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ON THE GROWTH OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS

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In the paper we study the growth of composite entire and meromorphic functions and improve some known results.

Key Words: Entire Function; Meromorphic Function; Composition; Growth

1. INTRODUCTION AND DEFINITIONS

Let f and g be two transcendental entire functions defined in the open complex plane $\mathbb{C}$. It is well known[3] that $\lim_{r\to\infty} \frac{T(r, fog)}{T(r,f)} = \infty$ and $\lim_{r\to\infty} \frac{T(r, fog)}{T(r, g)} = \infty$. In[11] Singh proved some comparative growth of log $T(r, fog)$ and $T(r,f)$. Further in[11] he raised the question of investigating the comparative growth of log $T(fog)$ and $T(r, g)$ which he was unable to solve. However, some results on the comparative growth of log $T(r, fog)$ and $T(r, g)$ are proved in[5]. In the paper we further investigate the above question of Singh[11] and improve some earlier results.

If f and g are of positive lower order then Song and Yang[13] proved that

$$\lim_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r,f)} = \lim_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r, g)} = \infty,$$

where

$$\log^{[k]} x = \log(\log^{[k-1]} x) \text{ for } k = 1, 2, 3, \ldots\ldots \text{ and } \log^{[0]} x = x.$$

Also in the sequel we use the following notation:

$$\exp^{[k]} x = \exp(\exp^{[k-1]} x) \text{ for } k = 1, 2, 3, \ldots \text{ and } \exp^{[0]} x = x.$$

Since $M(r,f)$ and $M(r, g)$ are increasing function of r, Singh and Baloria[12] asked whether for sufficiently large $R = R(r)$

$$\limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(R,f)} < \infty \quad \text{and} \quad \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(R, g)} < \infty.$$

Singh and Baloria[12], Lahiri and Sharma[6], Liao and Yang[7] worked on this question. Now it is natural to investigate this problem for Nevanlinna's characteristic functions instead of maximum modulus functions. In the paper we throw some light on this problem for composite entire and meromorphic functions. We also study the comparative growth of the composition of the form *hok* and *fog*, where *f*, *h* are meromorphic and *g*, *k* are entire which improves some known results.

Definition — The order ρ_f and lower order λ_f of a meromorphic function is defined as

$$\rho_f = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r}$$

and

$$\lambda_f = \liminf_{r \to \infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire, one can easily verify that

$$\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log r}$$

and

$$\lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

Definition 2 — The hyper order $\bar{\rho}_f$ and hyper lower order $\bar{\lambda}_f$ of a meromorphic function f is defined as

$$\bar{\rho}_f = \limsup_{r \to \infty} \frac{\log^{[2]} T(r,f)}{\log r} \text{ and } \bar{\lambda}_f = \liminf_{r \to \infty} \frac{\log^{[2]} T(r,f)}{\log r}.$$

If f is entire, then

$$\bar{\rho}_f = \limsup_{r \to \infty} \frac{\log^{[3]} M(r,f)}{\log r} \text{ and } \bar{\lambda}_f = \liminf_{r \to \infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

Definition 3 — A function $\lambda_f(r)$ is called a lower proximate order of a meromorphic function f if

(*i*) $\lambda_f(r)$ is nonnegative and continuous for $r \geq r_0$, say;

(*ii*) $\lambda_f(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\lambda_f'(r-0)$ and $\lambda_f'(r+0)$ exist;

(*iii*) $\lim_{r \to \infty} \lambda_f(r) = \lambda_f < \infty$;

(*iv*) $\lim_{r \to \infty} r\lambda_f'(r) \log r = 0$; and

(v) $\liminf\limits_{r \to \infty} \dfrac{T(r,f)}{r^{\lambda_f(r)}} = 1.$

We do not explain the standard notations and definitions of the theory of entire and meromorphic functions because those are available in[14 & 4].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 — Let f be a meromorphic function. Then for $\delta (> 0)$ the function $r^{\lambda_f + \delta - \lambda_f(r)}$ is ultimately an increasing function of r.

Proof : Since $\frac{d}{dr} r^{\lambda_f + \delta - \lambda_f(r)} = \{\lambda_f + \delta - \lambda_f(r) - r\lambda'(r) \log r\} \, r^{\lambda_f + \delta - \lambda_f(r)} > 0$ for all sufficiently large values of r, the lemma is proved.

Lemma 2[8] — Let f be an entire function of finite lower order. If there exist entire functions a_i $(i = 1, 2, ..., n; n \leq \infty)$ satisfying $T(r, a_i) = o\,\{T(r,f)\}$ and

$$\sum_{i=1}^{n} \delta(a_i, f) = 1 \text{ then } \lim_{r \to \infty} \frac{T(r,f)}{\log M(r,f)} = \frac{1}{\pi}.$$

Lemma 3[9] — Let f and g be two entire functions. If $M(r, g) > \frac{2+\varepsilon}{\varepsilon} |g(0)|$ for any ε (> 0) then

$$T(r, fog) < (1 + \varepsilon)\, T(M(r, g) f).$$

In particular, if $g(0) = 0$ then

$$T(r, fog) \leq T(M(r, g), f)$$

for all $r > 0$.

Lemma 4[1] — If f is meromorphic and g is entire then for all large values of r

$$T(r, fog) \leq \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f).$$

Lemma 5[2] — Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity

$$T(r, fog) \geq T(\exp(r)^{\mu}, f).$$

Lemma 6[9] — Let f and g be transcendental entire functions with $\rho_g < \infty$, η be a constant satisfying $0 < \eta < 1$ and α be a positive number. Then

$$T(r, fog) + O(1) \geq N(r, 0; fog)$$

$$\geq \left(\log \frac{1}{\eta} \right) \left[\frac{N\left(M\left((\eta\, r)^{\frac{1}{1+\alpha}}, g\right), 0; f\right)}{\log\left(M\left((\eta\, r)^{\frac{1}{1+\alpha}}, g\right) - O(1)\right.} - O(1) \right]$$

as $r \to \infty$ through all values.

3. Main Results

In this section we present the main results of the paper.

Theorem 1 — *Let f and g be two non constant entire functions such that λ_f and λ_g are finite. Then*

$$(i)\ \liminf_{r \to \infty} \frac{\log T(r, fog)}{T(r, g)} \leq 3\rho_f 2^{\lambda g}$$

and

$$(ii)\ \limsup_{r \to \infty} \frac{\log T(r, fog)}{T(r, g)} \geq \frac{\lambda_f}{4^{\lambda g}}.$$

Proof : If $\rho_f = \infty$ then (i) is obvious. So we suppose that $\rho_f < \infty$. For two entire functions f and g, following two inequalities are well known.

$$T(r, f) \leq \log^+ M(r, f) \leq 3T(2r, f) \qquad \dots (1)$$

{cf.[4], p. 18} and

$$M(r, fog) \leq M(M(r, g), f). \qquad \dots (2)$$

For $\varepsilon (> 0)$ we get from (1)and (2) for all large values of r

$$T(r, fog) \leq \log M(M(r, g), f) \leq \{M(r, g)\}^{\rho_f + \varepsilon}$$

i.e.,

$$\frac{\log T(r, fog)}{T(r, g)} \leq (\rho_f + \varepsilon)\ \frac{\log M(r, g)}{T(r, g)};$$

since ε (> 0) is arbitrary, we get from above

$$\limsup_{r \to \infty} \frac{\log T(r, fog)}{T(r, g)} \leq \rho_f \limsup_{r \to \infty} \frac{\log M(r, fog)}{T(r, g)}; \qquad \dots (3)$$

since $\liminf\limits_{r \to \infty} \dfrac{\log T(r, g)}{r^{\lambda g(r)}} = 1$, for given $\varepsilon (0 < \varepsilon < 1)$ we get for a sequence of values of r tending to infinity.

$$T(r, g) < (1 + \varepsilon)\, r^{\lambda g(r)} \qquad \dots (4)$$

and for all large values of r

$$T(r, g) > (1-\varepsilon)\, r^{\lambda g(r)}. \qquad \dots (5)$$

Therefore for a sequence of values of r tending to infinity we get for any δ (> 0).

$$\frac{\log M(r, g)}{T(r, g)} \leq \frac{3(1+\varepsilon)}{1-\varepsilon} \frac{(2r)^{\lambda g+\delta}}{(2r)^{\lambda_g+\delta-\lambda_g(2r)}} \frac{1}{r^{\lambda g(r)}}$$

$$\leq \frac{3(1+\varepsilon)}{1-\varepsilon}\, 2^{\lambda_g+\delta}.$$

because $r^{\lambda_g+\delta-\lambda_g(r)}$ is ultimately an increasing function of r.

Since $\varepsilon(>0)$ and $\delta(>0)$ are arbitrary, we get

$$\liminf_{r\to\infty} \frac{\log M(r, g)}{T(r, g)} \leq 3.2^{\lambda_g}. \qquad \dots (6)$$

Now (i) follows from (3) and (4).

If $\lambda_f = 0$ then (ii) is obvious. So we suppose that $\lambda_f > 0$. For all values of r we know that[10].

$$T(r, fog) \geq \frac{1}{3} \log M\left\{\frac{1}{8} M\left(\frac{r}{4}, g\right) + O(1), f\right\}. \qquad \dots (7)$$

For $\varepsilon\,(0<\varepsilon<\min\{\lambda_f, 1\})$ we get for all large values of r $\log M(r, f) > r^{\lambda_f-\varepsilon}$ and $\log M(r, g) > r^{\lambda_g-\varepsilon}$.

So from (7), we get for all large values of r

$$T(r, fog) \geq \frac{1}{3}\left\{\frac{1}{8} M\left(\frac{r}{4}, g\right) + O(1)\right\}^{\lambda_f-\varepsilon}$$

$$\geq \frac{1}{3}\left\{\frac{1}{9} M\left(\frac{r}{4}, g\right)\right\}^{\lambda_f-\varepsilon}$$

that is,

$$\log T(r, fog) \geq (\lambda_f-\varepsilon) \log M\left(\frac{r}{4}, g\right) + O(1) \qquad \dots (8)$$

$$\geq (\lambda_f-\varepsilon)\, T\left(\frac{r}{4}, g\right) + O(1). \qquad \dots (9)$$

From (5) and (9), we get for δ (> 0) and for all large values of r

$$\log T(r, fog) \geq (\lambda_f-\varepsilon)(1-\varepsilon)(1+O(1))$$

$$\frac{\left(\frac{r}{4}\right)^{\lambda_g+\delta}}{\left(\frac{r}{4}\right)^{\lambda_g+\delta-\lambda_g(r/4)}}.$$

Since $r^{\lambda_g+\delta-\lambda_g(r)}$ is ultimately an increasing function of r, it follows that

$$\log T(r, fog) \geq (\lambda_f - \varepsilon)(1-\varepsilon)(1+o(1)) \frac{r^{\lambda_g(r)}}{4^{\lambda_g+\delta}}$$

for all large values of r.

So by[4] we get for a sequence of values of r tending to infinity

$$\log T(r, fog) \geq (\lambda_f - \varepsilon)\frac{1-\varepsilon}{1+\varepsilon}(1+o(1)) \frac{T(r,g)}{r^{\lambda_g+\delta}}.$$

Since ε and δ are arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log T(r, fog)}{T(r,g)} \geq \frac{\lambda_f}{4\lambda_g}$$

which is (*ii*). This proves the Theorem 1.

Theorem 2 — *Let f and g be two nonconstant entire functions such that ρ_f and λ_g are finite. Also suppose that there exist entire functions a_i $(i = 1, 2, \ldots, n; n \leq \infty)$* such that

$$T(r, a_i) = o\,\{T(r, g)\} \text{ as } r \to \infty \; (i = 1, 2, \ldots, n) \text{ and } \sum_{i=1}^{n} \delta(a_i, g) = 1.$$

Then

$$\frac{\pi\lambda_f}{4^{\lambda_g}} \leq \limsup_{r\to\infty} \frac{\log T(r, fog)}{T(r,g)} \leq \pi\rho_f.$$

PROOF : Since the second inequality is proved in[5], we prove only the first inequality. If $\lambda_f = 0$, the first inequality is obvious. So we suppose that $\lambda_f > 0$. For $0 < \varepsilon < \min\{\lambda_f, 1\}$ we get from (8) for all large values of r that

$$\frac{\log T(r, fog)}{T(r,g)} \geq (\lambda_f - \varepsilon) \frac{\log M\left(\frac{r}{4}, g\right)}{T\left(\frac{r}{4}, g\right)} \frac{T\left(\frac{r}{4}, g\right)}{T(r,g)} + O(1). \quad \ldots (10)$$

From (4) and (5) we get for a sequence of values of r tending to infinity and for δ (> 0)

$$\frac{T\left(\frac{r}{4}, g\right)}{T(r, g)} > \frac{1-\varepsilon}{1+\varepsilon} \frac{\left(\frac{r}{4}\right)^{\lambda_g + \delta}}{\left(\frac{r}{4}\right)^{\lambda_g + \delta - \lambda_{g(r/4)}}} \frac{1}{r^{\lambda_g(r)}}$$

$$\geq \frac{1-\varepsilon}{1-\varepsilon} \frac{1}{4^{\lambda_g + \delta}},$$

because $r^{\lambda_g + \delta - \lambda_g(r)}$ is ultimately an increasing function of r.

Since ε (>0) and δ (> 0) are arbitrary, we get from Lemma 2, (10) and above that

$$\limsup_{r \to \infty} \frac{\log T(r, fog)}{T(r, g)} \geq \frac{\pi \lambda_f}{4^{\lambda_g}}.$$

This proves the Theorem 2.

Theorem 3 — *Let f and g be two nonconstant entire functions such that $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g \leq \rho_g < \infty$.*

Then

$$\frac{\overline{\lambda}_g}{\rho_g} \leq \liminf_{r \to \infty} \frac{\log^{[3]} T(r, fog)}{\log T(r, g^{(k)})} \leq \limsup_{r \to \infty} \frac{\log^{[3]} T(r, fog)}{\log T(r, g^{(k)})} \leq \frac{\overline{\rho}_g}{\lambda_g}$$

for $k = 0, 1, 2, \ldots$.

PROOF : For given $\varepsilon (0 < \varepsilon < \lambda_f)$, we get for all large values of r

$$\log M(r, f) > r^{\lambda_f - \varepsilon}. \qquad \ldots (11)$$

From (7) and (11), we get for all large values of r.

$$T(r, fog) \geq \frac{1}{3} \left\{ \frac{1}{9} M\left(\frac{r}{4}, g\right) \right\}^{\lambda_f - \varepsilon};$$

so we get for all large values of r

$$\frac{\log^{[3]} T(r, fog)}{\log T(r, g^{(k)})} \geq \frac{\log^{[3]} M\left(\frac{r}{4}, g\right)}{\log \frac{r}{4}} \frac{\log \frac{r}{4}}{\log T(r, g^{(k)})} + o(1). \qquad \ldots (12)$$

Since $\limsup_{r \to \infty} \frac{\log T(r, g^{(k)})}{\log r} = \rho_g$, for all large values of r

we obtain

$$\log T(r, g^{(k)}) < (\rho_g + \varepsilon) \log r. \qquad \ldots (13)$$

Since ε (> 0) is arbitrary, we get from (12) and (13)

$$\liminf_{r \to \infty} \frac{\log^{[3]} T(r, fog)}{\log T(r, g^{(k)})} \geq \frac{\overline{\lambda}_g}{\rho_g}. \quad \text{... (14)}$$

Again for given $\varepsilon (0 < \varepsilon < \lambda_g)$, it follows from (1) and (2) that for all large values of r

$$T(r, fog) \leq \log M(M(r, g), f) \leq \{M(r, g)\}^{\rho_f + \varepsilon}$$

i.e.,

$$\frac{\log^{[3]} T(r, fog)}{\log T(r, g^{(k)})} \leq \frac{\log^{[3]} M(r, g)}{\log T(r, g^{(k)})} + o(1). \quad \text{... (15)}$$

Since $\liminf_{r \to \infty} \dfrac{\log T(r, g^{(k)})}{\log r} = \lambda_g$, it follows for all large values of r

$$\log T(r, g^{(k)}) > (\lambda_g - \varepsilon) \log r. \quad \text{... (16)}$$

Since ε (> 0) is arbitrary, we get from (15) and (16)

$$\limsup_{r \to \infty} \frac{\log^{[3]} T(r, fog)}{\log T(r, g^{(k)})} \leq \frac{\overline{\rho}_g}{\lambda_g}. \quad \text{... (17)}$$

The theorem follows from (14) and (17). This proves the Theorem 3.

***Theorem* 4** — *Let f and g be two nonconstant entire functions such that $0 < \lambda_f \leq \rho_f < \infty$ and $\lambda_g < \infty$. Then*

$$\frac{\lambda_g}{\rho_g} \leq \liminf_{r \to \infty} \frac{\log^{[3]} T(r, fog)}{\log T(r, g)} \leq 1 \leq \limsup_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log T(r, g)} \leq \frac{\rho_g}{\lambda_g}.$$

PROOF : For given $\varepsilon (0 < \varepsilon < 1)$, we get from (1) and (2) for all large values of r

$$T(r, fog) \leq \{M(r, g)\}^{\rho_f + \varepsilon}$$

i.e.,

$$\log^{[2]} T(r, fog) \leq \log^{[2]} M(r, g) + O(1). \quad \text{... (18)}$$

Again from (1) and (4) we get for a sequence of values of r tending to infinity and for δ (> 0).

$$\log M(r, g) < 3(1 + \varepsilon)(2r)^{\lambda_g(2r)}$$
$$= 3(1 + \varepsilon) \frac{(2r)^{\lambda_g + \delta}}{(2r)^{\lambda_g + \delta - \lambda_g(2r)}}.$$

Since $r^{\lambda_g + \delta - \lambda_g(r)}$ is ultimately an increasing function of r, it follows for a sequence of values of r tending to infinity that

$$\log M(r, g) < 3(1+\varepsilon)\, 2r^{\lambda_{g+\delta} \cdot r^{\lambda_g(r)}} \qquad \text{... (19)}$$

Again by (5) and (19) we obtain for a sequence of values of r tending to infinity

$$\log M(r, g) < 3\,\frac{1+\varepsilon}{1-\varepsilon} \cdot 2^{\lambda_g + \delta}\, T(r, g)$$

i.e.,

$$\log^{[2]} M(r, g) < \log T(r, g) + o(1).$$

So from (18), we get for a sequence of values of r tending to infinity

$$\frac{\log^{[2]} T(r, fog)}{\log T(r, g)} \le 1 + o(1).$$

So

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log T(r, g} \le 1. \qquad \text{... (20)}$$

Again by (7) and (11) we get for all large values of r

$$\frac{\log^{[2]} T(r, fog)}{\log T(r, g)} \ge \frac{\log^{[2]} M\left(\frac{r}{4}, g\right)}{\log \frac{r}{4}} \, \frac{\log \frac{r}{4}}{\log T(r, g)}.$$

This implies that

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log T(r, g)} \ge \frac{\lambda_g}{\rho_g}. \qquad \text{... (21)}$$

From (18), we get for all large values of r

$$\frac{\log^{[2]} T(r, fog)}{\log T(r, g)} \le \frac{\log^{[2]} M(r, g)}{\log r} \, \frac{\log r}{\log T(r, g)}.$$

This implies that

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log T(r, g)} \le \frac{\rho_g}{\lambda_g}. \qquad \text{... (22)}$$

From (1) and (5) we obtain for all large values of r and for δ (> 0) and ε $(0 < \varepsilon < 1)$

$$\log M\left(\frac{r}{4}, g\right) > (1-\varepsilon)\ \frac{\left(\frac{r}{4}\right)^{\lambda_g+\delta}}{\left(\frac{r}{4}\right)^{\lambda_g+\delta-\lambda_g(r/4)}} \geq \frac{1-\varepsilon}{4^{\lambda_g+\delta}}\, r^{\lambda_g(r)},$$

because $r^{\lambda_g+\delta-\lambda_g(r)}$ is ultimately an increasing function of r.

So by (4), we get for a sequence of values of r tending to infinity

$$\log M\left(\frac{r}{4}, g\right) \geq \frac{1-\varepsilon}{1+\varepsilon}\ \frac{1}{4^{\lambda_g+\delta}}\, T(r, g)$$

that is,

$$\log M\left(\frac{r}{4}, g\right) \geq \log T(r, g) + o(1). \qquad \ldots (23)$$

Also by (7) and (11), we obtain for all large values of r

$$\log^{[2]} T(r, fog) \geq \log^{[2]} M\left(\frac{r}{4}, g\right) + O(1). \qquad \ldots (24)$$

From (23) and (24), we obtain

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(r, fog)}{\log T(r, g)} \geq 1. \qquad \ldots (25)$$

Now the theorem follows from (20), (21), (22) and (25). This proves the Theorem 4.

Corollary 1 — If in addition to the condition of Theorem 4, we suppose that g is of regular growth i.e., $\lambda_g = \rho_g$ then

$$\lim_{r\to\infty} \frac{\log^{[2]} T(r, fog)}{\log T(r, g)} = 1.$$

Remark 1 : The conditions $\lambda_f > 0$ and $\rho_f < \infty$ are necessary for Theorem 4 and Corollary 1 which are evident from the following two examples.

Example 1 — Let $f = z$, $g = \exp z$.

Then

$$\lambda_f = \rho_f = 0, \quad 0 < 1 = \lambda_g = \rho_g < \infty.$$

Since $T(r, fog) = T(r, g) = \frac{r}{\pi}$, it follows that

$$\frac{\log^{[2]} T(r, fog)}{\log T(r, g)} = \frac{\log^{[2]} r + O(1)}{\log r + O(1)}$$

and so

$$\lim_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log T(r, g)} = 0.$$

Example 2 — If $f = \exp^{[2]} z$ and $g = \exp z$, then

$$\lambda_f = \rho_f = \infty, \quad \lambda_g = \rho_g = 1.$$

Now $3T(2r, fog) \geq \log M(r, fog) = \exp^{[2]} r$

or, $\log^{[2]} T(r, fog) \geq \frac{r}{2} + o(1).$

Also $T(r, g) = \frac{r}{\pi}.$

Therefore

$$\lim_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log T(r, g)} = \infty.$$

Theorem 5 — *Let f and g be two entire functions such that* $0 < \lambda_f \leq \rho_f < \infty$ *and* $0 < \lambda_g \leq \rho_g < \infty$. *Then*

$$\frac{\lambda_g}{\rho_f} \leq \liminf_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log T(r, f^{(k)})} \leq \limsup_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log T(r, f^{(k)})} \leq \frac{\rho_g}{\lambda_f}$$

for $k = 0, 1, 2, \ldots$.

PROOF : By Lemma 3, we get for all large values of r

$$T(r, fog) < 2T(M(r, g), f)$$

and so

$$\log^{[2]} T(r, fog) < \log^{[2]} T(M(r, g), f) + o(1). \quad \ldots (26)$$

Also for ε (> 0), we see that for all large values of r

$$\log^2 M\left(\frac{r}{4}, g\right) > (\lambda_g - \varepsilon) \log \frac{r}{4}. \quad \ldots (27)$$

and

$$\log T(r, f^{(k)}) > (\lambda_f - \varepsilon) \log r. \quad \ldots (28)$$

From (26), (27) and (28), we get because ε (> 0) is arbitrary

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log T(r, f^{(k)})} \leq \frac{\rho_g}{\lambda_f}.$$

Again for given ε (> 0), we get for all large values of r

$$\log^{[2]} M\left(\frac{r}{4}, g\right) > (\lambda_g - \varepsilon) \log \frac{r}{4}$$

and

$$\log T(r, f^{(k)}) < (\rho_f + \varepsilon) \log r.$$

So, from (24), we get because ε (> 0) is arbitrary

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log T(r, f^{(k)})} \geq \frac{\lambda_g}{\rho_f}.$$

This proves the Theorem 5.

Theorem 6 — *Let f be a meromorphic function and g be an entire function such that $0 < \lambda_f \leq \rho_f < \infty$ and $\rho_g < \infty$. Then*

$$\lim_{r \to \infty} \frac{\log T(r, fog)}{T(\exp(r), f^{(k)})} = 0$$

for $k = 0, 1, 2, 3, \ldots$.

PROOF : By Lemma 4, we get for all sufficiently large values of r

$$\log T(r, fog) \leq \log T(r, g) + \log T(M(r, g), f) + o(1). \quad \ldots (29)$$

For given $\varepsilon (0 < \varepsilon < \lambda_f)$, we get for all large values of r

$$\log T(r, g) < (\rho_g + \varepsilon) \log r,$$

$$\log T(M(r, g), f) < (\rho_f + \varepsilon)\, r^{\rho_g + \varepsilon}$$

and

$$T(\exp(r), f^{(k)}) > e^{r(\lambda_f - \varepsilon)}$$

So, from (29), we get for all large values of r

$$\frac{\log T(r, fog)}{T(\exp(r), f^{(k)})} \leq \frac{(\rho_g + \varepsilon) \log r}{e^{r(\lambda_f - \varepsilon)}} + \frac{(\rho_f + \varepsilon)\, r^{\rho_g + \varepsilon}}{e^{r(\lambda_f - \varepsilon)}} + o(1)$$

and hence

$$\lim_{r \to \infty} \frac{\log T(r, fog)}{T(\exp(r), f^{(k)})} = 0.$$

This proves the Theorem 6.

Remark 2 : The following example shows that the condition $\rho_g < \infty$ in Theorem 6 is necessary.

Example 3 — Let $f = \exp z$ and $g = \exp^{[3]} z$.

So, $\lambda_f = \rho_f = 1$ and $\rho_g = \infty$.

Therefore, we obtain,

$$3T(2r, fog) \geq \log M(r, fog) = \exp^{[3]} r$$

or,

$$\log T(r, fog) \geq \exp^{[2]} r/2 + O(1)$$

and

$$T(\exp(r), f^{(k)}) = \frac{e^r}{\pi}.$$

Thus it follows that

$$\lim_{r \to \infty} \frac{\log T(r, fog)}{T(\exp(r), f^{(k)})} = \infty.$$

Theorem 7 — *Let f be a meromorphic function and g be an entire function such that $0 < \lambda_g \leq \rho_g < \infty$ and $\rho_f < \infty$. Then*

$$\lim_{r \to \infty} \frac{\log T(r, fog)}{T(\exp(r), g^{(k)})} = 0$$

for $k = 0, 1, 2 \ldots$.

The proof is similar to that of Theorem 6.

Zhen-Zhong Zhou[15] proved the following theorem.

Theorem A — *Let f and g be two entire functions of finite order satisfying $\rho_g \leq \lambda_f \leq \rho_f$ and* $g(0) = 0$. Then

$$\lim_{r \to \infty} \frac{\log T(r, fog)}{T(r, f)} = 0.$$

Following theorem improves Theorem A to composite meromorphic functions.

Theorem 8 — *Let f be a meromorphic function and g be an entire function such that $\rho_g < \lambda_f \leq \rho_f < \infty$. Then*

$$\lim_{r \to \infty} \frac{\log T(r, fog)}{T(r, f^{(k)})} = 0$$

for $k = 0, 1, 2 \ldots$.

PROOF : We choose an ε (> 0) such that $\rho_g + \varepsilon < \lambda_f - \varepsilon$. Then for all large values of r, we get

$$\log T(r, g) < (\rho_g + \varepsilon) \log r,$$

$$\log T(M(r, g), f) < (\rho_f + \varepsilon) r^{\rho_g + \varepsilon}$$

and

$$T(r, f^{(k)}) > r^{\lambda_f - \varepsilon}$$

So from (29), we get for all large values of r

$$\frac{\log T(r, fog)}{T(r, f^{(k)})} \leq \frac{(\rho_g + \varepsilon) \log r}{r^{\lambda_f - \varepsilon}} + \frac{(\rho_f + \varepsilon) r^{\rho_g + \varepsilon}}{r^{\lambda_f - \varepsilon}} + o(1)$$

and hence

$$\lim_{r \to \infty} \frac{\log T(r, fog)}{T(r, f^{(k)})} = 0$$

This proves the Theorem 8.

Remark 3 : The condition $\rho_g < \lambda_f$ is necessary for Theorem 8, which follows from the following two examples.

Example 4 — Considering $f = z$, $g = \exp z$, $k = 0$, we see that

$$\rho_f = \lambda_f = 0 \quad \text{and} \quad \rho_g = 1.$$

Since $T(r, fog) = \frac{r}{\pi}$ and $T(r, f) \leq \log M(r, f) = \log r$, it follows that

$$\liminf_{r \to \infty} \frac{\log T(r, fog)}{T(r, f)} \geq 1.$$

Example 5 — Let $f = g = \exp z$.

So, $\rho_f = \lambda_f = 1$ and $\rho_g = 1$.

Since, $T(r, f^{(k)}) = \frac{r}{\pi}$ and $T(r, fog) \sim \frac{e^r}{(2\pi^3 r)^{1/2}}$, we obtain

$$\frac{\log T(r, fog)}{T(r, f^{(k)})} \sim \frac{r - 1/2 \log r + O(1)}{\frac{r}{\pi}}$$

so that

$$\lim_{r \to \infty} \frac{\log T(r, fog)}{T(r, f^{(k)})} = \pi.$$

Theorem 9 — *Let f and g be two transcendental entire functions such that*

(*i*) $0 < \lambda_g \le \rho_g < \infty$

(*ii*) $\lambda_f > 0$;

and

(*iii*) $\delta(0;f) < 1$.

Then for any real number A,

$$\limsup_{r \to \infty} \frac{\log T(r, fog)}{\log T(r^A, g^{(k)})} = \infty$$

for $k = 0, 1, 2, 3, \ldots$.

PROOF : We suppose that $A > 0$ because otherwise the theorem is obvious.

For given $\varepsilon\,(0 < \varepsilon < 1 - \delta(0;f))$ there exists a sequence of values of r tending to infinity such that

$$N(r, 0; f) > (1 - \delta(0;f) - \varepsilon)\, T(r, f).$$

So from Lemma 6 we get for a sequence of values of r tending to infinity

$$T(r, fog) + O(1)) \ge \left(\log \frac{1}{\eta} \right)$$

$$\left[\frac{(1 - \delta(0;f) - \varepsilon)\, T\left\{ M\left((\eta r)^{\frac{1}{1+\alpha}}, g\right), f \right\}}{\log(M\left((\eta r)^{\frac{1}{1+\alpha}}, g\right) - O(1)} - O(1) \right]. \qquad \ldots (30)$$

Since for all large values of r, $\log M(r, g) < r^{\rho_g + \varepsilon}$, it follows from (30) that for a sequence of values of r tending to infinity

$$\log T(r, fog) + o(1) \ge O(\log r) + \log T\left\{ M\left((\eta r)^{\frac{1}{1+\alpha}}, g\right), f \right\}$$

$$+ \log 1 - \left[\frac{\log(M\left((\eta r)^{\frac{1}{1+\alpha}}, g\right) O(1)}{(1 - \delta(0;f) - \varepsilon)\, T\left\{ M\left((\eta r)^{\frac{1}{1+\alpha}}, g\right), f \right\}} \right].$$

Since f is transcendental, it follows that

$$\lim_{r \to \infty} \frac{\log\left(M\left((\eta r)^{\frac{1}{1+\alpha}}, g\right)\right)}{T\left\{M\left((\eta r)^{\frac{1}{1+\alpha}}, g\right), f\right\}} = 0.$$

So from above we get for a sequence of values of r tending to infinity

$$\log T(r, fog) \geq O(\log r) + \log T\left\{ M\left((\eta r)^{\frac{1}{1+\alpha}}, g\right), f\right\}. \qquad \ldots (31)$$

Also we see that for all large values of r

$$M(r, g) > \exp\left\{ (r)^{\frac{1}{2}\lambda_g} \right\}$$

$$\log T(r, f) > \frac{1}{2}\lambda_f \log r$$

and

$$T(r, g^{(k)}) < r^{\rho_g + 1}.$$

So, from (31), we obtain for a sequence of values of r tending to infinity

$$\frac{\log T(r, fog)}{\log T(r^A, g^{(k)})} > \frac{O(\log r)}{A(1+\rho_g)\log r} + \frac{\lambda_f}{2} \frac{(\eta r)^{\frac{\lambda_g}{2(1+\alpha)}}}{A(1+\rho_g)\log r}$$

which implies that $\limsup_{r \to \infty} \frac{\log T(r, fog)}{\log T(r^A, g^{(k)})} = \infty$. This proves the Theorem 9.

Recently Liao and Yang[7] proved the following result.

Theorem B — *Let k, g be entire functions and h meromorphic such that* $\lambda_h > 0, \rho_g < \rho_k$.

Then for every $v \left(1 \leq v < \frac{\rho_k}{\rho_g} \right)$ and every meromorphic function f of finite order, we have

$$\limsup_{r \to \infty} \frac{T(r, hok)}{T(r^v, fog) \log M(r^v, g)} = \infty.$$

If $\lambda_f > 0,\ v > \frac{\rho_k}{\rho_g}\ (0 < \rho_g < \rho_k < \infty)$ then

$$\liminf_{r \to \infty} \frac{T(r, hok)}{T(r^v, fog)} = 0.$$

Following theorem improves Theorem B.

***Theorem* 10** — *Let k, g be entire functions and h be meromorphic function such that* $\lambda_h > 0$, $\rho_g < \rho_k$. *Then for every* $v\left(1 \le v < \dfrac{\rho_k}{\rho_g}\right)$ *and every meromorphic function f of finite order,*

$$\limsup_{r \to \infty} \frac{T(r, hok)}{\left\{T(r^v, fog) \log M(r^v, g)\right\}^{+\alpha}} = \infty.$$

where $-\infty < \alpha < \infty$.

If $\lambda_f > 0$ and $v > \dfrac{\rho_k}{\rho_g}$ $(0 < \rho_g < \rho_k < \infty)$ then

$$\liminf_{r \to \infty} \frac{\left\{T(r, hok)\right\}^{1+\alpha}}{T(r^v, fog)} = 0$$

where $-\infty < \alpha < \infty$.

PROOF : Since if $1 + \alpha \le 0$, the theorem is obvious, we suppose that $1 + \alpha > 0$. By Lemma 5 we get for a sequence of values of r tending to infinity

$$T(r, hok) \ge T(\exp(r^{\mu}), h),$$

where $0 < v\rho_g < \mu < \rho_k$.

Since $\lambda_h > 0$, we obtain for a sequence of values of r tending to infinity

$$T(r, hok) \ge \exp\left(\frac{1}{2} \lambda_h r^{\mu}\right). \qquad \dots (32)$$

Again by Lemma 4, we get for ε (> 0) and for all large values of r

$$\{T(r^v, fog) \log M(r^v, g)\}^{1+\alpha}$$

$$< \{1 + o(1)\}^{1+\alpha} \,.\, r^{v(1+\alpha)(\rho_g+\varepsilon)} \,.\, \exp\{(1+\alpha)(\rho_f + \varepsilon) r^{v(\rho_g+\varepsilon)}\}. \qquad \dots (33)$$

From (32) and (33), we get for a sequence of values of r tending to infinity

$$\frac{T(r, hok)}{\left\{T(r^v, fog) \log M(r^v, g)\right\}^{1+\alpha}} \ge \frac{\exp\left\{1/2\, \lambda_f r^{\mu} - (1+\alpha)(\rho_f + \varepsilon) r^{v(\rho_g+\varepsilon)}\right\}}{\left\{1 + o(1)\right\}^{1+\alpha} r^{v(1+\alpha)(\rho_g+\varepsilon)}}.$$

Since we can choose ε (> 0) so small that $v(\rho_g + \varepsilon) < \mu$, it follows that

$$\limsup_{r \to \infty} \frac{T(r, hok)}{\left\{T(r^v, fog) \log M(r^v, g)\right\}^{1+\alpha}} = \infty.$$

Again by Lemma 4, we get for ε (> 0) and for all large values of r

$$\left\{T(r, hok)\right\}^{1+\alpha} \le \left\{1 + o(1)\right\}^{1+\alpha} \,.\, r^{(\rho_k - \lambda_k + 2\varepsilon)(1+\alpha)}$$

$$\exp\left[(\rho_h + \varepsilon)(1 + \alpha)\, r^{\rho_k - \varepsilon}\right] \qquad \dots (34)$$

On the other hand for a sequence of values of r tending to infinity we get from Lemma 5

$$T(r^{\nu}, fog) \geq T(e^{r^{\nu(\rho_g - \varepsilon)}}, f)$$

$$\geq \exp\left\{ (\lambda_f - \varepsilon)\, r^{\nu(\rho_g - \varepsilon)} \right\}. \qquad \dots (35)$$

Since we can choose ε (> 0) so small that $\nu(\rho_g - \varepsilon) > \rho_k + \varepsilon$, from (34) and (35) it follows that

$$\liminf_{r \to \infty} \frac{\{T(r, hok)\}^{1+\alpha}}{T(r^{\nu}, fog)} = 0.$$

This proves the Theorem 10.

Remark 4 : Following two examples show that the conditions $\lambda_h > 0$ and $\lambda_f > 0$ are necessary for Theorem 10.

Example 6 — Let $f = h = z$, $g = \exp z$, $k = \exp(z^2)$, $\nu = 1$, $\alpha = 0$. Then we obtain

$$\rho_k = 2, \rho_g = 1, \ \rho_f = 0 \ \text{ and } \ \lambda_h = 0.$$

Since $T(r, hok) \leq r^2$ and $T(r, fog) = \frac{r}{\pi}$, it follows that

$$\frac{T(r, hok)}{\left\{T(r^{\nu}, fog) \log M(r^{\nu}, g)\right\}^{1+\alpha}} \leq \frac{r^2}{\frac{r}{\pi} \cdot r} = \pi.$$

i.e.,

$$\limsup_{r \to \infty} \frac{T(r, hok)}{\left\{T(r^{\nu}, fog) \log M(r^{\nu}, g)\right\}^{1+\alpha}} \leq \pi.$$

Example 7 — Let $f = z$, $g = h = \exp(z^1)$, $k = \exp(z^2)$, $\nu = 3$, $\alpha = 0$.

$$\text{So, } \lambda_f = 0, \rho_g = 1, \ \rho_k = 2 \ \text{ and } \ \lambda_h = 1.$$

Since $T(r, hok) \geq 1/3\, e^{r^2/4}$ and $T(r, fog) = r/\pi$, we obtain

$$\frac{\{T(r, hok)\}^{1+\alpha}}{T(r^{\nu}, fog)} \geq \frac{\pi\, e^{r^2/4}}{3r^3}.$$

So,

$$\liminf_{r \to \infty} \frac{\{T(r, hok)\}^{1+\alpha}}{T(r^{\nu}, fog)} = \infty.$$

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