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SEVERAL TYPES OF FUZZY REGULAR SPACES

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We introduce r-regularized fuzzy (semi, weakly semi, regular) closed sets in Sostak's fuzzy topological spaces[13]. Moreover, we introduce several types of fuzzy regular spaces and investigate relations between them and fuzzy cluster operators.

Key Words : Fuzzy Topology; r-Regularized Fuzzy (Semi, Weakly Semi, Regular) Closed Sets; r-Fuzzy (s-Almost) Regular Spaces; r-θ (Semi, Weakly Semi, Regular) Cluster Point

1. INTRODUCTION

Sostak[13] introduced the fuzzy topology as an extension of Chang's fuzzy topology[2]. It has been developed in many directions[3,4,7-9,12]. Balasubramanian and Sundaram[1] gave the concept of generalized fuzzy closed sets in Chang's fuzzy topology as an extension of generalized closed sets of Levine[10] in topological spaces. Kim *et al.*[7-9,12] defined r-generalized fuzzy closed sets and several operators in Sostak's fuzzy topological spaces as an extension of those[1,5,6] in (fuzzy) topological spaces.

In this paper, we define r-generalized fuzzy (semi, weakly semi, regular) closed sets and r-fuzzy (s-, almost) regular spaces in Sostak's fuzzy topological spaces. We investigate some properties of them. Moreover, we obtain some properties of r-fuzzy (s-, almost) regular spaces and investigate relations between them and cluster operators.

Throughout this paper, let X be a nonempty set, $I = [0, 1]$ and $I_0 = (0, 1]$. For $\alpha \in I, \overline{\alpha}(x) = \alpha$ for all $x \in X$. For $x \in X$ and $t \in I_0$, a fuzzy point x_t is defined by

$$x_t(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$$

Let $Pt(X)$ be the family of all fuzzy points in X. For $\lambda, \mu \in I^X$, λ is called quasi-coincident with μ, denoted by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Otherwise, we denote $\lambda \overline{q} \mu$.

Definition 1.1[13] — A function $\tau : I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions:

(01) $\tau(\overline{0}) = \tau(\overline{1}) = 1$,

(O2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, for any $\mu_1, \mu_2 \in I^X$,

(O3) $\tau(\vee_{i \in \Gamma} \mu_i) \geq \wedge_{i \in \Gamma} \tau(\mu_i)$, for any $\{\mu_i\}_{i \in \Gamma} \subset I^X$.

The pair (X, τ) is called a fuzzy topological space (for short, fts).

Definition 1.2[7] — Let (X, τ) be a fts, $\lambda, \mu \in I^X$ and $r \in I_0$.We define operators as follows:

$$C_\tau(\lambda, r) = \wedge\{\mu \in I^X \mid \lambda \leq \mu, \tau(\overline{1} - \mu) \geq r\}$$

$$I_\tau(\lambda, r) = \vee\{\mu \in I^X \mid \mu \leq \lambda, \tau(\mu) \geq r\}.$$

(1) A fuzzy set λ is called r-semiopen if $\lambda \leq C_\tau(I_\tau(\lambda, r), r)$.

(2) A fuzzy set λ is called r-regular open if $\lambda = I_\tau(C_\tau(\lambda, r), r)$.

(3) A fuzzy set λ is called r-semiclosed (resp. r-regular closed) if $\overline{1} - \lambda$ is r-semiopen (resp. r-regular open).

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***Theorem* 1.3**[7] — *Let (X, τ) be a fts.*

(1) Any union of r-semiopen sets is r-semiopen.

(2) Any intersection of r-semiclosed sets is r-semiclosed.

Notation 1.4[8] — Let (X, τ) be a fts and $x_t \in Pt(X)$. We denote

$$Q_\tau(x_t, r) = \left\{\mu \in I^X \mid x_t\, q\, \mu,\ \tau(\mu) \geq r\right\}$$

$$\mathcal{R}_\tau(x_t, r) = \left\{\mu \in I^X \mid x_t\, q\, \mu,\ \mu = I_\tau(C_\tau(\mu, r), r)\right\}.$$

Definition 1.5[8] — Let (X, τ) be a fts, $\lambda \in I^X, x_t \in P_t(X), r \in I_o$.

(1) x_t is called a r-cluster point of λ if for every $\mu \in Q_\tau(x_t, r)$, we have $\mu\, q\, \lambda$.

(2) x_t is called a r-δ-cluster point of λ if for every $\mu \in \mathcal{R}_\tau(x_t, r)$, we have $\mu\, q\, \lambda$..

(3) x_t is called a r-θ-cluster point of λ if for every $\mu \in Q_\tau(x_t, r)$, we have $C_\tau(\mu\, r)\, q\, \lambda$..

Theorem 1.6[8] — *Let (X, τ) be a fts. For each $\lambda, \mu, \rho \in I^X$ and $r \in I_o$, let (X, τ) be a fts. For each $r \in I_0$, $\lambda \in I^X$, we define operators as follows:*

$$SC_\tau(\lambda, r) = \wedge\left\{\mu \in I^X \mid \lambda \leq \mu,\ \mu \text{ is } r\text{–semiclosed}\right\}$$

$$SI_\tau(\lambda, r) = \vee\left\{\mu \in I^X \mid \mu \leq \lambda, \ \mu \text{ is } r\text{–semiopen}\right\}$$

$$D_\tau(\lambda, r) = \vee\left\{x_t \in Pt(X) \mid x_t \text{ is a } r\text{–}\delta\text{–cluster point of } \lambda\right\}$$

$$T_\tau(\lambda, r) = \vee\left\{x_t \in Pt(X) \mid x_t \text{ is a } r\text{–}\theta\text{–cluster point of } \lambda\right\}.$$

For each $\lambda, \mu, \rho \in I^X$ *and* $r, s \in I_0$, *it satisfies the following properties:*

(1) $I_\tau(\overline{1} - \lambda, r) = \overline{1} - C_\tau(\lambda, r)\ SI_\tau(\overline{1}\ \lambda, r) = \overline{1} - SC_\tau(\lambda, r)$.

(2) $T_\tau(\lambda, r) = \wedge\left\{\mu \in I^X \mid \lambda \leq I_\tau(\mu, r),\ \tau(\overline{1} - \mu) = r\right\}$.

(3) $D_\tau(\lambda, r) = \wedge\left\{\mu \in I^X \mid \lambda \leq \mu,\ \mu = \ C_\tau(I_\tau(\mu, r), = r\right\}$.

(4) $D_\tau(\lambda, r) = \vee\left\{x_t \in Pt(X) \mid x_t \text{ is a } r\text{–cluster point of } \lambda\right\}$.

(5) x_t is a $r-\theta$-cluster point of λ iff $x_t \in T_\tau(\lambda, r)$.

(6) x_t is a $r-\delta$-cluster point of λ iff $x_t \in D_\tau(\lambda, r)$.

(7) x_t is a r-cluster point of λ iff $x_t \in C_\tau(\lambda, r)$.

(8) If $\rho = C_\tau(I_\tau(\rho, r), r)$, then $D_\tau(\rho, r) = \rho$.

(9) $\lambda \leq C_\tau(\lambda, r) \leq D_\tau(\lambda, r) \leq T_\tau(\lambda, r)$.

(10) If $\tau(\lambda) \geq r$, then $C_\tau(\lambda, r) = D_\tau(\lambda, r) = T_\tau(\lambda, r)$.

(11) $C_\tau(I_\tau(\lambda, r), r) = C_\tau(I_\tau(C_\tau(I_\tau(\lambda, r), r), r), r)$.

(12) $I_\tau(C_\tau(\lambda, r), r) = I_\tau(C_\tau(I_\tau(C_\tau(\lambda, r), r), r), r)$.

(13) $W(\lambda \vee \mu, r), r) = W(\lambda, r) \vee W(\mu, r)$, for each $W \in \{C_\tau, D_\tau, T_\tau\}$.

(14) $W(W(\lambda, r), r) = W(\lambda, r)$, for each $W \in \{C_\tau, D_\tau, SC_\tau\}$.

2. R-Regularized Fuzzy Closed (Semiclosed, Regular) Sets

Definition 2.1. — Let (X, τ) has a fts, $\lambda, \mu \in I^X$ and $r \in I_0$. A fuzzy set λ is called:

(1) r-regularized fuzzy closed (for short, $r$gfc) if $C_\tau(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and $\tau(\mu) \geq r$.

(2) r-generalized fuzzy semiclosed (for short, r-gfsc) if $C_\tau(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and μ is r-semiopen.

(3) r-generalized fuzzy weakly semiclosed (for short, r-gfwsc) if $C_\tau > SC_\tau$ whenever $\lambda \leq \mu$

and μ is r-regular open.

(4) r-generalized fuzzy regular closed (for short, r-gfrc) if $C_\tau(\lambda, r) \le \mu$ whenever $\lambda \le \mu$ and μ is r-regular open.

(5) r-*gfo* (resp. r-gfso, r-gfwso, r-gfro) if $\overline{1} - \lambda$ is r-gfc (resp. r-gfsc, r-gfwsc, r-gfrc).

The following theorem easily obtain from the above definition.

Theorem 2.2 — *Let* (X, τ) *be a fts. We have the following implications:*

$$r - gfsc \Rightarrow r - gfc \Rightarrow r - gfrc, \text{ moreover, } r - gfsc \Rightarrow r - gfws.$$

Theorem 2.3 — *Let* (X, τ) *be a fts.*

(1) If λ_1 *and* λ_2 *are r-gfc (resp. r-gfsc, r-gfrc) sets, then* $\lambda_1 \vee \lambda_2$ *is r-gfc (resp. r-gfsc, r-gfrc).*

(2) If λ *is r-gfwsc and* $\lambda \le \mu \le SC_\tau(\lambda, r)$, *then* μ *is r-gfwsc.*

(3) If λ *is r-gfc (resp. r-gfsc, r-gfrc) and* $\lambda \le \mu \le C_\tau(\lambda, r)$, *then* μ *is r-gfc (resp. r-gfsc, r-gfrc).*

(4) If $\tau(\overline{1} - \lambda) \ge r$ *and* $r \in I_0$, *then* λ *is r-gfc, r-gfsc, r-gfrc and r-gfwsc.*

(5) If λ *is r-semiclosed, then* λ *is r-gfwsc.*

PROOF : (1) Let λ_1 and λ_2 be r-gfsc sets and $\lambda_1 \vee \lambda_2 \le \mu$ such that μ is r-semiopen. For $i \in \{1, 2\}$, $\lambda_i \le \mu$ such that μ is r-semiopen, we have $C_\tau(\lambda, r) \le \mu$. By Theorem 1.6[13], it implies

$$C_\tau(\lambda_1 \vee \lambda_2, r) = C_\tau(\lambda_1, r) \vee C_\tau(\lambda_2, r) \le \mu.$$

Hence $\lambda_1 \vee \lambda_2$ is r-gfsc. Other cases are similarly proved.

(2) For $\mu \le \rho$ such that ρ is r-semiopen, since λ is r-gfwsc set, $\lambda \le \rho$ implies $SC_\tau(\lambda, r) \le \rho$. By Theorem 1.6[14], $\mu \le SC_\tau(\lambda, r)$ implies

$$SC_\tau(\lambda, r) \le SC_\tau\ (SC_\tau(\lambda, r), r) = SC_\tau(\lambda, r) \le \rho.$$

Hence μ is r-gfwsc.

(3) It is similarly proved as in (2).

(4) and (5) are easily proved from $C_\tau(\lambda, r) = \lambda$ and $C_\tau(\lambda, r) = \lambda$, respectively.

Example 2.4 — Let X - $\{a, b, c, d\}$ be a set. We define a fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2}, & \text{if } \lambda \in \{\chi_{\{a\}}, \chi_{\{b\}}, \chi_{\{a,b\}}, \chi_{\{a,b,c\}}, \chi_{\{a,b,d,f\}}\} \\ 0, & \text{otherwise.} \end{cases}$$

(1) For $0 < r \leq \frac{1}{2}$, $\mu = \chi_{\{a,b,d\}} \vee c_{0.5}$ is r-gfc set from $\mu \leq \overline{1}$ and $C_\tau(\mu, r) = \overline{1}$. But, since $\mu = \chi_{\{a,b,d\}} \vee c_{0.5}$ is r-semiopen from $\mu \leq C_\tau(I_\tau(\mu, r), r) = \overline{1}$, μ is neither r-gfsc nor r-gfwsc because

$$\mu \leq \mu,\ SC_\tau(\mu, r) = C_\tau(\mu, r) = \overline{1} \not\leq \mu.$$

(2) For $0 < r \leq \frac{1}{2}$, since $\chi_{\{a,b\}}$ is r-semiopen, then $\chi_{\{a,b\}}$ is neither r-gfwsc nor r-gfc set from:

$$\chi_{\{a,b\}} \leq \chi_{\{a,b\}},\ C_\tau(\chi_{\{a,b\}}, r) = SC_\tau(\chi_{\{a,b\}}, r) = \overline{1} \not\leq \chi_{\{a,b\}}.$$

But $\chi_{\{a,b\}}$ is r-gfrc because $\chi_{\{a,b\}} \leq \overline{1} = I_\tau(C_\tau(\overline{1}, r)$.

(3) Since $a_{0.3} \leq I_\tau(C_\tau(\chi_{\{a\}}, r),\ r) = \chi_{\{a\}}$ where $\chi_{\{a\}}$ is r-regular open, r-semiopen and $SC_\tau(a_{0.3}, r) = \chi_{\{a\}}$ for $0 < r \leq \frac{1}{2}$, then $a_{0.3}$ is r-gfwsc but not r-gfrc because $SC_\tau(a_{0.3}, r) = \chi_{\{a,c,d\}} \not\leq \chi_{\{a\}}$.

(4) From (2)and (3), the notions of r-gfwsc and r-gfrc sets are independent.

(5) In Theorem 2.3(1), the union of any two r-gfwsc sets need not be r-gfwsc. For $0 < r \leq \frac{1}{2}$, $SC_\tau(\chi_{\{a\}}, r) = \chi_{\{a\}}$ and $SC_\tau(\chi_{\{b\}}, r) = \chi_{\{b\}}$, then $\chi_{\{a\}}$ and $\chi_{\{b\}}$ is r-gfwsc. But $\chi_{\{a,b\}}$ is not r-gfwsc from (2).

The intersection of any two r-gfc (resp. r-gfsc, r-gfrc) sets need not be r-gfc (resp. r-gfsc, r-gfrc) from the following example.

Eample 2.5 — Let $X = \{a, b\}$ be a set. We define a fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2}, & \text{if } \lambda = \overline{0.3} \\ 0, & \text{otherwise.} \end{cases}$$

Let $\lambda_1, \lambda_2 \in I^X$ be given as follows

$$\lambda_1(a) = 0.2,\ \lambda_1(b) = 0.8,\ \lambda_2(a) = 0.9,\ \lambda_2(b) = 0.2.$$

For $0 < r \leq \frac{1}{2}$, λ_1 and λ_2 are r-gfc, r-gfsc and r-gfrc.

But $\overline{0.2} = \lambda_1 \wedge \lambda_2$ is neither r-gfc nor r-gfrc. Because for $0 < r \leq \frac{1}{2}$,

$$\overline{0.3} \leq \lambda \leq \overline{0.7} \text{ is } r\text{-semiopen, } \overline{0.2} \text{ is not } r\text{-gfsc.}$$

The following theorem is easily proved from Theorems 1.6(1) and 2.3.

***Theorem* 2.6** — *Let (X, τ) be a fts, $\lambda \Rightarrow \lambda, \mu \in I^X$ and $r \in I_0$.*

(1) The intersection of any two r-gfo (resp. r-gfso, r-gfro) sets is r-gfo (resp. r-gfso, r-gfro).

(2) If λ is r-gfwso and $SI_\tau(\lambda_1 = \lambda, r) \leq \mu \leq \lambda$, then μ is r-gfwso.

(3) If $I_\tau(\lambda, r) \leq \mu \leq \lambda$ and λ is r-gfo (resp. r-gfso, r-gfro), then μ is r-gfo (resp. r-gfso, r-gfro).

(4) If $\tau(\lambda) \geq r$ and $r \in I_0$, then λ is r-gfo, r-gfso and r-gfro.

(5) If λ is r-semiopen, then λ is r-gfwso.

***Theorem* 2.7** — *Let (X, τ) be a fts. For each $r \in I_0$ and $\lambda \in I^X$, we define operators $GC_\tau, GR_\tau GS_\tau, GW_\tau : I^X \times I_0 \to I^X$ as follows:*

$$GC_\tau(\lambda, s) = \wedge \left\{ \mu \in I^X \mid \lambda \leq \mu, \ \mu \text{ is } r\text{–gfc} \right\}$$

$$GR_\tau(\lambda, s) = \wedge \left\{ \mu \in I^X \mid \lambda \leq \mu, \ \mu \text{ is } r\text{–gfrc} \right\}$$

$$GS_\tau(\lambda, s) = \wedge \left\{ \mu \in I^X \mid \lambda \leq \mu, \ \mu \text{ is } r\text{–gfsc} \right\}$$

$$GW_\tau(\lambda, s) = \wedge \left\{ \mu \in I^X \mid \lambda \leq \mu, \ \mu \text{ is } r\text{–gfwsc} \right\}.$$

For $\lambda, \mu \in I^X$ and $r \in I_0$, it holds the following properties.

(1) $\lambda \leq GR_\tau(\lambda, r) \leq GC_\tau(\lambda, r) \leq GS_\tau(\lambda, r) \leq C_\tau(\lambda, r)$ and

$$\lambda \leq GW_\tau(\lambda, r) \leq GS_\tau(\lambda, r) \leq C_\tau(\lambda, r).$$

(2) For each operator $W \in \{GC_\tau, GS_\tau, GR_\tau\}$,

$$W(\lambda, r) \vee W(\mu, r) = W(\lambda \vee \mu, r).$$

(3) For each operator $W \in \{GC_\tau, GS_\tau, GR_\tau, GW_\tau\}$,

$$W(W(\lambda, r), r) = W(\lambda, r).$$

(4) For each operator $W \in \{GC_\tau, GS_\tau, GR_\tau, GW_\tau\}$,

$$C_\tau(W(\lambda, r), r) = C_\tau(\lambda, r) = W(C_\tau(\lambda, r), r).$$

PROOF : (1) It is easily proved from Theorem 2.2.

(2) For $W = GR_\tau$, since $\lambda, \mu \leq \lambda \vee \mu$, we have

$$GR_\tau(\lambda, r) \vee GR_\tau(\mu, r) \leq GR_\tau(\lambda \vee \mu, r).$$

Suppose $GR_\tau(\lambda, r) \vee GR_\tau(\mu, r) \not\geq GR_\tau(\lambda \vee \mu, r)$. There exists $x \in X$ and $t \in (0, 1)$ such that

$$GR_\tau(\lambda, r)(x) \vee GR_\tau(\mu, r)(x) < t < GR_\tau(\lambda \vee \mu, r)(x). \quad \text{... } (A)$$

Since, $GR_\tau(\lambda, r)(x) < t$ and $GR_\tau(\mu, r)(x) < t$, there exist r-gfrc sets λ_1, μ_1 with $\lambda \leq \lambda_1$ and $\mu \leq \mu_1$ such that $\lambda_1(x) < t$ and $\mu_1(x) < t$. Since $\lambda \vee \mu \leq \lambda_1 \vee \mu_1$ and $\lambda_1 \vee \mu_1$ is r-gfrc from Theorem 2.3(1), we have $GR_\tau(\lambda \vee \mu, r)(x) \leq \lambda_1 \vee \mu_1(x) < t$. It is a contradiction for (A). Thus, $GR_\tau(\lambda, r) \vee GR_\tau(\mu, r) = GR_\tau(\lambda \vee \mu, r)$.

Other cases are similarly proved.

(3) For $W = GR_\tau$, we only show $GR_\tau(\lambda, r) \geq GW_\tau(GW_\tau(\lambda, r), r)$. Suppose $GW_\tau(\lambda, r) \not\geq GW_\tau(GW_\tau(\lambda, r), r)$. There exists $x \in X$ and $t \in (0, 1)$ such that

$$GW_\tau(\lambda, r)(x) < t < GW_\tau(GW_\tau(\lambda, r), r)(x). \quad \text{... } (B)$$

Since $GW_\tau(\lambda, r)(x) < t$, there exists r-gfwsc set λ_1 with $\lambda \leq \lambda_1$ such that

$$GW_\tau(\lambda, r)(x) \leq \lambda_1(x) < t.$$

Since $\lambda \leq \lambda_1$, we have $GW_\tau(\lambda, r) \leq \lambda_1$. Again $GW_\tau(GW_\tau(\lambda, r), r) \leq GW_\tau(\lambda_1, r) = \lambda_1$. Hence, $GW_\tau(GW_\tau(\lambda, r), r)(x) \leq \lambda_1(x) < t$. It is a contradiction for B). Thus, $GW_\tau(\lambda, r) = GW_\tau(GW_\tau(\lambda, r), r)$.

Other cases are similarly proved.

(4) By Theorem 2.3(4), since $\tau(\overline{1} - C_\tau(\lambda, r)) \geq r$, we have, for each $W \in \{GC_\tau, GS_\tau, GR_\tau, GW_\tau\}$, $W(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.

From (1), we only show that $C_\tau(GS_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.

Since $\lambda \leq GS_\tau(\lambda, r)$, we have $C_\tau(GS_\tau(\lambda, r), r) \geq C_\tau(\lambda, r)$.

Suppose $C_\tau(GS_\tau(\lambda, r), r) \not\leq C_\tau(\lambda, r)$. There exists $x \in X$ and $t \in (0, 1)$ such that

$$C_\tau(GS_\tau(\lambda, r), r)(x) > t \Rightarrow t > C_\tau(\lambda, r)(x).$$

Since $C_\tau(\lambda, r)(x) < t$, by the definition of C_τ, there exists $\rho \in I^X$ with $\lambda \leq \rho$ and $\tau(\bar{1} - \rho) \geq r$ such that

$$C_\tau(GS_\tau(\lambda, r), r)(x) > t > \rho(x) \geq C_\tau(\lambda, r)(x).$$

On the other hand, since $\rho = C_\tau(\rho, r)$ is r-gfsc, $\lambda \leq \rho$ implies

$$GS_\tau(\lambda, r) \leq GS_\tau(\rho, r) = GS_\tau(C_\tau(\rho, r), r) = C_\tau(\rho, r) = \rho.$$

Thus $C_\tau(GS_\tau(\lambda, r), r) \leq \rho$. It is a contradiction, Hence

$$C_\tau(GS_\tau(\lambda, r), r) \leq C_\tau(\lambda, r).$$

Example 2.8 — Ler X and τ be as in Example 2.4. For $0 < r \leq \frac{1}{2}$, since $GW_\tau(\chi_{\{a\}}, r) = \chi_{\{a\}}$, $GW_\tau(\chi_{\{b\}}, r) = \chi_{\{b\}}$ and $GW_\tau(\chi_{\{a,b\}}, r) = 1$, then

$$\bar{1} = GW_\tau(\chi_{\{a,b\}}, r) \neq GW_\tau(\chi_{\{a\}}, r) \vee GW_\tau(\chi_{\{b\}}, r) = \chi_{\{a,b\}}.$$

***Theorem* 2.9** — *Let (X, τ) be a fts. For each $r \in I_0$ and $r \in I^X$, we define an operator GI_τ (resp. GIS_τ, $GIR_\tau GIW_\tau$) as follows:*

$$GI_\tau(\lambda, r) = \vee \left\{\mu \in I^X \mid \mu \leq \lambda, \ \ \mu \text{ is } r\text{–}gfo\right\},$$

(resp. μ is r-gfso, r-gfro, r-gfwso). Then $O(\bar{1} - \lambda, r) = \bar{1} - P(\lambda, r)$ where

$$O \in \{GI_\tau, GIR_\tau GIS_\tau GIW_\tau\}$$

and

$$P \in \{GC_\tau, GR_\tau GS_\tau GW_\tau\}, \text{ respectively.}$$

PROOF : For each $r \in I_0$ and $r \in I^X$, $O = GIW_\tau$ and $P = GW_\tau$, we have

$$GIW_\tau(\bar{1} - \lambda, r) = \vee \left\{\mu \in I^X \mid \mu \in \bar{1} - \lambda, \ \ \mu \text{ is } r\text{–}gfwso\right\}$$

$$= \bar{1} - \wedge \left\{\bar{1} - \mu \in I^X \mid \bar{1} - \mu \geq \lambda, \bar{1} - \mu, \text{ is } r\text{–}gfwsc\right\}$$

$$= \bar{1} - GW_\tau(\lambda, r).$$

3. SEVERAL TYPES OF FUZZY REGULARITIES

Definition 3.1 — Let (X, τ) be a fts.

(1) X is called r-fuzzy regular if for each $\tau(\mu) \geq r$, there exists a family $\{\nu_i \in I^X \mid \tau(\nu_i) \geq r\}$ such that $\mu = \vee_{i \in \Gamma} \nu_i$ with $C_\tau(\nu_i, r) \leq \mu$.

(2) X is called r-fuzzy s-regular if for each r-semi open μ, there exists a family $\{\nu_i \in I^X \mid \nu_i$ is r-semiopen$\}$ such that $\mu = \vee_{i \in \Gamma} \nu_i$ with $SC_\tau(\nu_i, r) \leq \mu$.

(3) X is called r-fuzzy weakly s-regular if for each r-semi open μ, there exists a family $\{\nu_i \in I^X \mid \nu_i$ is r-semiopen$\}$ such that $\mu = \vee_{i \in \Gamma} \nu_i$ with $C_\tau(\nu_i, r) \leq \mu$.

(4) X is called r-fuzzy almost regular if for each r-regular open μ, there exists a family $\{\nu_i \in I^X \mid \nu_i$ is r-regular open$\}$ such that $\mu = \vee_{i \in \Gamma} \nu_i$ with $C_\tau(\nu_i, r) \leq \mu$.

Notation 3.2 — Let (X, τ) be a fts and $x_t \in Pt(X)$. We denote

$$S_\tau(x_t, r) = \{\mu \in I^X \mid x_t q \mu, \quad \mu \text{ is } r\text{-semiopen}\}.$$

We easily prove the following lemma.

Lemma 3.3 — For $\lambda, \lambda_i, \mu \in I^X$ and $x_t \in Pt(X)$, we have

(1) $\lambda \leq \mu$ iff $x_t q \lambda$ implies $x_t q \mu$.

(2) $x_t q \vee_{i \in \Gamma} \lambda_i$ iff there exists $i \in \Lambda$ such that $x_t q \lambda_i$.

Theorem 3.4 — *Let (X, τ) be a fts and $r \in I_0$. Then the following statements are equivalent.*

(1) X is r-fuzzy weakly s-regular.

(2) For all $\mu \in S_\tau(x_t, r)$, there exists $\nu \in S_\tau(x_t, r)$ with $SC_\tau(\nu, r) \leq \mu$.

(3) For each $x_t \in Pt(X)$ and each r-semiclosed $\lambda \in I^X$ with $x_t \notin \lambda$, there exists $\nu \in S_\tau(x_t, r)$ and r-semiopen $\mu \in I^X$ such that $\lambda \leq \mu$ and $\mu \bar{q} \nu$.

(4) For each $x_t \in Pt(X)$ and each r-semiclosed $\lambda \in I^X$ with $x_t \notin \lambda$, there exists $\nu \in S_\tau(x_t, r)$ and r-gfwso μ such that $\lambda \leq \mu$ and $\mu \bar{q} \nu$.

(5) For each r-semiclosed $\lambda \in I^X$,

$$\lambda = \wedge \{SC_\tau(\nu, r) \mid \lambda \leq \nu, \quad \nu \text{ is } r\text{-gfwso}\}.$$

(6) For each $\lambda \in I^X$ with r-semiclosed λ and $\rho \not\leq \lambda$, there exist $\nu \in S_\tau(x_t, r)$ and r-gfwso μ such that $\lambda \leq \mu$, $pq\nu$ and $\mu \bar{q} \nu$.

PROOF : (1) $\Rightarrow$ (2). Let $\mu \in \mathcal{S}_\tau(x_t, r)$ be given. Since (X, τ) is r-fuzzy weakly s-regular, there exists a family $\{\nu_i \mid \nu_i$ is r-semiopen$\}$ such that $\mu = \vee_{i \in \Gamma} \nu_i$ with $SC_\tau(\nu_i, r) \le \mu$. Since $x_t q\left(\mu = \vee_{i \in \Gamma} \nu_i\right)$, by Lemma 3.3(2), there exists a $i \in \Gamma$ such that $\nu_i \in \mathcal{S}_\tau(x_t, r)$ with $SC_\tau(\nu_i, r) \le \mu$.

(2) $\Rightarrow$ (1). Let $\mu \in \mathcal{S}_\tau(x_t, r)$, there exists $\nu_i \in \mathcal{S}_\tau(x_t, r)$ such that $SC_\tau(\nu_i, r) \le \mu$. Let $\{\nu_i \in \mathcal{S}_\tau(x_t, r) \mid i \in \Lambda,\ SC_\tau(\nu_i, r) \le \mu\}$ be the family satisfying the above condition. Trivially, $\vee_{i \in \Gamma} \nu_i$.

We only show that, by Lemma 3.3(1), $x_t\ q = \vee_{i \in \Gamma} \nu_i$ for each $x_t q \mu$. For each $\mu \in \mathcal{S}_\tau(x_t, r)$, by (2), there exists $\nu \in \mathcal{S}_\tau(x_t, r)$ such that $SC_\tau(\nu_i, r) \le \mu$. So, $x_t q\, \nu_i$ implies x_t $q \vee_{i \in \Gamma} \nu_i$. Thus $\mu = \vee_{i \in \Lambda} \nu_i$ such that $SC_\tau(\nu_i, r) \le \mu$.

(2) $\Rightarrow$ (3). Let $x_t \notin \lambda$ with r-semiclosed λ. Then $\bar{1} - \lambda \in \mathcal{S}_\tau(x_t, r)$. By (2), there exists $\nu \in \mathcal{S}_\tau(x_t, r)$ such that $SC_\tau(\nu, r) \le \bar{1} - \lambda$. Put $\mu = \bar{1} - SC_\tau(\nu, r)$. By Theorem 1.3(1), μ is r-semiopen such that $\lambda \le \mu$ and $\mu \bar{q}\, \nu$.

(3) $\Rightarrow$ (4). Since every r-semiopen is r-gfwso from Theorem 2.6(5), it is easily proved.

(4) $\Rightarrow$ (5). Suppose there exists r-semiclosed $\lambda \in I^X$ such that

$$\lambda \ngeq \wedge \{SC_\tau(\nu, r) \mid \lambda \le \nu, \quad \nu \text{ is } r\text{-gfwso}\}.$$

Then there exist $x \in X$ and $t \in I_0$ such that

$$\lambda(x) < t < \wedge \{SC_\tau(\nu, r)(x) \mid \lambda \le \nu, \quad \nu \text{ is } r\text{-gfwso}\}. \qquad \ldots (C)$$

Since $x_t \notin \lambda$, (4), there exists $\mu \in \mathcal{S}_\tau(x_t, r)$ and r-gfwso ν such that $\lambda \le \nu$ and $\mu \bar{q}\, \nu$. By the definition of SC_τ, we have $SC_\tau(\nu, r)(x) \le \bar{1} - \mu(x) < t$. It is contradiction for (C). Thus

$$\lambda = \wedge \{SC_\tau(\nu, r) \mid \lambda \le \nu, \quad \nu \text{ is } r\text{-gfwso}\}.$$

(5) $\Rightarrow$ (6). Let $\lambda \in I^X$ be r-semiclosed with $\rho \nleq \lambda$. Then $x_t \in Pt(X)$ such that $x_t \in \rho$ and $t > \lambda(x)$. But (5), there exists r-gfwso μ such that $\lambda \le \mu$ and $SC_\tau(\mu, r)(x) < t$. Put

$\nu = \overline{1} - SC_\tau(\mu, r)$. By Theorem 1.3(1), ν is r-semiopen, that is, $\nu \in \mathcal{S}_\tau(x_t, r)$ such that $\lambda \leq \mu$, $\rho\, q\, \nu$ and $\mu\, \overline{q}\, \nu$.

(6) $\Rightarrow$ (2). For all $\mu \in \mathcal{S}_\tau(x_t, r)$, $t > \overline{1} - \mu(x)$. So $x_t \not\leq \overline{1} - \mu$ and $\overline{1} - \mu$ is r-semiclosed. By (6), there exists $\nu \in \mathcal{S}_\tau(x_t, r)$ and r-gfwso ρ such that $\overline{1} - \mu \leq \rho$ and $\rho\, \overline{q}\, \nu$. Thus $\nu \leq \overline{1} - \rho \leq \mu$. Since $\overline{1} - \rho$ is r-gfwsc and μ is r-semiopen, $SC_\tau(\overline{1} - \rho, r) \leq \mu$. It implies $\nu \in \mathcal{S}_\tau(x_t, r)$ such that $SC_\tau(\nu, r) \leq \mu$.

The following two theorems are similarly proved as in Theorem 3.4.

Theorem 3.5 — *Let (X, τ) be a fts and $r \in I_0$. Then the following statements are equivalent.*

(1) X is r-fuzzy regular.

(2) For all $\mu \in Q(x_t, r)$, there exists $\nu \in Q(x_t, r)$ with $C_\tau(\nu, r) \leq \mu$.

(3) For each $x_t \in Pt(X)$ and each $\lambda \in I^X$ with $\tau(\overline{1} - \lambda) \geq r$ and $x_t \notin \lambda$, there exists $\nu \in Q_\tau(x_t, r)$ and $\mu \in I^X$ with $\tau(\mu) \geq r$ such that $\lambda \leq \mu$ and $\mu\, \overline{q}\, \nu$.

(4) For each $x_t \in Pt(X)$ and each $\lambda \in I^X$ with $\tau(\overline{1} - \lambda) \geq r$ and $x_t \notin \lambda$, there exists $\nu \in Q_\tau(x_t, r)$ and r-gfo μ such that $\lambda \leq \mu$ and $\mu\, \overline{q}\, \nu$.

(5) For each $\lambda \in I^X$ with $\tau(\overline{1} - \lambda) \geq r$,

$$\lambda = \wedge \{C_\tau(\nu, r) \mid \lambda \leq \nu, \quad \nu \text{ is } r\text{-gfo}\}.$$

(6) For each $\lambda \in I^X$ with $\tau(\overline{1} - \lambda) \geq r$ and $\rho \not\leq \lambda$, there exist $\nu \in Q_\tau(x_t, r)$ and r-gfo μ such that $\lambda \leq \mu$, $\rho\, q\, \nu$ and $\mu\, \overline{q}\, \nu$.

Theorem 3.6 — *Let (X, τ) be a fts and $r \in I_0$. Then the following statements are equivalent.*

(1) X is r-fuzzy s-regular.

(2) For all $\mu \in \mathcal{S}_\tau(x_t, r)$, there exists $\nu \in \mathcal{S}_\tau(x_t, r)$ with $C_\tau(\nu, r) \leq \mu$.

(3) For each $x_t \in Pt(X)$ and each r-semiclosed $\lambda \in I^X$ with $x_t \notin \lambda$, there exists $\nu \in \mathcal{S}_\tau(x_t, r)$ and $\mu \in I^X$ with $\tau(\mu) \geq r$ such that $\lambda \leq \mu$ and $\mu\, \overline{q}\, \nu$.

(3) For each $x_t \in Pt(X)$ and each r-semiclosed $\lambda \in I^X$ with $x_t \notin \lambda$, there exists $\nu \in \mathcal{S}_\tau(x_t, r)$ and $\mu \in I^X$ with $\tau(\mu) \geq r$ such that $\lambda \leq \mu$ and $\mu \bar{q} \nu$.

(4) For each $x_t \in Pt(X)$ and each r-semiclosed $\lambda \in I^X$ with $x_t \notin \lambda$, there exists $\nu \in \mathcal{S}_\tau(x_t, r)$ and r-gfso μ such that $\lambda \leq \mu$ and $\mu \bar{q} \nu$.

(5) For each r-semiclosed $\lambda \in I^X$,

$$\lambda = \wedge \{SC_\tau(\nu, r) \mid \lambda \leq \nu, \quad \nu \text{ is r-gfso}\}.$$

(6) For each r-semiclosed $\lambda \in I^X$ with $\rho \not\leq \lambda$, there exist $\nu \in \mathcal{S}_\tau(x_t, r)$ and r-gfso μ such that $\lambda \leq \mu$, $\rho q \nu$ and $\mu \bar{q} \nu$.

Theorem 3.7 — *Let (X, τ) be a fts and $r \in I_0$. Then the following statements are equivalent.*

(1) X is r-fuzzy almost regular.

(2) For all $\mu \in \mathcal{R}_\tau(x_t, r)$, there exists $\nu \in \mathcal{R}_\tau(x_t, r)$ with $C_\tau(\nu, r) \leq \mu$.

(3) For each $x_t \in Pt(X)$ and each r-regular closed $\lambda \in I^X$ with $x_t \notin \lambda$, there exists $\nu \in \mathcal{R}_\tau(x_t, r)$ and $\mu \in I^X$ with $\tau(\mu) \geq r$ such that $\lambda \leq \mu$ and $\mu \bar{q} \nu$.

(4) For each $x_t \in Pt(X)$ and each r-regular closed $\lambda \in I^X$ with $x_t \notin \lambda$, there exists $\nu \in \mathcal{R}_\tau(x_t, r)$ and r-gfro μ such that $\lambda \leq \mu$ and $\mu \bar{q} \nu$.

(5) For each r-regular closed $\lambda \in I^X$,

$$\lambda = \wedge \{C_\tau(\nu, r) \mid \lambda \leq \nu, \quad \nu \text{ is r-gfro}\}.$$

(6) For each r-regular closed $\lambda \in I^X$ with $\rho \not\leq \lambda$, there exist $\nu \in Q_\tau(x_t, r)$ and r-gfro μ such that $\lambda \leq \mu$, $\rho q \nu$ and $\mu \bar{q} \nu$.

PROOF : (5) $\Rightarrow$ (6). Let $\lambda \in I^X$ be r-regular closed with $\rho \not\leq \lambda$. Then $x_t \in Pt(X)$ such that $x_t \in \rho$ and $t > \lambda(x)$. By (5), there exists r-gfro μ such that $\lambda \leq \mu$ and $C_\tau(\mu, r)(x) < t$. Put $\nu = \bar{1} - C_\tau(\mu, r)$. By Definition 1.2, $\tau(\nu) \geq r$. So, $\nu \in Q_\tau(x_t, r)$ such that $\lambda \leq \mu$, $\rho q \nu$ and $\mu \bar{q} \nu$.

(6) $\Rightarrow$ (2). Let all $\mu \in \mathcal{R}_\tau(x_t, r)$, $t > \overline{1} - \mu(x)$, i.e., $x_t \not\leq \overline{1} - \mu(x)$ and $\overline{1} - \mu$ is r-regular closed. By (6), there exists $\nu \in Q_\tau(x_t, r)$ and r-gfro ρ such that $\overline{1} - \mu \leq \rho$ and $\rho \overline{q} \nu$. Thus $\nu \leq \overline{1} - \rho \leq \mu$. Since $\overline{1} - \rho$ is r-gfrc and μ is r-regular open, $C_\tau(\overline{1} - \rho, r) \leq \mu$. It implies $\omega = I_\tau(C_\tau(\nu, r), r) \in \mathcal{R}_\tau(x_t, r)$ such that $C_\tau(\omega, r) \leq \mu$.

Other cases are similarly proved as in Theorem 3.4.

***Theorem* 3.8** — *Let (X, τ) be a fts and $r \in I_0$.*

(1) If X is r-fuzzy regular, then it is r-fuzzy almost regular.

(2) If X is r-fuzzy s-regular, then it is r-fuzzy weakly s-regular.

PROOF : For all $\mu \in \mathcal{R}_\tau(x_t, r)$, there exists $\rho \in Q_\tau(x_t, r)$ such that $C_\tau(\rho, r) \leq \mu$. By Theorem 1.6 (11-12), $I_\tau(C_\tau(\rho, r), r) \in \mathcal{R}_\tau(x_t, r)$ such that

$$C_\tau(I_\tau(C_\tau(\rho, r), r), r) = C_\tau(\rho, r) \leq \mu.$$

Thus, X is r-fuzzy almost regular.

(1) Since $SC_\tau(\lambda, r) \leq C_\tau(\lambda, r)$, it is trivial.

The converse of the above theorem need not be true from the following example.

Example 3.9 — Let $X = \{a, b, c\}$ be a set and $a_{0.6} \in Pt(X)$. We define fuzzy topologies $\tau, \eta : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \{\overline{0}, \overline{1}\} \\ \frac{1}{2}, & \text{if } \lambda = \{\chi_{\{a\}}, \chi_{\{b,c\}}\} \\ \frac{1}{2}, & \text{if } \lambda = \{a_{0.6}, a_{0.6} \vee \chi_{\{b,c\}}\} \\ 0, & \text{otherwise,} \end{cases}$$

$$\eta(\lambda) = \begin{cases} 1, & \text{if } \lambda = \{\overline{0}, \overline{1}\} \\ \frac{1}{2}, & \text{if } \lambda = \overline{0.4} \\ 0, & \text{otherwise.} \end{cases}$$

(1) For $0 < r \leq \frac{1}{2}$, since $\chi_{\{a\}}$ and $\chi_{\{b,c\}}$ are r-regular open and r-regular closed sets,

$C_\tau(\chi_{\{a\}}, r) = \chi_{\{a\}}$ and $C_\tau(\chi_{\{b,c\}}, r) = \chi_{\{b,c\}}$, then (X, τ) is r-fuzzy almost regular.

(2) For $a_{0.6} \in Q_\tau\left(a_{0.7}, \frac{1}{2}\right)$, for all $\mu \in Q_\tau\left(a_{0.7}, \frac{1}{2}\right)$ we have $C_\tau\left(\mu, \frac{1}{2}\right) \not\leq a_{0.6}$. So, (X, τ) is not a $\frac{1}{2}$-fuzzy regular. Moreover, for $a_{0.6} \in S\left(a_{0.5}, \frac{1}{2}\right)$ and for all $\mu \in S\left(a_{0.5}, \frac{1}{2}\right)$, we have $C_\tau\left(\mu, \frac{1}{2}\right) \not\leq a_{0.6}$ and $SC_\tau\left(\mu, \frac{1}{2}\right) \not\leq a_{0.6}$. Thus (X, τ) is neither a $\frac{1}{2}$-fuzzy s-regular nor a $\frac{1}{2}$-fuzzy weakly s-regular.

(3) For $0 < r \leq \frac{1}{2}$ and $\overline{0.4} \leq \lambda \leq \overline{0.6}$, λ are r-semiopen and r-semiclosed sets on (X, τ), that is $SC_\eta(\lambda, r) = \lambda$, then (X, η) is r-fuzzy weakly s-regular. Since $\overline{0.3} \leq \overline{0.4}$ and $\overline{0.6} = C_\eta(\overline{0.3}, r) \not\leq \overline{0.4}$. So (X, η) is not r-fuzzy s-regular. For $\overline{0.4} \in \mathcal{R}_\eta\left(a_{0.7}, \frac{1}{2}\right)$, for all $\mu \in \mathcal{R}_\eta\left(a_{0.7}, \frac{1}{2}\right)$ we have $C_\tau\left(\mu, \frac{1}{2}\right) \not\leq \overline{0.4}$. Hence (X, η) is not a $\frac{1}{2}$-fuzzy almost regular.

Definition 3.10 — Let (X, τ) be a fts, $\lambda \in I^X$, $x_t \in P_t(X)$ and $r \in I_0$. A fuzzy point x_t is called:

(1) a r-semicluster point of λ if $\mu \, q \, \lambda$, for every $\mu \in \mathcal{S}_\tau(x_t, r)$,

(2) a r-θ-semicluster point of λ if $C_\tau(\mu, r) \, q \lambda$ for every $\mu \in \mathcal{S}_\tau(x_t, r)$,

(3) a r-θ-weakly semicluster point of λ if $SC_\tau(\mu, r) \, q \lambda$ for every $\mu \in \mathcal{S}_\tau(x_t, r)$,

(4) a r-θ-regular cluster point of λ if $C_\tau(\mu, r) \, q \lambda$ for every $\mu \in \mathcal{R}_\tau(x_t, r)$.

We define mappings $ST_\tau(SW_\tau, RT_\tau) : I^X \times I_o \to I^X$ as follows:

$$ST_\tau(\lambda, r) = \vee \{x_t \in P_t(X) \mid x_t \text{ is a } r\text{-}\theta\text{-semicluster point of } \lambda\}$$

$$SW_\tau(\lambda, r) = \vee \{x_t \in P_t(X) \mid x_t \text{ is a } r\text{-}\theta\text{-weakly semicluster point of } \lambda\}$$

$$RT_\tau(\lambda, r) = \vee \{x_t \in P_t(X) \mid x_t \text{ is a } r\text{-}\theta\text{-regular cluster point of } \lambda\}.$$

Theorem 3.11 — *Let (X, τ) be a fts. For each $\lambda, \mu, \rho \in I^X$ and $r \in I_o$, we have the following properties:*

(1) $SC_\tau(\lambda, r) = \vee \{x_t \in P_t(X) \mid x_t$ *is a r-semicluster cluster point of* $\lambda\}$.

(2) $RT_\tau(\lambda, r) = \wedge \{\mu \in I^X \mid \lambda \leq I_\tau(\mu, r), \quad C_\tau(I_\tau(\mu, r), r) \leq \mu\}$.

(3) $ST_\tau(\lambda, r) = \wedge \{\mu \in I^X \mid \lambda \leq I_\tau(\mu, r), \quad I_\tau(C_\tau(\mu, r), r) \leq \mu\}$.

(4) $SW_\tau(\lambda, r) = \wedge \{\mu \in I^X \mid \lambda \leq SI_\tau(\mu, r), \quad I_\tau(C_\tau(\mu, r), r) \leq \mu\}$.

(5) x_t *is a r-θ-semicluster cluster point of* λ *iff* $x_t \in ST_\tau(\lambda, r)$.

(6) x_t *is a r-semicluster cluster point of* λ *iff* $x_t \in SC_\tau(\lambda, r)$.

(7) x_t *is a r-θ-weakly semicluster cluster point of* λ *iff* $x_t \in SW_\tau(\lambda, r)$.

(8) x_t *is a r-θ-regular cluster point of* λ *iff* $x_t \in RT_\tau(\lambda, r)$.

(9) $\mathcal{R}_\tau(x_t, r) \subset \mathcal{Q}_\tau(x_t, r) \subset \mathcal{S}_\tau(x_t, r)$.

(10) $SC_\tau(\lambda, r) \leq SW_\tau(\lambda, r) \leq ST_\tau(\lambda, r)$ *and* $D_\tau(\lambda, r) \leq RT_\tau(\lambda, r)$.

(11) $ST_\tau(\lambda, r) \leq T_\tau(\lambda, r) \leq RT_\tau(\lambda, r)$.

(12) If $\rho \leq I_\tau(C_\tau(\rho, r), r)$ *then*

$$ST_\tau(\rho, r) \leq C_\tau(\rho, r) = D_\tau(\rho, r) = T_\tau(\rho, r) = RT_\tau(\rho, r).$$

PROOF : (1) Put $\rho = \vee \{x_t \in P_t(X) \mid x_t$ is a r-semicluster point of $\lambda\}$.

Suppose $SC_\tau(\lambda, r) \not\leq \rho$. Then there exist $x \in X$ and $t \in (0, 1)$ such that $SC_\tau(\lambda, r)(x) > t > \rho(x)$. Then x_t is not a r-semicluster point of λ. So, there exists $\mu \in \mathcal{S}_\tau(x_t, r)$, $\lambda \leq \bar{1} - \mu$ and $\bar{1} - \mu$ is r-semiclosed. By the definition of SC_τ in Theorem 1.6, $SC_\tau(\lambda, r)(x) \leq (\bar{1} - \mu)(x) < t$. It is a contradiction. Thus, $SC_\tau(\lambda, r) \leq \rho$.

Suppose $SC_\tau(\lambda, r) \not\geq \rho$. Then there exists a r-semicluster point $y_s \in Pt(X)$ of λ such that $SC_\tau(\lambda, r)(y) < s \leq \rho(y)$. By the definition of SC_τ, there exists r-semiclosed μ with $\lambda \leq \mu$ such that

$SC_\tau(\lambda, r)(y) < \mu(y) < s < \rho(y)$. Then $\bar{1} - \mu \in \mathcal{S}_\tau(y_s, r)$ and $\lambda \bar{q}\, \bar{1} - \mu$. Hence y_s is not a r-semicluster point of λ. It is a contradiction. Thus $SC_\tau(\lambda, r) \geq \rho$.

(2) Put $\gamma = \wedge \{\mu \in I^X \mid \lambda \leq I_\tau(\mu, r), \quad C_\tau(I_\tau(\mu, r), r) = \mu\}$. Suppose $RT_\tau(\lambda, r) \not\geq \gamma$. Then there exists $x \in X$ and $t \in (0, 1)$ such that $RT_\tau(\lambda, r)(x) < t < \gamma(x)$. Then x_t is not a r-θ-regular cluster point of λ. So, there exists $\mu \in \mathcal{R}_\tau(x_{ts}, r)$, $C_\tau(\mu, r) \leq \bar{1} - \lambda$. Thus,

$$\lambda \leq \bar{1} - C_\tau(\mu, r) = I_\tau(\bar{1} - \mu, r), C_\tau(I_\tau(\bar{1} - \mu, r), r) = \bar{1} - \mu.$$

Hence $\gamma(x) \leq (\bar{1} - \mu(x) < t$. It is a contradiction. Thus $RT_\tau(\lambda, r)(x) \geq \gamma$.

Suppose $RT_\tau(\lambda, r) \not\leq \gamma$. Then there exists r-θ-regular cluster point of y_s of λ such that $RT_\tau(\lambda, r)\, y \geq s > \gamma(y)$. By the definition of γ, there exists μ with $\lambda \leq I_\tau(\mu, r)$, $C_\tau(I_\tau(\mu, r), r) = \mu$ such that $RT_\tau(\lambda, r)(y) \geq s > \mu(y) \geq \gamma(y)$. Then μ is r-regular closed and $\bar{1} - \mu \in \mathcal{R}_\tau(y_s, r)$. Furthermore, $\lambda \leq \mu$ implies $\lambda \bar{q}\, \bar{1} - I_\tau(\mu, r) = C_\tau(\bar{1} - \mu, r)$. Hence, y_s is not a r-θ-regular cluster point of λ. It is a contradiction. Thus $RT_\tau(\lambda, r) \leq \gamma$.

(3) It is similarly proved as in (1) and (2).

(4) Put $\delta = \wedge \{\mu \in I^X \mid \lambda \leq SI_\tau(\mu, r), \quad I_\tau(C_\tau(\mu, r), r) \leq \mu\}$.

Suppose $SW_\tau(\lambda, r) \not\geq \delta$. Then there exists $x \in X$ and $t \in (0, 1)$ such that $SW_\tau(\lambda, r)(x) < t < \delta(x)$. Then x_t is not a r-θ-weakly semicluster point of λ. So, there exists $\mu \in \mathcal{S}_\tau(x_t, r)$, $SC_\tau(\mu, r) \leq \bar{1} - \lambda$. Thus $\bar{1} - \mu$ is r-semiclosed and

$$\lambda \leq \bar{1} - SC_\tau(\mu, r) = SI_\tau(\bar{1} - \mu, r).$$

Hence $\gamma(x) \leq (\bar{1} - \mu)(x) < t$. It is a contradiction. Thus $SW_\tau(\lambda, r) \geq \delta$.

Suppose $SW_\tau(\lambda, r) \not\leq \delta$. Then there exists r-θ-weakly semicluster point y_s of λ such that $SW_\tau(\lambda, r)(y) \geq s > \delta(y)$. By the definition of δ, there exists μ with $\lambda \leq SI_\tau(\mu, r)$ and $I_\tau(C_\tau(\mu, r), r) \leq \mu$ such that

$$SW_\tau(\lambda, r)(y) \geq s > \mu(y) \geq \delta(y).$$

Then μ is r-semiclosed and $\overline{1} - \mu \in \mathcal{S}_\tau(y_s, r)$. So, $\lambda \leq SI_\tau(\mu, r) = \overline{1} - SC_\tau(\overline{1} - \mu, r)$ implies $\lambda \bar{q}\, SC_\tau(\overline{1} - \mu, r)$. Hence, y_s is not a r-θ-weakly semicluster point of λ. It is a contradiction. Thus $SW_\tau(\lambda, r) \leq \delta$.

(5) $\Rightarrow$ It is trivial.

$\Leftarrow$ Suppose that x_t is not a r-θ-semi cluster point of λ. Then there exists $\mu \in \mathcal{S}_\tau(x_t, r)$ scuh that $C_\tau(\mu, r) \leq \overline{1} - \lambda$. Thus,

$$\lambda \leq \overline{1} - C_\tau(\mu, r) = I_\tau(\overline{1} - \mu, r).$$

By (3), we have $ST_\tau(\lambda, r)(x) \leq (\overline{1} - \mu)(x) < t$. Hence $x_t \notin ST_\tau(\lambda, r)$.

(6-11) are similarly proved as in (5).

(9-11) are easily proved from definitions of operators.

(12) Let $\rho \leq I_\tau(C_\tau(\rho, r), r)$ be given. Since $C_\tau(\rho, r)$ is r-semiclosed, by (3), $ST_\tau(\rho, r) \leq C_\tau(\rho, r)$. Moreover, since

$$C_\tau(\rho, r) \leq C_\tau(I_\tau(C_\tau(\rho, r), r)) \leq C_\tau(\rho, r)$$

then $C_\tau(\rho, r)$ is r-regular closed. By (2), $RT_\tau(\rho, r) \leq C_\tau(\rho, r)$. From (10) and Theorem 1.6 (8-10),

$$C_\tau(\rho, r) = D_\tau(\rho, r) = T_\tau(\rho, r) = RT_\tau(\rho, r).$$

Theorem 3.12 — *A fts (X, τ) is r-fuzzy weakly s-regular iff $SW_\tau(\lambda, r) = SC_\tau(\lambda, r)$ for each $\lambda \in I^X$.*

PROOF : We only show that $SW_\tau(\lambda, r) \leq SC_\tau(\lambda, r)$. Suppose there exists $\lambda \in I^X$ and $r \in I_o$ such that $SW_\tau(\lambda, r) \not\leq SC_\tau(\lambda, r)$. Then there exists $x \in X$ and $t \in I_o$ such that

$$SW_\tau(\lambda, r)(x) > t > SC_\tau(\lambda, r)(x).$$

Since $SC_\tau(\lambda, r)(x) < t$, x_t is not r-semicluster point of λ. Then there exists $\mu \in \mathcal{S}_\tau(x_t, r)$ such that $\lambda \leq \overline{1} - \mu$. Since (X, τ) is r-fuzzy weakly s-regular, for $\mu \in \mathcal{S}_\tau(x_t, r)$, there exists $\rho \in \mathcal{S}_\tau(x_t, r)$ such that $SC_\tau(\rho, r) \leq \mu$. Thus $\lambda \leq \overline{1} - \mu \leq \overline{1} - SC_\tau(\rho, r)$. It follows $SC_\tau(\rho, r)\, \bar{q}\, \lambda$. Thus x_t is not r-θ-weakly semicluster point of λ. So, $\mu \in \mathcal{S}_\tau(x_t, r) < t$. It is a contradiction.

Conversely, for each $\mu \in \mathcal{S}_\tau(x_t, r)$, $t > (\overline{1} - \mu)(x) = SC_\tau(\overline{1} - \mu)(x)$. Since $SW_\tau(\overline{1} - \mu, r) = SC_\tau(\overline{1} - \mu, r)$, then x_t is not r-fuzzy θ-weakly semicluster point of $\overline{1} - \mu$. Then there exists

$\rho \in \mathcal{S}_\tau(x_t, r)$ such that $SC_\tau(\rho, r) \leq \mu$. Hence (X, τ) is r-fuzzy weakly s-regular.

The following corollaries are similarly proved as the above theorem

Corollary 3.13. A fts (X, τ) is r-fuzzy weakly regular iff $ST_\tau(\lambda, r) = SC_\tau(\lambda, r)$ for each $\lambda \in I^X$.

Corollary 3.14. A fts (X, τ) is r-fuzzy weakly s-regular iff $T_\tau(\lambda, r) = C_\tau(\lambda, r)$ for each $\lambda \in I^X$.

Theorem 3.15 — *A fts (X, τ) is r-fuzzy almost regular iff $RT_\tau(\lambda, r) = D_\tau(\lambda, r)$ for each $\lambda \in I^X$.*

PROOF : We only show that $RT_\tau(\lambda, r) \leq D_\tau(\lambda, r)$. Suppose there exist $\lambda \in I^X$ and $t \in I_o$ such that $RT_\tau(\lambda, r) \not\leq D_\tau(\lambda, r)$. Then there exists $\lambda \in I^X$ and $t \in I_o$ such that

$$RT_\tau(\lambda, r) > t > D_\tau(\lambda, r)(x).$$

Since $D_\tau(\lambda, r)(x) < t$, x_t is not r-δ-cluster point of λ. Then there exists $\mu \in \mathcal{R}_\tau(x_t, r)$ such that

$$\lambda \leq \bar{1} - \mu = C_\tau(I_\tau(\bar{1} - \mu, r), r).$$

Since (X, τ) is r-fuzzy almost regular, for $\mu \in \mathcal{R}_\tau(x_t, r)$, there exists $\rho \in \mathcal{R}_\tau(x_t, r)$ such that $C_\tau(\rho, r) \leq \mu$. Thus $\lambda \leq \bar{1} - \mu \leq \bar{1} - C_\tau(\rho, r)$. It follows $C_\tau(\rho, r)\, \bar{q}\, \lambda$. Thus x_t is not r-θ-regular cluster point of λ. So, $RT_\tau(\lambda, r)(x) < t$. It is a contradiction.

Conversely, for each $\mu \in \mathcal{R}_\tau(x_t, r)$, we have

$$t > (\bar{1} - \mu)(x) = C_\tau(I_\tau(\bar{1} - \mu, r), r)\ (x).$$

Since, by Theorem 1.6 (8),

$$RT_\tau(\bar{1} - \mu, r) = D_\tau(\bar{1} - \mu, r) = \bar{1} - \mu,$$

we have $RT_\tau(\bar{1} - \mu, r)(x) = \bar{1} - \mu(x) < t$. So, x_t is not r-θ-regular cluster point of $\bar{1} - \mu$. Then there exists $\rho \in \mathcal{R}_\tau(x_t, r)$ such that $C_\tau(\rho, r) \leq \mu$. Hence (X, τ) is r-fuzzy almost regular.

Definition 3.16 — Let (X, τ) be a fts, $\lambda \in I^X$ and $r \in I_o$. A fuzzy set λ is called:

(1) r-fuzzy compact if for each a family $\{v_i \mid i \in \Gamma, \tau(v_i) \geq r\}$ such that $\lambda \leq \vee_{i \in \Gamma} v_i$, there exists a finite subset Γ_0 of Γ such that $\lambda \leq \vee_{i \in \Gamma_0} v_i$.

(2) r-fuzzy s-compact if for each a family $\{v_i \mid i \in \Gamma, v_i \text{ is } r\text{-semiopen}\}$ such that $\lambda \leq \vee_{i \in \Gamma} v_i$, there exists a finite subset Γ_0 of Γ such that $\lambda \leq \vee_{i \in \Gamma_0} v_i$.

(3) r-fuzzy almost compact if for each a family $\{v_i \mid i \in \Gamma, v_i \text{ is } r\text{-regular open}\}$ such that $\lambda \leq \vee_{i \in \Gamma} v_i$, there exists a finite subset Γ_0 of Γ such that $\lambda \leq \vee_{i \in \Gamma_0} v_i$.

Remark 3.17 : Let (X, τ) be a fts. We have:

r-fuzzy s-compact $\Rightarrow$ r-fuzzy compact $\Rightarrow$ r-fuzzy almost compact.

Theorem 3.18 — *If (X, τ) is r-fuzzy s-regular and λ is r-fuzzy s-compact, then λ is r-gfsc.*

PROOF : Let $\lambda \leq \mu$ with r-semiopen μ. Since (X, τ) is r-fuzzy s-regular, there exists a family $\{v_i \mid v_i \ r\text{-semiopen}\}$ such that $\mu \leq \vee_{i \in \Gamma} v_i$ with $C_\tau(v_i, r) \leq \mu$. Since λ is r-fuzzy s-compact, there exists a finite subset Γ_0 of Γ such that $\lambda \leq \vee_{i \in \Gamma_0} v_i$. Moreover, by Theorem 1.6 (13),

$$\bigvee_{i \in \Gamma_0} C_\tau(v_i, r) = C_\tau\left(\bigvee_{i \in \Gamma_0} v_i, r\right).$$

It implies

$$C_\tau(\lambda, r) \leq C_\tau\left(\bigvee_{i \in \Gamma_0} v_i, r\right) \leq \mu.$$

Hence, λ is $r$gfsc.

The following corollaries are similarly proved as in Theorem 3.18.

Corollary 3.19 — If (X, τ) is r-fuzzy regular and λ is -r-fuzzy compact, then λ is r-fgc.

Corollary 3.20 — If (X, τ) is r-fuzzy almost regular and λ is -r-fuzzy almost compact, then λ is r-gfrc.

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