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ON THE GENERALISED LAGRANGE SPACE AND CORRESPONDING LAGRANGE SPACE ARISING FROM THE METRIC TENSOR $g_{ij}(x, y) + (1/c^2)\, y_i\, y_j$

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This paper has been devoted to the study of some aspects of generalised Lagrange space L^n with metric tensor $G_{ij}(x, y) = g_{ij}(x, y) + (1/c^2)\, y_i\, y_j$ ($g_{ij}(x, y)$) being metric tensor of a Finsler space F^n and $y_i = g_{ij}(x, y)\, y^j$) and corresponding Lagrange space L^{*n} arising from the Lagrangian $L^* = (g_{ij}(x, y)\, y^i\, y^j)^{1/2}$. The properties of L^n and L^{*n} have been discussed in special cases when the associated Finsler space F^n is (a) Landsberg or (b) Berwald or (c) Minkowskian.

Key Words: Generalised Lagrange; Lagrange Spaces; Berwald; Landsberg; Minkowskian Spaces

1. INTRODUCTION

The geometrical properties of Lagrange and generalised Lagrange spaces have been studied by Miron and Anastasiei[5], Kawaguchi and Miron[2, 3], Singh[7, 8] and several other authors. An important class of generalised Lagrange space is given by $L^n = (M^n, a_{ij}(x, y))$, where M^n is a differentiable manifold of dimension n,

$$a_{ij}(x, y) = \gamma_{ij}(x) + (1/c^2)\, y_i\, y_j, \qquad \text{... (1.1)}$$

$y = (y^i)$ is the element of support $\gamma_{ij}(x)$ is the metric tensor of a Riemannian space and $y_i = \gamma_{ij}(x, y)\, y^j$. The metric tensor $a_{ij}(x, y)$ given above was used by Beil[1] in the study of electrodynamics, where constant c occurring in (1.1) was given a physical meaning as velocity of light. In present paper the metric tensor given by (1.1) has been generalised by replacing $\gamma_{ij}(x)$ as the right hand side of (1.1) by a Finsler metric tensor $g_{ij}(x, y)$.

2. GENERALISED LAGRANGE SPACE AND CORRESPONDING LAGRANGE SPACE ASSOCIATED WITH A FINSLER METRIC TENSOR

Consider a generalised Lagrange space $L^n = (M^n, G_{ij}(x, y))$ with the metric tensor

$$G_{ij}(x, y) = g_{ij}(x, y) + (1/c^2)\, y_i\, y_j, \qquad \text{... (2.1)}$$

where $g_{ij}(x, y)$ is metric tensor of a Finsler space $F^n, y_i = g_{ij}(x, y) y^j$ and c is a constant usually called the velocity of light.

The reciprocal tensor, G^{ij} of G_{ij} is obtained as

$$G^{ij} = g^{ij} - \frac{1}{a_1 c^2} y^i y^j, \qquad \text{... (2.2)}$$

where g^{ij} is the tensor reciprocal to g_{ij}, $a_\sigma = 1 + \frac{\sigma}{c^2} F^2$, for the integer σ and $F^2 = g_{ij}(x, y) y^i y^j$.

The d-tensor field $\underline{C}_{jhk}$, of L^n, is defined by

$$\underline{C}_{jhk} = (1/2)(\partial G_{jh}/\partial y^k + \partial G_{hk}/\partial y^j - \partial G_{jk}/\partial y^h). \qquad \text{... (2.3)}$$

Using the eq. (2.1) and the relation $\partial y_i/\partial y^k = g_{ik}$, we get

$$\underline{C}_{jhk} = C_{jhk} + (1/c^2) g_{jk} y_h, \qquad \text{... (2.4)}$$

where

$$C_{jhk} = (1/2)(\partial g_{jh}/\partial y^k + \partial g_{hk}/\partial y^j / \partial g_{jk}/\partial y^h) = (1/2) \partial g_{jh}/\partial y^k$$

is the d-tensor field of the associated Finsler space F^n.

A simple calculation based on eqs. (2.2) and (2.4) will give

$$\underline{C}^i_{jk} \overset{\text{def}}{=} G^{ih} \underline{C}_{jhk} = C^i_{jk} + \frac{1}{a_1 c^2} g_{jk} y^i, \qquad \text{... (2.5)}$$

where we have used the definition $C^i_{jk} = g^{ih} C_{jhk}$ and the relation $C_{jhk} y^h = 0$.

The metric tensor G_{ij} is used in defining the Lagrangian L^*, given by the relation:

$$L^{*2} = G_{ij} y^i y^j. \qquad \text{... (2.6)}$$

This Lagrangian gives a metric tensor

$$G^*_{ij} = (1/2) \partial^2 L^{*2} / \partial y^i \partial y^j, \qquad \text{... (2.7)}$$

of a Lagrange space $L^{*n} = (M^n G^*_{ij})$, corresponding to the given generalised Lagrange space L^n.

The eqs. (2.1) and (2.6) yield

$$L^{*2} = F^2 + (1/c^2) F^4. \qquad \text{... (2.8)}$$

A direct calculation based on eqs. (2.7), (2.8) and the fact $g_{ij} = (1/2) \partial^2 F^2(x, y)/\partial y^i \partial y^j$ gives

$$G^*_{ij} = a_2\, g_{ij} + (4/c^2)\, y_i\, y_j, \qquad \text{... (2.9)}$$

where we have used the relations $\partial F^2/\partial y^i = 2y_i$ and $C_{ijk}\, y^i = 0$.

The contravariant components, G^{*ij} of G^*_{ij}, are given by

$$G^{*ij} = (1/a_2)\left\{ g^{ij} - \frac{4}{c^2\, a_6}\, y^i\, y^j \right\}. \qquad \text{... (2.10)}$$

Defining d-tensor field $\underline{C}^*_{jhk}$, of L^{*n}, in a manner analogous to (2.3) and using the relation $\partial a_2/\partial y^k = (4/c^2)\, y_k$, we get

$$\underline{C}^*_{jhk} = a_2\, C_{jhk} + (2/c^2)\,(g_{hk}\, y_j + g_{jk}\, y_h + g_{jh}\, y_k). \qquad \text{... (2.11)}$$

A calculation based on eqs. (2.10) and (2.11) gives

$$\underline{C}^{*i}_{jk} = G^{*\,ih}\, \underline{C}^*_{jhk} = C^i_{jk} + \frac{2}{a_2\, c^2}\left(\delta^i_k\, y_j + \delta^i_j\, y_k + \frac{a_2}{a_6}\, y^i\, g_{j_k} - \frac{8}{c^2\, a_6}\, y^i\, y_j\, y_k \right), \qquad \text{... (2.12)}$$

where we have used the relation $C_{jh_k}\, y^h = 0$.

Theorem 2.1 — *The d-tensor fields* $\underline{C}_{jhk}$ *and* $\underline{C}^*_{jhk}$ *of* L^n *and* L^{*n} *respectively, are not indicatory tensor fields.*

PROOF : In view of relation (2.4), $\underline{C}_{jhk}\, y^j = 0$ implies $y_h\, y_k = 0$ and $\underline{C}_{jhk}\, y^h = 0$ implies $(F^2/c^2)\, g_{jk} = 0$, which are impossible. Hence $\underline{C}_{jhk}$ is not an indicatory tensor field.

Again the relation (2.11) and the condition $\underline{C}^*_{jhk}\, y^j = 0$ give

$$g_{hk} + 2\, l_h\, l_k = 0, \qquad \text{... (2.13)}$$

where

$$l^h = y^h/F \text{ and } l_h = g_{hr}\, l^r. \qquad \text{... (2.14)}$$

The eq. (2.13) yields $g_{hk} = -2\, l_h\, l_k$, which is again impossible. Therefore $\underline{C}^*_{jhk}$ is not an indicatory tensor field.

3. GEODESICS IN L^{*n}

The equations of a geodesic of L^{*n} are given by

$$d^2 x^i/dt^2 + 2\, G^{*i}(x, \dot{x}) = 0 \qquad \text{... (3.1)}$$

where $\dot{x}^i = dx^i/dt = y^i$ and $G^{*\,i}(x, y)$ is given by Kawaguchi and Miron[2] in the form

$$G^{*i} = (1/4)\, G^{*ij} \left\{ (\partial^2 L^{*2}/\partial y^j \partial x^k)\, y^k - \partial L^{*2}/\partial x^j \right\}. \qquad \dots (3.2)$$

A simple calculation based on equations $L^{*2} = a_1 F^2$ and $a_1 = 1 + F^2/c^2$ gives

$$\partial L^{*2}/\partial x^k = a_2\, \partial F^2/\partial x^k. \qquad \dots (3.3)$$

Differentiating this equation with respect to y^j and transvecting the resulting equation by y^k we get

$$(\partial^2 L^{*2}/\partial y^j \partial x^k)\, y^k = (4/c^2)\, y_j\, (\partial g_{rs}/\partial x^k)\, y^r y^s y^k + 2a_2\, (\partial g_{rj}/\partial x^k)\, y^r y^k, \qquad \dots (3.4)$$

where we have used the relations

$$(a)\ F^2 = g_{rs}\, y^r y^s,\ (b)\ \partial F^2/\partial x^k = (\partial g_{rs}/\partial x^k)\, y^r y^s,\ (c)\ \partial a_2/\partial y^j = (4/c^2)\, y_j \qquad \dots (3.5)$$

and the homogeneity property of g_{rs} in the calculation.

We proceed to introduce Christoffel's symbols

$$\gamma_{jhk} = (1/2)\ \{(\partial g_{jh}/\partial x^k + \partial g_{hk}/\partial x^j - \partial g_{jk}/\partial x^h), \qquad \dots (3.6)$$

$$\gamma^i_{jk} = g^{ih}\, \gamma_{jhk} \qquad \dots (3.7)$$

of the associated Finsler space. The eq. (3.6) will yield

$$\partial g_{rs}/\partial x^k = \gamma_{rsk} + \gamma_{srk}. \qquad \dots (3.8)$$

A direct calculation based on eqs. (3.2), (3.4), (3.5)b, (2.10) and (3.8) will give

$$G^{*i} = (1/2)\, \gamma^i_{jk}\, y^j y^k = G^i, \qquad \dots (3.9)$$

where we have used the identity $\dfrac{2}{a_2 c^2} - \dfrac{8F^2}{c^4 a_2 a_6} - \dfrac{2}{c^2 a_6} = 0$ and G^i is the quantity corresponding to G^{*i} for the associated Finsler space F^n.

Since a geodesic of F^n is given by $d^2 x^i/dt^2 + 2G^i(x, \dot{x}) = 0$, we have the following :

***Theorem* 3.1** — *The geodesics of the Lagrange space L^{*n} are given by the same equations as the geodesics of the associated Finsler space F^n.*

Also we have

***Theorem* 3.2** — *The space F^n and L^{*n} are in geodesic correspondence.*

4. Non-Linear Connection N^{*i}_j, Canonical Metrical Connection L^i_{jk} and Berwald Connection $\underline{G}^i_{jk}$ of L^n

The non-linear connection N^{*i}_j is defined by the relation

$$N^{*\,i}_{j} = \partial G^{*\,i}/\partial y^{j}. \qquad \text{... (4.1)}$$

The eq. (3.9) shows that

$$N^{*\,i}_{j} = \partial G^{i}/\partial y^{j} = F^{i}_{jk}\, y^{k}, \qquad \text{... (4.2)}$$

where the last relation in (4.2) has been quoted from Rund[6] and F^{i}_{jk} stand for Cartan's connection coefficients of the associated Finsler space F^{n}.

The canonical metrical connection L^{i}_{jk}, of L^{n} is defined by

$$L^{i}_{jk} = (1/2)\, G^{ih}\, (\delta G_{jh}/\delta x^{k} + \delta G_{hk}/\delta x^{j} - \delta G_{jk}/\delta x^{h}), \qquad \text{... (4.3)}$$

where

$$\delta/\delta x^{k} = (\partial/\partial x^{k} - N^{*r}_{k}\, \partial/\partial y^{r}).$$

A simple calculation based on eqs. (4.4), (4.2) and the relations

$$\partial g_{jh}/\partial y^{s} = 2C_{jhs} \qquad \text{... (4.5)}$$

and

$$\partial g_{jh}/\partial x^{k} = F_{jhk} + F_{hjk} + 2C_{jhs}\, F^{s}_{rk}\, y^{r} \qquad \text{... (4.6)}$$

(Rund[6]), for $F_{jhk} = g_{ih}\, F^{i}_{jk}$, yields

$$\delta g_{jk}/\delta x^{h} = F_{jkh} + F_{kjh}, \qquad \text{... (4.7)}$$

which will give

$$\delta g_{jh}/\delta x^{k} + \delta g_{hk}/\delta x^{j} - \delta g_{jk}/\delta x^{h} = 2F_{jhk}. \qquad \text{... (4.8)}$$

The relation $y_{j} = g_{jr}\, y^{r}$ and eqs. (4.4) and (4.8) give

$$\delta y_{j}/\delta x^{k} = F_{jrk}\, y^{r}. \qquad \text{... (4.9)}$$

Using eqs. (2.1) and (4.3) we find

$$L^{i}_{jk} = (1/2)\, G^{ih}\, [(\delta g_{jh}/\delta x^{k} + \delta g_{hk}/\delta x^{j} - \delta g_{jk}/\delta x^{h})$$

$$+ (1/c^{2})\, \{\delta(y_{j}\, y_{h})/\delta x^{k} + \delta(y_{h}\, y_{k})/\delta x^{j} - \delta(y_{j}\, y_{k})/\delta x^{h}\}]. \qquad \text{... (4.10)}$$

Substituting from (1.2), (4.8) and (4.9) in (4.10) we obtain

$$L^{i}_{jk} = F^{i}_{jk}, \qquad \text{... (4.11)}$$

where we have used the identity $(1/c^2) - 1/(c^2 a_1) - F^2/(a_1 c^4) = 0$.

Hence we have the following:

Theorem* 4.1— *For the generalised Lagrange space L^n, the components $\overset{*}{N}{}^i_j$, L^i_{jk} and $\underline{C}^i_{jk}$ of the Lagrange connection are given by (4.2), (4.11) and (2.5) respectively.

In order to obtain the Berwald's connection coefficients $\underline{G}^i_{jk}$, of L^n, we consider the eq. (3.9). This gives

$$\underline{G}^i_{jk} = \partial^2 G^{*i}/\partial y^j \partial y^k = \partial^2 G^i/\partial y^j \partial y^k = G^i_{jk} = F^i_{jk} + C^i_{jk \mid l} y^l, \quad \dots (4.12)$$

where the last relation has been taken from Matsumoto[4], G^i_{jk} stands for Berwald's connection coefficients of F^n and symbol $_\mid$ stands for Cartan's $h\_$ derivative in F^n. We have thereby proved the following:

***Theorem* 4.2 — *The Berwald's connection of L^n coincides with that of the associated Finsler space F^n and it is given by* (4.12).**

5. Canonical Metrical Connection $\overset{*}{L}{}^i_{jk}$ of L^{*n}

The connection $\overset{*}{L}{}^i_{jk}$, of L^{*n}, is given by

$$\overset{*}{L}{}^i_{jk} = (1/2)\ G^{*ih}\,(\delta \overset{*}{G}_{jh}/\delta x^k + \delta \overset{*}{G}_{hk}/\delta x^j - \delta \overset{*}{G}_{jk}/\delta x^h), \quad \dots (5.1)$$

where $\delta/\delta x^k$ and $\overset{*}{G}_{jk}$ are given by eqs. (4.4) and (2.9) respectively.

A straight forward calculation based on eqs. (4.2), (4.7) and relations

$$a_2 = 1 + (2/c^2)\, g_{rs}\, y^r y^s, \qquad \delta y^i/\delta x^k = -F^i_{kr} y^r,$$

gives

$$\delta a_2/\delta x^k = 0. \quad \dots (5.2)$$

Using this equation along with eqs. (1.9), (4.7) and (4.9) we get

$$\delta \overset{*}{G}_{jk}/\delta x^h = a_2(F_{jkh} + F_{kjh}) + (4/c^2)\,(F_{krh}\, y^r y_j + F_{jrh}\, y^r y_k). \quad \dots (5.3)$$

A calculation based on eqs. (5.1), (5.3) and (2.10) will give

$$\overset{*}{L}{}^i_{jk} = F^i_{jk}, \quad \dots (5.4)$$

where we have used the identity $(1/a_2) - (1/a_6) - 4F^2/(c^2 a_2 a_6) = 0$.

Hence we have proved the following:

Theorem 5.1 — *In the space* L^{*n}, *the components* N^{*i}_{j}, L^{*i}_{jk} *and* $\underline{C}^{*i}_{jk}$ *of the Lagrange connection are given by eqs. (4.2), (5.4) and (2.12) respectively.*

The Berwald's connection coefficients G^{*i}_{jk}, of L^{*n}, are obtained from G^{*i} given by (3.9). Therefore Theorem 4.2 yields the following:

Theorem 5.2 — *The Berwald's connection coefficients of the generalised Lagrange space* L^n *coincide with those of associated Finsler space* F^n *or those of the corresponding Lagrange space* L^{*n} *i.e.*

$$\underline{G}^{*i}_{jk} = \underline{G}^{i}_{jk} = G^{i}_{jk} = F^{i}_{jk} + C^{i}_{jk\,|\,l}\, y^{l}, \qquad \text{... (5.5)}$$

where $G^{i}_{jk}, \underline{G}^{i}_{jk}$ and $\underline{G}^{*i}_{jk}$ stand for coefficients of Berwald's connection in F^n, L^n and L^{*n} respectively.

6. Derivatives in F^n, L^n and L^{*n}

If $K^{i}_{j}(x, y)$ is a mixed tensor field of the manifold M^n, then its v, h and Berwald's covariant derivatives in L^n, are defined by

$$K^{i}_{j\,\underline{|}\,h} = \partial K^{i}_{j}/\partial y^{h} + K^{r}_{j}\,\underline{C}^{i}_{rh} - K^{i}_{r}\,\underline{C}^{r}_{jh}, \qquad \text{... (6.1)}$$

$$K^{i}_{j\,|\,h} = \delta K^{i}_{j}/\delta x^{h} + K^{r}_{j}\,L^{i}_{rh} - K^{i}_{r}\,L^{r}_{jh}, \qquad \text{... (6.2)}$$

and

$$K^{i}_{j(h)} = \delta K^{i}_{j}/\delta x^{h} + K^{r}_{j}\,\underline{G}^{i}_{rh} - K^{i}_{r}\,\underline{G}^{r}_{jh}, \qquad \text{... (6.3)}$$

respectively.

Similar definitions for these derivatives in F^n and L^{*n} can be given.

Remark : Since $L^{i}_{jk} = L^{*i}_{jk} = F^{i}_{jk}$; $\underline{G}^{i}_{jk} = \underline{G}^{*i}_{jk} = G^{i}_{jk}$ and $N^{*i}_{j} = \partial G^{*i}/\partial y^{j} = \partial G^{i}/\partial y^{j}$, the h-covariant and Berwald's covariant derivatives, in L^n, as given by (6.2) and (6.3) are identical with the corresponding derivatives in F^n or in L^{*n}. Therefore same symbols are used for these two derivatives in all the three spaces. The v-covariant derivatives in F^n and L^{*n} will be denoted by symbols | and v = *| respectively.

We now proceed to prove the following:

Theorem 6.1 — *In the space* L^n, *the covariant derivatives as given by eqs. (6.1), (6.2) and (6.3) satisfy the following identities:*

$$(a)\ (i)\ G_{ij|h}=0\ (ii)\ a_{\sigma|h}=0\ (iii)\ \overset{*}{L}_{|h}=0\ (iv)\ \overset{1}{L}_{|h}=0, \qquad \text{... (6.4)}$$

$$(b)\ (i)\ G_{ij}\underline{|}_h=0\ (ii)\ a_\sigma\underline{|}h=(2\sigma/c^2)\,y_h\ (iii)\ L^*\underline{|}_h=(a_2/a_1)\,L_h$$

$$(iv)\ L^i\underline{|}_h=(1/L^*)\,(\delta^i_h-l^i\,l_h),$$

$$(c)\ (i)\ G_{ij(h)}=g_{ij(h)}=-2C_{ijh\,|\,0}\ (ii)\ a_{\sigma(h)}=0\ \ (iii)\ \overset{*}{L}_{(h)}=0\ \textit{and}\ (iv)\ \overset{i}{L}_{(h)}=0,$$

where

$$(a)\ L^i=y^i/L^*\quad (b)\ L_h=G_{ih}\,L^i \qquad \text{... (6.5)}$$

and suffix 0 stands for transvection with respect to y^i.

PROOF : (a) The eq. (2.1), $g_{ij|h}=0$, $y_{i|h}=0$ give $G_{ij|h}=0$. Further, $a_\sigma = 1 + (\sigma F^2/c^2)$ and $F_{|h}=0$ will give $a_{\sigma|h}=0$. The identity $\overset{*}{L}_{|h}=0$ will follow immediately from the eqs. (2.6), $G_{ij|h}=0$ and $\overset{i}{y}_{|h}=0$. The identity $\overset{i}{L}_{|h}=0$ is an immediate consequence of eqs. (6.5)a and $\overset{i}{y}_{|h}=0$, $\overset{*}{L}_{|h}=0$.

(b) The eqs. (6.1) and $G_{ir}\underline{C}^r_{j\,h}=\underline{C}_{jih}$ will give

$$G_{ij|h}=\partial G_{ij}/\partial y^h-\underline{C}_{jih}-\underline{C}_{ijh} \qquad \text{... (6.6)}$$

A simple calculation based on eqs. (2.1), (2.4), (6.6) and relation $\partial y_i/\partial y^h=g_G$ will give $\underline{G}_{ij}\underline{|}_h=0$.

The relation $a_\sigma = 1 + (\sigma F^2/c^2)$, for a natural number σ and $F\underline{|}_h=\partial F/\partial y^h=l_h$ will prove $a_\sigma\underline{|}_h=(2\sigma/c^2)\,y_h$.

Further, the relation $L^*=(a_1)^{1/2}F, F\underline{|}_h=l_h$ and identity (6.4)b (ii) proved above give $L^*\underline{|}_h=(a_2/a_1)\,L_h$, where we have used the identity $L_h=(a_1)^{1/2}\,l_h$.

The eqs. (6.1), (2.5) and the identity $C^i_{r\,h}\,y^r=0$ will give

$$Y^i\underline{|}_h=\delta^i_h+1/(a_1\,c^2)\,y^i\,y_h.$$

A direct calculation based on eqs. (6.5)a, (6.4)b (iii), (6.7) and relations

$$L_h=(a_1)^{1/2}\,l_h, y^i=Fl^i, L^*=(a_1)^{1/2}\,F$$

and

$$\left\{ F^2/(a_1 c^2) - (a_2/a_1) \right\} = -1$$

will prove (6.4)*b* (*iv*).

In order to establish the identities given in (6.4)*c* we shall use the remark mentioned just before Theorem (6.1).

The relation (2.1) and identity $y_{i(h)} = 0$ give

$$G_{ij(h)} = g_{ij(h)} = -2\, C_{ijh\,|\,0} \text{ (Rund[6]).}$$

The relations $a_\sigma = 1 + (\sigma F^2/c^2)$, $L^* = (a_1)^{1/2} F$ and $F_{(h)} = 0$ will give $a_{\sigma(h)} = 0$ and $\overset{*}{L}_{(h)} = 0$.

The identity $\overset{i}{L}_{(h)} = 0$ is an immediate consequence of (6.5)*a*, $y^{i}_{(h)} = 0$, $\overset{*}{L}_{(h)} = 0$.

We shall now prove the following:

Theorem 6.2 — *In the space L^{*n}, the covariant derivatives corresponding to eqs. (6.1), (6.2) and (6.3) satisfy the following identities*:

$$(a)\ \overset{*}{G}_{ij} \overset{*}{|}_{h} = 0 \quad (b)\ \overset{*}{G}_{ij\,|\,h} = 0 \qquad \text{... (6.8)}$$

and

$$(c)\ \overset{*}{G}_{ij\,(h)} = a g_{ij(h)} = -2 a_2\, C_{ijh\,|\,0}.$$

PROOF : (*a*) By definition

$$\overset{*}{G}_{ij} \overset{*}{|}_{h} = \partial \overset{*}{G}_{ij}/\partial y^{h} - \underline{\overset{*}{C}}_{ijh} - \underline{\overset{*}{C}}_{ijh}. \qquad \text{... (6.9)}$$

The eqs. (2.9), (2.11), (6.9) and relations

$$\partial a_2/\partial y^h = (4/c^2)\, y_h,\ \ \partial y_i/\partial y^h = g_{ih}$$

will prove $\overset{*}{G}_{ij} \overset{*}{|}_{h} = 0$.

Part (*b*) and (*c*) of this theorem follow immediately from the remark given before Theorem 6.1 and the eqs. (2.9), (6.4)*a ii*, (6.4)*c* (*ii*) and facts $y_{i\,|\,h} = 0$, $y_{i(h)} = 0$.

7. GENERALISED LAGRANGE SPACE (M^n, G_{ij}) AND CORRESPONDING LAGRANGE SPACE $(M^n, \overset{*}{G}_{ij})$ ARISING FROM SOME SPECIAL FINSLER SPACES (M^n, g_{ij})

The notions of Landsberg and Berwald spaces have been given by Matsumoto[4]. Using these concepts we give the following:

Definition 7.1 — The space F^n or L^n or L^{*n} is a Landsberg space if the Berwald connection is *h*-metrical in the space concerned.

According to this definition F^n or L^n or L^{*n} is a Landsberg space iff $g_{ij(k)} = 0$ or $G_{ij(k)} = 0$ or $\overset{*}{G}_{ij(k)} = 0$ respectively.

From the eqs. (6.4)*c* (*i*) and (6.8)*c* we have the following:

Theorem 7.1 — *If anyone of the following three conditions holds then the remaining two also hold.*

(i) The generalised Lagrange space L^n is Landsberg.

(ii) The corresponding Lagrange space L^{*n} is Landsberg.

(iii) The associated Finsler space F^n is Landsberg.

We now give the following:

Definition 7.2 — The space F^n or L^n or L^{*n} is said to be a Berwald space if the Berwald's connection of the space concerned is independent of directional element y^i.

The eq. (5.5) proves the following:

Theorem 7.2 — *If anyone of the following three conditions is satisfied then the remaining two are also satisfied.*

(i) The generalised Lagrange space L^n is is a Berwald space.

(ii) The corresponding Lagrange space L^{*n} is a Berwald space.

(iii) The associated Finsler space F^n is a Berwald space.

It has been shown by Matsumoto[4] that a Berwald Finsler space is a Landsberg space. The Theorems (7.1) and (7.2), therefore, yield the following:

Theorem 7.3 — If any one of the three conditions given in Theorem 7.2 is satisfied then the remaining five conditions mentioned in Theorems 7.1 and 7.2 are also satisfied.

Consider a Finsler space $F^n = (M^n, g_{ij})$, the generalised Lagrange space $L^n = (M^n, G_{ij})$ and corresponding Lagrange space $L^{*n} = (M^n, \overset{*}{G}_{ij})$ where $G_{ij}, \overset{*}{G}_{ij}$ are defined by (2.1) and (2.9) respectively. Suppose that M^n undergoes a coordinate transformation $\bar{x}^i = \bar{x}^i(x^j)$ which results in the transformation of g_{ij}, G_{ij} and $\overset{*}{G}_{ij}$ into $\bar{g}_{ij}, \bar{G}_{ij}$ and $\overset{*}{\bar{G}}_{ij}$ respectively. The following lemma follows immediately from (2.1) and (2.9).

Lemma 7.1 — If M^n undergoes coordinate transformation given by $\bar{x}^i = \bar{x}^i(x^j)$ then

$$\bar{G}_{ij} = \bar{g}_{ij} + (1/c^2)\,\bar{y}_i\bar{y}_j \text{ and } \overset{*}{\bar{G}}_{ij} = a_2\,\bar{g}_{ij} + (4/c^2)\,\bar{y}_i\bar{y}_j$$

where $\bar{y}_i = \bar{g}_{ij}\,\bar{y}^j.$

We now give the following:

Definition 7.3 — The space F^n is said to be Minkowskian (Matsumoto[4]) if there exists a coordinate system in which the metric tensor g_{ij}, of F^n, is a function of directional element y^i only i.e. $\partial g_{ij}/\partial x^k = 0$. Similar definitions for L^n or L^{*n} to be Minkowskian can be given.

We shall now prove the following:

Theorem 7.4 — *The generalised Lagrange space L^n is Minkowskian if and only if associated Finsler space F^n is Minkowskian.*

PROOF : The condition F^n is Minkowskian will imply that for a given coordinate system g_{ij} and $y_i = g_{ij} y^j$ are independent of x^k. Therefore by eq. (2.1), G_{ij} is independent of x^k.

Conversely, if L^n is Minkowskian then Lemma 7.1, eq. (2.1) and condition $\partial G_{ij}/\partial x^k = 0$ will give

$$\partial g_{ij}/\partial x^k + (1/c^2)(y_i \partial y_j/\partial x^k + y_j \partial y_i/\partial x^k) = 0. \qquad \text{... (7.1)}$$

Transvecting this equation by y^j we get

$$a_1 \partial y_i/\partial x^k + y_i \partial F^2/\partial x^k = 0. \qquad \text{... (7.2)}$$

Again transvecting this equation by y^i we find

$$a_2 \partial F^2/\partial x^k = 0. \qquad \text{... (7.3)}$$

Eqs. (7.3), (7.2) and (7.1) yield $\partial g_{ij}/\partial x^k = 0$. Therefore, F^n is Minkowskian.

Also we have the following:

Theorem 7.5 — *The Lagrange space L^{*n} is Minkowskian if and only if associated Finsler space F^n is Minkowskian.*

PROOF : The condition F^n is Minkowskian implies that for a certain coordinate system g_{ij}, y_i and a_2 are independent of x^k. Therefore, G^*_{ij} is independent of x^k.

Conversely, if L^{*n} is Minkowskian then by using Lemma 7.1, the eq. (2.9) and condition $\partial G^*_{ij}/\partial x^k = 0$, for a certain coordinate system, we get

$$a_2 \partial g_{ij}/\partial x^k + (2/c^2)\, \partial F^2/\partial x^k\, g_{ij}$$
$$+ (4/c^2)(y_i \partial y_j/\partial x^k + y_j \partial y_i/\partial x^k) = 0. \qquad \text{... (7.4)}$$

where we have used the relation $\partial a_2/\partial x^k = (2/c^2)\,\partial F^2/\partial x^k$.

Transvecting (7.4), by y^j we get

$$a_6\,\partial y_i/\partial x^k + (6/c^2)\,y_i\,\partial F^2/\partial x^k = 0. \qquad \text{... (7.5)}$$

Transvecting this relation by y^i, we find

$$a_{12}\,\partial F^2\,\partial x^k = 0. \qquad \text{... (7.6)}$$

The eqs. (7.6), (7.5) and (7.4) yield $\partial F^2/\partial x^k = 0$, $\partial y_i/\partial x^k$ and $\partial g_{ij}/\partial x^k = 0$.

Therefore, F^n is Minkowskian.

Theorems 7.4 and 7.5 establish the following:

Theorem 7.6 — *If any one of the following three conditions is true then the remaining two are also true.*

(i) The generalised Lagrange space L^n is Minkowskian.

(ii) The corresponding Lagrange space L^{*n} is Minkowskian.

(iii) The associated Finsler space F^n is Minkowskian.

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REFERENCES

1. R. G. Beil, *Intl. J. Theor. Phys.*, **26** (1987), 189-97.
2. T. Kawaguchi and R. Miron, *Tensor N.S.*, **48** (1989), 52-63.
3. T. Kawaguchi and R. Miron, *Tensor N.S.*, **48** (1989), 152-68.
4. M. Matsumoto, *Foundations of Finsler Geometry and special Finsler Spaces*, Kaiseisha Press, 1986.
5. R. Miron and M. Anastasiei, *Vector bundle, Lagrange Spaces, Applications in Relativity*, Edtura Academiei, R. S. Romania, 1987.
6. H. Rund, *The differential geometry of Finsler Spaces*, Springer Verlag, 1959.
7. U. P. Singh, *J. Nat. Acad. Math.*, **13** (1999), 41-52.
8. U. P. Singh, *Varahmihir J. Math. Sci.*, **2** (2002), 23-32.