \int_c^b [\lambda(t)\, y''^2(t) - Q(t)\, y'(t)\, k(y'(t))$$

$$- P(t)\, y'(t)\, y^{\alpha}(t) + F(t)\, y'(t)]\, dt > 0$$

a contradiction.

If possible, let $y(t)$ be oscillatory. Let a, b and a' $(T_1 \leq a < b < a')$ be any three consecutive zeros of $y(t)$ such that $y'(a) \leq 0$, $y'(b) \geq 0$, $y'(a') \leq 0$, $y(t) < 0$ for $t \in (a, b)$ and $y(t) > 0$ for $t \in (b, a')$. So there exists points $c \in (a, b)$ and $c' \in (b, a')$ such that $y'(c) = 0$, $y'(c') = 0$ and $y'(t) > 0$ for $t \in (c, c')$. We consider two cases, viz., (*i*) $y''(b) \leq 0$ and (*ii*) $y''(b) > 0$. Suppose that $y''(b) \leq 0$. Integrating (3) from c to b, we obtain

$$0 \geq \lambda(b)\, y'(b)\, y''(b) = \int^b [\lambda(t)\, y''^2(t) - Q(t)\, y'(t)\, k(y'(t))$$

$$- P(t)\, y'(t)\, y^{\alpha}(t) + F(t)\, y'(t)]\, dt > 0$$

a contradiction. Let $y''(b) > 0$. Integrating of (3) from b to c', we get

$$-\lambda(b)\, y'(b)\, y''(b) = \int_b^{c'} [\lambda(t)\, y''^2(t) - Q(t)\, y'(t)\, k(y'(t))$$

$$- P(t)\, y'(t)\, y^{\alpha}(t) + F(t)\, y'(t)]\, dt.$$

We proceed as in non-negative z-type to conclude that $0 \geq \lambda(b)\, y'(b)\, y''(b) > 0$. This is a contradiction. So $y(t)$ is nonoscillatory. This completes the proof of the theorem.

Remark 2.1 : If $F(t) \equiv 0$ in Theorem 2.1, then $P(t) \equiv 0$ and hence Theorem 2.1 is not applicable to the equation

$$(\lambda(t)\, y'')' + Q(t)\, k(y') + P(t)\, y^{\alpha} = 0.$$

***Theorem* 2.6 —** *If $Q(t)$ does not change sign for large t, $F(t) > 0$ and $\lim_{t \to \infty} \frac{F(t)}{|P(t)|} = \infty$, then all bounded solutions of (2) are nonoscillatory.*

PROOF : Let $Q(t) \leq 0$ for $t \geq t_0$. Let $y(t)$ be a bounded solution of (2) such that $|y(t)| \leq M$. From the given function it follows that there exists a $t_1 \geq t_0$ such that $F(t) \geq M^{\alpha} |P(t)|$ for $t \geq t_1$. If possible, let $y(t)$ be oscillatory or z-type for $t \in (t_2, t_3)$. Now integrating (2) from t_2 to t_3, we obtain

$$0 > \lambda(t)\, y''(t) \Big|_{t_1}^{t_3} = \int_{t_2}^{t_3} F(t)\, dt - \int_{t_2}^{t_3} Q(t)\, k(y'(t))\, dt - \int_{t_2}^{t_3} P(t)\, y^{\alpha}(t)\, dt$$

$$\geq \int_{t_2}^{t_3} F(t)\, dt - \int_{t_2}^{t_3} |P(t)\, y^{\alpha}(t)|\, dt$$

$$\geq \int_{t_2}^{t_3} (F(t) - M^{\alpha} |P(t)|)\, dt > 0$$

a contradiction. So $y(t)$ is nonoscillatory.

If $Q(t) \geq 0$ for $t \geq t_0$, then we can choose t_2 and t_3 $(t_1 < t_2 < t_3)$ such that $y'(t_2) = 0$, $y'(t_3) = 0$ and $y'(t) \leq 0$ for $t \in (t_2, t_3)$ to get a contradiction.

3.

In this section we reduce eq. (1) to an equation of the form (2). Let $\int_0^{\infty} \frac{du}{r(u)} = \infty$. Set $R(s) = \int_0^{s} \frac{du}{r(u)}$. So $R : [0, \infty) \to [0, \infty)$ is an increasing function such that $s \to \infty$ implies that $R(s) \to \infty$. Consequently R^{-1} exists. Put $t = R(s)$, that is $s = R^{-1}(t)$. Setting $y(t) = x(R^{-1}(t)) = x(s)$, we get

$$r(s)\frac{dx}{ds} = \frac{dy}{dt}$$

and

$$\sigma(s)\frac{d}{ds}\left(r(s)\frac{dx}{ds}\right) = \frac{\sigma(s)}{r(s)}\frac{d^2y}{dt^2}.$$

So

$$\frac{d}{ds}\left[\sigma(s)\frac{d}{ds}\left(r(s)\frac{dx}{ds}\right)\right] = \frac{1}{r(s)}\frac{d}{dt}\left[\frac{\sigma(s)}{r(s)}\frac{d^2y}{dt^2}\right].$$

Putting $\lambda(t) = \dfrac{\sigma(R^{-1}(t))}{r(R^{-1}(t))}$, (1) is reduced to (2) with $Q(t) = r(R^{-1}(t))\ q(R^{-1}(t))$, $P(t) = r(R^{-1}(t))\ p(R^{-1}(t))$ and $F(t) = r(R^{-1}(t))\ f(R^{-1}(t))$. In view of Theorem 2.1, 2.2, 2.3, 2.4, 2.5,and 2.6, we have following theorems for eq. (1).

Theorem 3.1 — *Let* $p(s) \geq 0, q(s) \leq 0$ *and* $R(s) = \int_0^s \frac{du}{r(u)}$. *If* $q(s) + R^{\alpha}(s)\,p(s) \leq 0$ *for large s, then all solutions of (1) with* $k(rx') = (rx')^{\alpha}$ *are nonoscillatory.*

Theorem 3.2 — *Let* $p(s) \geq 0$ *and* $q(s) \leq 0$. *Let q(s) be once continuously differentiable. If* $q'(s) \geq 0$ *such that* $q'(s) + p(s) > 0$ *for large s and* $\lim_{s\to\infty} \frac{q'(s)}{p(s)} = \infty$, *then all bounded solutions of (1) with* $\alpha \geq 1$ *and* $k(rx') = x'$ *are nonoscillatory.*

Theorem 3.3 — *Let* $p(s) \geq 0$ *and* $q(s) \leq 0$. *If* $f(s) \geq M^{\alpha} p(s)$ *for large s, where* $M > 0$, $\beta > 0$ *is a ratio of odd integers, then all solutions of (1) with* $k(rx') = (rx')^{\beta}$ *which are bounded by M in any interval where* $x(s) > 0$ *are nonoscillatory.*

Theorem 3.4 — *Let* $p(s) \geq 0$ *and* $q(s) \leq 0$. *Let rp and rf be once continuously differentiable functions. If* $(rp)'(s) \geq 0$, $(rf)'(s) \geq 0$ *and* $(\alpha+1)f(s) + p(s) \geq 0$, $\beta > 0$ *is a ratio of odd integers, then all solutions x(s) of (1) with* $k(rx') = (rx')^{\beta}$ *for which* $|x(s)| \leq 1$ *ultimately are nonoscillatory.*

Theorem 3.5 — *Let* $p(s) \geq 0$ *and* $q(s) \leq 0$. *Let rp and rf be once continously differentiable functons. If* $(rp)'(s) \geq 0$, $(rf)'(s) \leq 0$ *and* $(\alpha+1)f(s) \geq p(s)$, *then all solutions x(s) of (1) for which* $|x(s)| \leq 1$ *ultimately are nonoscillatory.*

Example — Consider

$$\frac{d}{ds}\left[\frac{s}{2}\frac{d}{ds}\left(\frac{1}{s}\frac{dx}{ds}\right)\right]-\left(\frac{s^{13}}{2^7}-\frac{5s^{11}}{2^5}\right)\left[\frac{1}{s^3}\left(\frac{dx}{ds}\right)^3+\frac{2}{s^5}\left(\frac{dx}{ds}\right)^5\right]$$

$$+\left(s-\frac{2}{s}\right)x^3=\frac{s}{2}+\frac{2}{s}+\frac{10.2^5}{s^9}+\frac{24}{s^3},\quad s\geq 5.$$

Clearly, the equation satisfies conditions of Theorem 3.5 and in particular $x(s)=1-\frac{2}{s^2}$ is a nonoscillatory solution of equation.

***Theorem* 3.6** — *If $q(s)$ does not change sign for large s, $f(s)>0$ and $\lim_{s\to\infty}\frac{f(s)}{|p(s)|}=\infty$, then all bounded solutions of (1) are nonoscillatory.*

REFERENCES

1. G. D. Birkhoff, *Ann. Math.*, **12** (1911), 103-27.
2. M. Cecchi, Z. Dosla and M. Marini, *J. Math. Anal. Appl.*, **197**(3) (1996), 749-66.
3. L. Erbe, *Pacific J. Math.*, **64**(2) (1976), 369-85.
4. M. Gregus, *Mat. Fiz. Casopis Slovensk, Akad. Vied.*, **5** (1955), 73-85.
5. M. Gregus, *Third Order Linear Differential Equations*, D. Reidel Publishing Company, Dordrecht, Boston, Lancaster, 1987.
6. M. Hanan, *Pacific J. Math.*, **11** (1961), 919-44.
7. J. W. Heidel, *Pacific J. Math.*, **27** (1968), 507-26.
8. N. Parhi, *Applicable Analysis*, **12**(4) (1981), 273-85.
9. N. Parhi and S. Parhi, *Bulletin Inst. Math. Academia Sinica*, **11**(2) (1983), 125-39.
10. N. Parhi and S., Parhi, *Bulletin Inst. Math. Academia Sinica*, **13**(4) (1985), 367-84.
11. N. Parhi and S. Parhi, *Annales Polonici Math.*, **47**(2) (1986), 137-48.
12. N. Parhi and S. Parhi, *Riv. Math. Univ. Parma*, **13**(4) (1987), 201-10.
13. N. Parhi, *Acta Math. Hung.*, **54**(1-2) (1989), 79-88.
14. Ch. G. Philos, *Bulletin Inst. Math. Academia Sinica*, **11**(2) (1983), 141-60.
15. P. Temtek and A. Tiryaki, *Applied Mathematics and Mechanics*, **23**(10) (2002), 10.
16. C. A. Swanson, *Comparison and Oscillation Theory of Linear Differential Equations*, Acad. Press, New York, 1968.