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CHARACTERIZATIONS OF CERTAIN SUB (SUPER) CLASSES OF HAUSDORFF SPACES AND A FACTORIZATION OF REGULARITY

J. K. KOHLI* AND A. K. DAS**

**Department of Mathematics, Hindu College, University of Delhi, Delhi* 110 007, *India*
***Department of Mathematics, University of Delhi, Delhi* 110 007, *India*

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Veličko's notion of θ-closure of a set is used to obtain characterizations of Hausdorff spaces, Urysohn spaces, θ-Hausdorff spaces and R_1-spaces. This improves/strengthens several known results in the literature. In addition, a decomposition of regularity is obtained.

Key Words: **θ-Hausdorff Space; R_0-Space; R_1-Space; Functionally Hausdorff Space; θ-Regular Space; θ-Closure, θ-Closed (θ-open) Set; θ-Limit Point; $u\theta$-Cluster Point; $u\theta$-Limit Point; Regularly Open Set; N-Closed Set**

1. INTRODUCTION

In[15] Veličko introduced the notions of a θ-limit point of a set and θ-closed set to study the important class of H-closed spaces. Since, then the notion of θ-closure and θ-limit point has been utilized by host of authors in a wide variety of topological situations[4, 5, 6, 7 & 8]. In this paper we use the notion of a θ-closure of a set to obtain characterizations of Hausdorff spaces, Urysohn spaces, θ-Hausdorff spaces[11] and R_1-spaces[3]. In addition, we obtain a factorization of regularity and show that a space is regular if and only if it is both an R_0-space[3] and a θ-regular space[9]. The results obtained in the process improve/strengthen certain results of Dickman and Porter[4, 5], Carnahan[1], Janković[7] and others.

2. DEFINITIONS AND PRELIMINARIES

Definition 2.1[15] — Let X be a topological space and let $A \subset X$. A point $x \in X$ is called a θ-limit point of A if every closed neighbourhood of x intersects A. Let $cl_\theta A$ denote the set of all θ-limit points of A. The set A is called θ-closed if $A = cl_0 A$.

The complement of a θ-closed set will be referred to as a θ-open set.

Lemma 2.2[9] — A subset A of a topological space X is θ-open if and only if for each $x \in A$, there is an open set U such that $x \in U \subset \overline{U} \subset A$.

In general the θ-closure operator is not a Kuratowski closure operator, since θ-closure of a set might not be θ-closed[8]. However, the following modification in[10] yields a Kuratowski closure operator.

Definition 2.3[10] — Ler X be a topological space and let $A \subset X$. A point $x \in X$ is called a $u\theta$-limit point of A if every θ-open U containing x intersects A. Let $A_{u\theta}$ denote the set of all $u\theta$-limit point of A. The set A is called $u\theta$-closed if $A = A_{u\theta}$.

It turns out that $A_{u\theta}$ is the smallest θ-closed set containing A.

Definition 2.4[10] — Let $\Im$ be a filter in X. A point $x \in X$ is said to be $u\theta$-cluster point of $\Im$ if every θ-open set U containing x intersects every member of the filter $\Im$. The filter $\Im$ is said to $u\theta$-converge to x if every θ-open set U containing x belongs to $\Im$. In symbols $\Im \xrightarrow{u\theta} x$.

Throughout the paper interior of a set A will be denoted by int$A$ and closure of a set A will be denoted by either $\overline{A}$ or clA. A subset A of a topological space is said to be regularly open if $A = \text{int } \overline{A}$. The complement of a regularly open set will be referred to as a regularly closed set.

3. Characterizations

Definition 3.1[1] — A subset F of X is said to be N-closed relative to X if every cover of F by regularly open sets in X has a finite subcollection which covers F.

Dickman and Porter[4] introduced the notion of a θ-rigid set. it turns out that the notions of θ-rigid set and N-closed set relative to X are equivalent notions[5].

Theorem 3.2 — *For a topological space X, the following statements are equivalent.*

(a) X is Hausdorff.

(b) Every N-closed set relative to X is θ-closed.

(c) Every compact set in X is θ-closed.

(d) Every singleton in X is θ-closed.

(e) For every pair of distinct points x and y in X there exist θ-open sets U and V such that $x \in U, y \notin U$ *and* $y \in V, x \notin V$.

(f) For every pair of distinct points $x, y \in X$, $cl_{u\theta}\{x\} \cap cl_{u\theta}\{y\} = \phi$.

(g) For every pair of distinct points there exists a θ-open set containing one but not the other.

PROOF : To prove the assertion $(a) \Rightarrow (b)$, let X be a Hausdorff space and let A be an N-closed set relative to X and suppose that $x \notin A$. Then for every $y \in A$ there exist disjoint open sets U_y and V_y containing x and y, respectively. Since U_y and V_y are disjoint, $U_y \cap \text{int } \overline{V}_y = \phi$. Now $\{\text{int } \overline{V}_y : y \in A\}$ is a regularly open cover of A and so has a finite subcover $\{\text{int } \overline{V}_{y_i} : i = 1, ..., n\}$. Thus $U = \bigcap_{i=1}^{n} U_{y_i}$ and $V = \bigcup_{i=1}^{n} \text{int } \overline{V}_{y_i}$ are disjoint open set containing x and A, respectively. Now, $\overline{U} \cap V = \phi$ and hence $\overline{U} \cap A = \phi$. Thus $x \notin cl_\theta A$ and so A is θ-closed.

The implications $(b) \Rightarrow (c) \Rightarrow (d)$ and $(f) \Rightarrow (g)$ are obvious. To prove $(d) \Rightarrow (e)$, let x and y be two distinct points in X. Since every singleton in X is θ-closed, X-$\{x\}$ and $X - \{y\}$ are θ-open sets satisfying the condition (e).

To prove the assertion $(e) \Rightarrow (f)$, assume contrapositive. That is $cl_{u\theta}\{x\} \cap cl_{u\theta}\{y\} \neq \phi$. Then there exists $z \in X$ such that $z \in cl_{u\theta}\{x\} \cap cl_{u\theta}\{y\}$. Thus every θ-open set containing z contains x as well as y, which contradicts the condition (e). Hence, $cl_{u\theta}\{x\} \cap cl_{u\theta}\{y\} = \phi$.

To prove the implication $(g) \Rightarrow (s)$, let x and y be distinct points of X. By the hypothesis there exists a θ-open set U such that $x \in U$ and $y \notin U$. By Lemma 2.2. there exists an open set V such that $x \in V \subset \overline{V} \subset U$. Thus V and $X - \overline{V}$ are disjoint open sets containing x and y, respectively and so X is Hausdorff.

Remark 3.3. : The equivalence of $(a) - (d)$ are due to Dickman and Porter[4, 5].

Theorem 3.4 — *For a topological space X the following statements are equivalent.*

(a) X is Urysohn.

(b) For every pair of disjoint N-closed sets A and B relative to X, there exist open sets U and V containing A and B, respectively such that $\overline{U} \cap \overline{V} = \phi$.

(c) The diagonal Δ *is θ-closed in* $X \times X$.

PROOF : To prove $(a) \Rightarrow (b)$, let X be an Urysohn space and let A, B be disjoint N-closed sets relative to X. For each $a \in A$ and for each $b \in B$ there exist, open sets U_b and V_b containing a and b, respectively such that $\overline{U}_b \cap \overline{V}_b = \phi$. Then int $\overline{U}_b$ and int $\overline{V}_b$ are disjoint regularly open sets containing a and b, respectively. Now, $\{\text{int } \overline{V}_b : b \in B\}$ is a cover of B by regularly open sets in X, and so it has a finite subcollection, say $\{\text{int } \overline{V}_{b_i} : i = 1, 2, \ldots, n\}$ which cover B. Then $U = \bigcap_{i=1}^{n} U_{b_i}$ and $V = \bigcup_{i=1}^{n} \overline{V}_{b_i}$ are disjoint open sets containing a and B respectively, such that $\overline{U} \cap \overline{V} = \phi$. Therefore, for each $a \in A$, there exist disjoint open sets W_a and O_a containing a and B respectively such that $\overline{W}_a \cap \overline{O}_a = \phi$. Now $\{\text{int } \overline{W}_a : a \in A\}$ is a regularly open cover of A and so has a finite subcover $\{\text{int } W_{a_i} : i = 1, \ldots, n\}$. Thus, $W = \bigcup_{i=1}^{n} \text{int } \overline{W}_{a_i}$ and $O = \bigcap_{i=1}^{n} O_{a_i}$ are disjoint open sets containing A and B, respectively, whose closures are disjoint.

To prove $(b) \Rightarrow (c)$, let $(x, y) \in X \times X - \Delta$. Then $\{x\}$ and $\{y\}$ are disjoint N-closed sets relative to X. So there exist open sets U and V containing x and y respectively, such that $\overline{U} \cap \overline{V} = \phi$. Hence, $\overline{U} \times \overline{V} = \subset X \times X - \Delta$. Thus, .$(x, y) \in U \times V \subset \overline{U \times V} \subset X \times X - \Delta$ and so by Lemma 2.2. $X \times X - \Delta$ is θ-open. Hence Δ is θ-closed.

To prove $(c) \Rightarrow (a)$, let $x \neq y$. Then $(x, y) \in X \times X - \Delta$.. Since $X \times X - \Delta$ is θ-open by Lemma 2.2 there exists an open set $U \times V$ such that $(x, y) \in U \times V \subset \overline{U \times V} \subset X \times X - \Delta$. Thus U and V are disjoint open sets containing x and y, respectively, such that $\overline{U} \cap \overline{V} = \phi$. Hence X is Uryshon.

Remark 3.5 : The equivalence of (a) and (c) is due to Janković[7], Theorem 4.5.

Since every nearly compact Hausdorff space is almost regular[13] (Theorem 2.4) and since every almost regular Hausdorff space is Uryshon[12] (Theorem 3.2), the following result is immediate from Theorem 3.4.

Corollary 3.6 — (Carnahan[1], Theorem 2.8). Let X be a nearly compact Hausdorff space and let K and M be disjoint N-closed sets relative to X. Then K and M are contained in open sets with disjoint closures.

Definition 3.7[11] — A topological space X is said to be θ-Hausdorff if every pair of distinct points can be separted by disjoint θ-open sets.

Recall that a space X is functionally Hausdorff if for every pair of distinct points x and y in X there exists a continuous function $f: X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$.

It is easy to verify that every functionally Hausdorff space as well as every T_0-regular space is θ-Hausdorff. However, the converse statement is not true as is well exhibited by the following examples.

Example 3.8 — A θ-Hausdorff space which is not regular.

Consider the countable complement extension topology[14], where X is the real line; $\mathfrak{I}_1$ is the Euclidean topology on X and $\mathfrak{I}_2$ is the topology of countable complements, on X. Let $\mathfrak{I}$ be the smallest topology generated by $\mathfrak{I}_1 \cup \mathfrak{I}_2$. Since open sets of $\mathfrak{I}_1$ are open in $\mathfrak{I}$ and since every member of $\mathfrak{I}_1$ is θ-open, $\mathfrak{I}$ is θ-Hausdorff. However, the space X is not regular as the closed set Q and a point outside it can not be separated by disjoint open sets.

Example 3.9 — A θ-Hausdorff space which is not functionally Hausdorff.

The example of Hewitt's condensed Corkscrew[14] is a T_1-regular space and hence a θ-Hausdorff space. However, it is not functionally Hausdorff.

Theorem 3.10 — *For a topological space X, the following statements are equivalent.*

(a) X is θ-Hausdorff.

(b) No filter on X has more than one $u\theta$-limt point

PROOF : To prove $(a) \Rightarrow (b)$, let X be a θ-Hausdorff space and let $\mathfrak{I}$ be a filter on X, where $\mathfrak{I} \xrightarrow{u\theta} x$. Now for $y \neq x$, there exist disjoint θ-open sets U and V containing x and y respectively. Since $\mathfrak{I} \xrightarrow{u\theta} x$, $U \in \mathfrak{I}$. Thus y is not a $u\theta$-cluster point of $\mathfrak{I}$. Since every $u\theta$-limit point of a filter is also a $u\theta$-cluster point, no filter on X has more than one $u\theta$-limit point.

To prove $(b) \Rightarrow (a)$, suppose, no filter on X has more than one $u\theta$-limit point. Now let $y \neq x$. Suppose that every θ-open set U containing x intersects every θ-open set V containing y. Then $\{U \cap V : x \in U, y \in V\}$ is a filterbase which has both x and y as its $u\theta$-limit points. This contradiction proves that X is θ-Hausdorff.

Definition 3.11[7] — If X is a topological space, $A \subset X$ and $x \in X$, then Ker $(A) = \bigcap \{U : A \subset U$ and U is open in $X\}$.

Definition 3.12[7] — If X is a topological space and $A \subset X$, then $\text{Ker}_\theta(A) = \{x \in X : cl_\theta \{x\} \cap A \neq \phi\}$.

It is observed in [7] that $A \subset \text{Ker}(A) \subset \text{Ker}_\theta(A) \subset cl_\theta(A)$.

Lemma 3.13 — Let X be a topological space. Then for every N-closed set A relative to X, $\text{Ker}_\theta(A) = cl_\theta(A)$

PROOF : Since $\text{Ker}_\theta(A) \subset cl_\theta(A)$, it is sufficient to show that $cl_\theta A \subset \text{Ker}_\theta(A)$, for each N-closed set A relative to X. Let $x \in cl_\theta A$ and suppose $x \notin \text{Ker}_\theta(A)$, then $cl_\theta\{x\} \cap A = \phi$. It follows that for each $y \in A$, there exist disjoint open sets U_y and V_y containing x and y, respectively such that $U_y \cap \text{int } \overline{V}_y = \phi$. Since $\{\text{int } \overline{V}_y : y \in A\}$ is a regularly open cover of A, there exists a finite subcollection say $\{\overline{V}_{y_i} : i = 1, ..., n\}$ which cover A. Now, $U = \bigcap_{i=1}^{n} U_{y_i}$ and $V = \bigcup_{i=1}^{n} \overline{V}_{y_i}$ are disjoint open sets containing x and A respectively, such that $\overline{U} \cap V = \phi$. Hence, $x \notin cl_\theta A$, which is a contradiction. Thus $x \in \text{Ker}_\theta(A) = cl_\theta A$.

Corollary 3.14 — Janković[7], Lemma 3.5(d). Let X be a topological space, then for every compact set $A \subset X$, $\text{Ker}_\theta(A) = cl_\theta A$.

Definition 3.15[3] — A topological space X is said to be an R_1-space if $y \notin \{\overline{x}\}$ implies that x and y have disjoint neighbourhoods.

R_1-sapces are referred to as S_2-spaces in[2].

Lemma 3.16 — (Janković[7]). For a topological space X the following statements are equivalent.

(*a*) X is an R_1-space.

(*b*) For each $x \in X$, $cl_\theta\{x\} = cl\{x\}$.

(*c*) If $x \in U$, then $cl_\theta\{x\} \subset U$ for every open set U in X.

(*d*) For each $x \in X$, $cl_\theta\{x\} = \mathrm{Ker}\{x\}$.

Theorem 3.17 — *For a topological space X, the following statements are equivalent.*

(a) X is an R_1*-space.*

(b) For each N-closed set A relative to X, $\mathrm{Ker}_\theta(A) = cl\,A$.

(c) For each N-closed set A relative to X, $cl_\theta A = cl\,A$.

(d) For each N-closed set A relative to X, $cl_\theta A = Ker\ A$.

PROOF : To prove $(a) \Rightarrow (b)$, let A be an N-closed set relative to X. By Lemma 3.13 $\mathrm{Ker}_\theta(a) = cl_\theta A$. Thus $cl\,A \subset \mathrm{Ker}_\theta(A)$. To prove $\mathrm{Ker}_\theta(a) = cl\,A$. Let $x \in \mathrm{Ker}_\theta A$ and suppose $x \notin cl\,A$. Then there exists an open set U containing x such that $U \cap A = \phi$. Since the space is R_1, for every $y \in A$, there exist disjoint open sets U_y and V_y containing x and y respectively. Thus for every $y \in A$, $\overline{V}_y \cap \{x\} = \phi$ and so $cl_\theta\{x\} \cap A = \phi$, which is a contradiction to the fact that $x \in \mathrm{Ker}_\theta A$.

Therefore, $\mathrm{Ker}_\theta \subset cl\,A$.

The assertion $(b) \Rightarrow (c)$ is immediate in view of Lemma 3.13. To prove $(c) \Rightarrow (a)$, let $x \in X$, then $cl_\theta\{x\} = cl\{x\}$. Thus X is an R_1-space in view of Lemma 3.16.

To prove $(a) \Rightarrow (d)$, let A be an N-closed set relative toX. By Lemma 3.13 $cl_\theta A = \mathrm{Ker}_\theta A$.Since X is an R_1-space, for each set $A \subset X$, $\mathrm{Ker}_\theta A = \mathrm{Ker}(A)$. Hence, $cl_\theta A = \mathrm{Ker}(A)$.

To prove $(d) \Rightarrow (a)$. Let $x \in X$. Since $\{x\}$ is N-closed, relative, to X, $cl_\theta\{x\} = \mathrm{Ker}\{x\}$. By Lemma 3.16 X is an R_1-space.

Corollary 3.18 — [Janković[7], Theorem 3.6]. For a topological space X the following statements are equivalent.

(*a*) X is an R_1-space.

(*b*) For each compact set $A \subset X$, $\mathrm{Ker}_\theta(A) = cl\,A$.

(*c*) For each compact set $A \subset X$, $cl_\theta A = cl\,A$.

(*d*) For each compact set $A \subset X$, $cl_\theta A = \mathrm{Ker}$ (A).

Lemma 3.19[6] — If A is an N-closed set relative to X, then $cl_\theta A = \bigcup_{x \in A} cl_\theta \{x\}$.

Theorem 3.20 — *Let X be an R_1-space. Then*

(a) If K is an N-closed set relative to X contained in an open set U, then $cl_\theta K \subseteq U$.

(b) If K is compact, then $cl_\theta K$ is compact.

PROOF : (a) Let K be an N-closed set relative to X, contained in an open set U. Then in view of Lemma 3.16 and Lemma 3.19, it is immediate that $cl_\theta K \subset U$.

(b) Let K be a compact subset of X. Then any open cover of $cl_\theta K$ is an open cover of K, which has a finite subcover. Thus by (a), the same finite subcollection also cover $cl_\theta K$.

Theorem 3.21 — *In an R_1-sapce, if $K \subset K_1 \subset cl_\theta K$. Then K is compact if and only if K_1 is compact.*

PROOF : Suppose K is compact and $\mathcal{U}$ be an open cover of K_1. Then $\mathcal{U}$ covers K which has a finite subcover. Thus by Theorem 3.20 this subcollection covers $cl_\theta K$ and hence cover K_1. Conversely, suppose K_1 is compact and let $\mathcal{U}$ be an open cover of K. Again by Theorem 3.20, $\mathcal{U}$ covers $cl_\theta K$, which covers K_1. Thus $\mathcal{U}$ has a finite subcollection which clearly covers K.

Corollary 3.22[2] — In an R_1-space, if K is compact, then $\overline{K}$ is compact.

Definition 3.23[9] — A topològical space X is said to be θ-regular if for every open set U containing a closed set F there exist a θ-open set V such that $F \subset V \subset U$.

Definition 3.24[3] — A space X is an R_0-space if for every open set U in $X, x \in U$ implies $\{\overline{x}\} \subset U$.

R_0-spaces are called S_1-spaces in[2].

The following result besides giving a factorization of regularity, strengthens an observation of[9], wherein it is stated that in the class of T_1-spaces the notions of regularity and θ-regularity coincide.

Theorem 3.25 — *For a topological X, the following statements are equivalent.*

(a) X is regular.

(b) X is θ-regular and an R_1-sapce.

(c) X is θ-regular and an R_0-sapce.

PROOF : The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are obvious.

To prove the assertion $(c) \Rightarrow (a)$, let A be a closed set in X and let $x \notin A$. Then $X - A$ is an open set containing x. Since X is an R_0-space, $\{\overline{x}\} \subset X - A$. Thus by θ-regularity of X, there exists a θ-open set U such that $\{\overline{x}\} \subset U \subset X - A$. By Lemma 2.2, there exists an open set V such that that $x \in V \subset \overline{V} \subset U$. Hence, V and $X - \overline{V}$ are disjoint open sets containing x and A, respectively.

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