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AN INEQUALITY FOR TWO SIMPLICES AND TWO POINTS

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We establish an inequality involving two simplices and two points, which combines the distances between any point in n-dimensional Euclidean space E^n to the vertices of one simplex with the distances from an interior point of another simplex to its facets and prove some applications thereof.

Key Words: Simplices; Volumes; Mass-Point Systems; Inequalities

1. INTRODUCTION

In Yang and Zhang[1] generalied the well-known Neuberg-Pedoe inequality to E^n. Following Yang and Zhang, a number of inequalities for two simplices have been established[2-6]. Specially, in [2&7], the authors established some inequalities for two simplices. In this paper, we established an inequality for two simplices and two points which combine the distances between any point of E_n to vertices of a simplex with the distances from an interior point of another simplex to its facets. It may be the first inequality for two simplices and two points.

Let Ω and Ω' be two n-dimensional simplices in the n-dimensional Euclidean space E^n, We use the following notations. $\tau = \{A_0, A_1, ..., A_n\}$ is the vertex set of Ω, V the volume of Ω. Ω_i denotes the $(n-1)$-dimensional facet spanned by the vertex set $\{A_0, ..., A_{i-1}, A_{i+1}, ..., A_n\}$ (i = 0, 1, ..., n). For the second simplex Ω', we use the analogous notations. For example, V' is the volume of Ω'.

Our main result is the following theorem:

Theorem 1 — *Let Ω and Ω' be two n-dimensional simplices, P an arbitrary point in E^n and $D_i = |PA_i|$, d_i' the distance from an interior point P' of Ω' to the $(n-1)$-dimensional facet Ω_i', i = 0, 1, 2, ..., n. Then*

$$\sum_{i=0}^{n} \frac{D_i^2}{d_i'} \geq C(n) \left(\frac{V^2}{V'}\right)^{1/n}, \qquad \text{... (1.1)}$$

where $C(n) = n^{3/2}\, n!^{1/n} (n+1)^{(n-1)/2n}$, and equality is valid if Ω and Ω' are regular, and the *points P and P′ are the centre of Ω and Ω′ respectively.*

By applying the well-known mass-point systems theorem[8] and sine theorem for simplex[9], we will prove Theorem 1 in Section 2. In Section 3, we will give its some applications.

2. Proof of Theorem 1

Suppose that $e_0, e_1, \ldots, e_n$ are the unit outer normal vectors on $n + 1$ facet of Ω, and

$$D_i = \det(e_0, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n).$$

Then $\theta_i = \arcsin |D_i|$ is called the vertex angle at A_i of the simplex Ω[9].

Lemma 1 — (Sine theorem for simplices[9]). Let S_i be the $(n - 1)$-dimensional content of $(n - 1)$-dimensional facet Ω_i of Ω, θ_i the vertex angle at A_i of the simplex Ω. Then

$$V^{n-1} = \frac{(n-1)!}{n^{n-1}} \left(\prod_{\substack{j=0 \\ j \neq i}}^{n} S_j \right) \sin \theta_i. \qquad \ldots (2.1)$$

Lemma 2[10] — Let $\theta_0, \theta_1, \ldots, \theta_n$ be the vertex angles of Ω, and $\alpha_0, \alpha_1, \ldots, \alpha_n$ real positive constants. Then

$$\sum_{i=0}^{n} \alpha_i \sin^2 \theta_i \leq \frac{1}{n^n} \left(\prod_{j=0}^{n} \alpha_j \right) \left(\sum_{i=0}^{n} \frac{1}{\alpha_i} \right)^n, \qquad \ldots (2.2)$$

and equality is true if Ω is regular and $\alpha_0 = \alpha_1 = \cdots = \alpha_n$.

Denote by $\sigma_m^{[k]}[x_1, x_2, \ldots, x_m]$ the kth symmetric means of the m real positive numbers $x_1, x_2, \ldots, x_m$, namely,

$$\sigma_m^{[k]}[x_1, x_2, \ldots, x_m] = \left[1\Big/\binom{m}{k} \sum_{1 < i_1 < i_2 < \ldots < i_k \leq m} \left(\prod_{j=0}^{n} x_{i_j} \right) \right]^{1/k}$$

$(k = 1, 2, \ldots, m)$.

Lemma 3 — Let M be an interior point of Ω, d_i denote the distance from M to the $(n - 1)$-dimensional facet Ω_i of Ω. Then

$$\sigma_{n+1}^{[n]}[d_0, d_1, \ldots, d_n] \leq \frac{(n!)^{1/n}}{(n+1)^{(n+1)/2n}\, n^{1/2}} V^{1/n}, \qquad \ldots (2.3)$$

and equality is true if Ω is regular.

PROOF : Let $x_0, x_1, ..., x_n$ be $n + 1$ real positive constants. Taking $\alpha_i = \prod_{\substack{j=0 \\ j \neq i}}^{n} x_j$ in inequality (2.2), we can obtain

$$\sum_{i=0}^{n} \left(\prod_{\substack{j=0 \\ j \neq i}}^{n} x_j \right) \sin^2 \theta_i \leq \frac{1}{n^n} \left(\sum_{i=0}^{n} x_i \right)^n . \qquad \text{... (2.4)}$$

Applying Cauchy's inequality and (2.4), we get

$$\sum_{i=0}^{n} \left(\prod_{\substack{j=0 \\ j \neq i}}^{n} x_j \right) \sin \theta_i \leq \left[\sum_{i=0}^{n} \left(\prod_{\substack{j=0 \\ j \neq i}}^{n} x_j \right) \right]^{1/2} \left[\sum_{i=0}^{n} \left(\prod_{\substack{j=0 \\ j \neq i}}^{n} x_j \right) \sin^2 \theta_i \right]^{1/2}$$

$$\leq \frac{1}{n^{n/2}} \left(\sum_{i=0}^{n} x_i \right)^{n/2} \left[\sum_{i=0}^{n} \left(\prod_{\substack{j=0 \\ j \neq i}}^{n} x_j \right) \right]^{1/2} \qquad \text{... (2.5)}$$

$$= \frac{(n+1)^{1/2}}{n^{n/2}} \left(\sum_{i=0}^{n} x_i \right)^{n/2} \left\{ \sigma_{n+1}^{[n]} [x_0, x_1, ..., x_n] \right\}^{n/2}$$

For the right-hand *Sin* α of (2.5), using the well-known result

$$\sigma_{n+1}^{[n]} [x_0, x_1, ..., x_n] \leq \sigma_{n+1}^{[1]} [x_0, x_1, ..., x_n] = \frac{1}{(n+1)} \sum_{i=0}^{n} x_i ,$$

we have

$$\sum_{i=0}^{n} \left(\prod_{\substack{j=0 \\ j \neq i}}^{n} x_j \right) \sin \theta_i \leq \frac{1}{(n+1)^{(n-1)/2} n^{n/2}} \left(\sum_{i=0}^{n} \right)^n \qquad \text{... (2.6)}$$

We get from (2.1) and (2.6)

$$V^{n-1}\sum_{i=0}^{n}\left(\prod_{\substack{j=0\\ j\neq i}}^{n} x_j\right)S_i \le \frac{(n-1)!}{(n+1)^{(n-1)/2}\,n^{(3n-2)/2}}$$

$$\left(\sum_{i=0}^{n} x_i\right)^n\left(\prod_{i=0}^{n} S_i\right). \qquad \text{... (2.7)}$$

Taking $x_i = d_i S_i$ in (2.7), we obtain

$$V^{n-1}\sum_{i=0}^{n}\left(\prod_{\substack{j=0\\ j\neq i}}^{n} d_j\right) \le \frac{(n-1)!}{(n+1)^{(n-1)/2}\,n^{(3n/2)/2}}\left(\sum_{i=0}^{n} d_i S_i\right)^n. \qquad \text{... (2.8)}$$

Noting the obvious geometric fact

$$nV = \sum_{i=0}^{n} d_i S_i, \qquad \text{... (2.9)}$$

and substituting (2.9) into the right-hand side of (2.8) and rearranging it, we obtain inequality (2.3). □

Remark[7] : Gerber proved the following inequality:

$$\sigma_{n+1}^{[n+1]}[d_0, d_1, ..., d_n] \le \frac{n!^{1/n}}{(n+1)^{(n+1)/2}\,n^{1/2}}\,V^{1/n}. \qquad \text{... (2.10)}$$

Recently, Chen conjectured that inequality (2.10) can be improved as

$$\sigma_{n+1}^{[n-1]}[d_0, d_1, ..., d_n] \le \frac{n!^{1/n}}{(n+1)^{(n+1)/2}\,n^{1/2}}\,V^{1/n}. \qquad \text{... (2.11)}$$

Shan and Chen negated inequality (2.11) and conjectured again that inequality (2.10) can be sharpened as[11].

$$\sigma_{n+1}^{[n]}[d_0, d_1, ..., d_n] \le \frac{n!^{1/n}}{(n+1)^{(n+1)/2}\,n^{1/2}}\,V^{1/n}.$$

The proof of Lemma 3 above shows that their conjecture is true.

Lemma 4 — Let $a_{ij}\,(0 \le i < j \le n)$ be the length-edge of Ω, that is, $a_{ij} = |A_i A_j|$, P any point in E^n, $D_i = |PA_i|$ and $x_0, x_1, ..., x_n$ real positive constants. Then

$$\sum_{0\le i<j\le n} x_i x_j a_{ij}^2 \le \left(\sum_{i=0}^{n} x_i\right)\left(\sum_{i=0}^{n} x_i X_i^2\right). \qquad \text{... (2.12)}$$

and equality is valid if and only if

$$(x_0, x_1, ..., x_n) \cdot (\overrightarrow{PA_0}, \overrightarrow{PA_1}, ..., \overrightarrow{PA_n}) = 0.$$

PROOF : Let $\overrightarrow{PQ} = \sum_{i=0}^{n} x_i \overrightarrow{PA_i} / \sum_{i=0}^{n} x_i$, then

$$0 \le \left(\sum_{i=0}^{n} x_i \right)^2 |\overrightarrow{PQ}|^2 = \left| \sum_{i=0}^{n} x_i \overrightarrow{PA_i} \right|^2$$

$$= \sum_{i=0}^{n} x_i^2 |\overrightarrow{PA_i}|^2 + 2 \sum_{0 \le i < j \le n} x_i x_j \overrightarrow{PA_i}\, \overrightarrow{PA_j}$$

$$= \sum_{i=0}^{n} x_i^2 |\overrightarrow{PA_i}|^2 + \sum_{0 \le i < j \le n} x_i x_j (|\overrightarrow{PA_i}|^2 \, |\overrightarrow{PA_j}|^2 - |\overrightarrow{A_i A_j}|^2)$$

$$= \sum_{i=0}^{n} x_i^2 |\overrightarrow{PA_i}|^2 + \sum_{0 \le i \ne j \le n} x_i x_j |\overrightarrow{PA_i}|^2 - \sum_{0 \le i < j \le n} x_i x_j a x_{ij}^2$$

$$= \left(\sum_{i=0}^{n} x_i \right) \left(\sum_{i=0}^{n} x_i |\overrightarrow{PA_i}|^2 \right) - \sum_{0 \le i < j \le n} x_i x_j a x_{ij}^2.$$

Hence, the inequality (2.12) is true. □

Lemma 5 — (Zhang and Yang[8]). Let $\Gamma = \{A_i (m_i), i = 0, 1, ..., N\}$ be a mass-points system in E^n. The points A_i are endowed with weight $m_i > 0$ ($i = 1, 2, ..., N$) and $N > n$, respectively. Denote by $V_{i_0, i_1, ..., i_k}$ the volume of k-dimensional simplex spanned by the vertex set $\{A_{i_0}, A_{i_1}, ..., A_{i_k}\}$ $(0 \le i_i < i_1 < ... < i_k \le N)$. Put

$$M_k = \sum_{0 \le i_0 < i_1 < \ldots < i_k \le N} m_{i_0} m_{i_1} ... m_k V^2_{i_0, i_1, ..., i_k} \quad (1 \le k \le n),$$

$$M_0 = \sum_{i=0}^{N} m_i .$$

If $k, l \in \{1, 2, ..., n\}$, $k < l$, then

$$\frac{M_k^l}{M_l^k} \ge \frac{[(n-l)!\,(l\ 1)^3]^k}{[(n-k)!\,(k\,!)^3]^l} (n!\, M_0)^{l-k}, \quad \text{... (2.13)}$$

and equality holds if and only if the inertia ellipsoid of Γ with respect to its barycenter is a hypershphere.

For the proof of Lemma 5 the reader is referred to[8], see also [12, XX, 9, 17].

Lemma 6 — Let P be any point in E^n D_i denote the distance from P to vertex A_i of Ω, that is, $D_i = |PA_i|$ and $x_0, x_1, ..., x_n$ real positive constants. Then

$$\sum_{i=0}^{n} x_i D_i^2 \geq \left\{ n \left(\frac{n!^2}{(n+1)} \right)^{1/n} / \sigma_{n+1}^{[n]} \left[x_0^{-1}, x_1^{-1}, ..., x_n^{-1} \right] \right\} V^{2/n}, \qquad ... (2.14)$$

and equality is valid if Ω is regular and point P is centre of Ω and $x_0 = x_1 = \cdots = x_n$.

PROOF : Taking $N = n$, $k = 1$, $l = n$, $m_i = x_i > 0$ in Lemma 5, and noting

$$M_1 = \sum_{0 \leq i < j \leq n} x_i x_j a_{ij}^2, \quad M_n = \left(\prod_{i=0}^{n} x_i \right) V^2.$$

we have

$$\left(\sum_{0 \leq i < j \leq n} x_i x_j a_{ij}^2 \right)^n \geq n^n n!^2 \left(\sum_{i=0}^{n} x_i \right)^{n-1} \left(\prod_{i=0}^{n} x_i \right) V^2 \qquad ... (2.15)$$

From (2.12) and (2.15), we get

$$\left(\sum_{i=0}^{n} x_i \right) \left(\sum_{i=0}^{n} x_i D_i^2 \right)^n \geq n^n n!^2 \left(\prod_{i=0}^{n} x_i \right) V^2.$$

Rearranging, this, we obtain the inequality (2.14). □

$$\sum_{i=0}^{n} \frac{D_i^2}{d_i'} \geq \left\{ n \left(\frac{n!^2}{(n+1)} \right)^{1/n} / \sigma_{n+1}^{[n]} \left[d_0', d_1', ..., d_n' \right] \right\} V^{2/n}, \qquad ... (2.16)$$

Applying Lemma 3 to Ω', it follows that

$$\sigma_{n+1}^{[n]} [d_0', d_1', ..., d_n'] \leq \frac{(n!)^{1/n}}{(n+1)^{(n+1)/2n} n^{1/2}} (V')^{1/n}. \qquad ... (2.17)$$

From (2.16) and (2.17), we can obtain the required inequality (1.1). □

3. SOME APPLICATIONS

We keep the notations of the previous two sections. Moreover, let m_i be median of Ω from vertex A_i, R and r denote the circum radius and the inradius of Ω respectively.

We prove the following two theorems and some corollaries by using (1.1).

Theorem 2 — *Let* Ω *and* Ω' *be two n-dimensional simplices in* E^n, *and P any point in E* $D_i = |PA_i|$. *Then*

$$\sum_{i=0}^{n} S_i' D_i^2 \geq \frac{n^{5/2}(n!)^{1/n}}{(n+1)^{(n+1)/2n}} V^{2/n} V'^{(n-1)/n}, \qquad \ldots (3.1)$$

and equality is true if Ω *and* Ω' *are regular and P is the centre of* Ω.

PROOF : Let h_i' denote the altitude of Ω' from vertex A_i', $d_i'(G')$ the distance from the centroid G' of Ω' to $(n-1)$-dimensional facets Ω_i', B_i' be the intersection point of the line $A_i' G'$ with Ω_i'. Taking P' for the centroid G' of Ω' in the inequality (1.1), and noting that geometric fact

$$\frac{d_i'(G')}{h_i'} = \frac{|G'B_i'|}{|A_i'B_i'|} = \frac{1}{n+1},$$

we can obtain the inequality

$$\sum_{i=0}^{n} \frac{D_i^2}{h_i'} \geq \frac{n^{3/2}(n!)^{1/n}}{(n+1)^{n+1/2n}} \left(\frac{V^2}{V'}\right)^{1/n}. \qquad \ldots (3.2)$$

According to well-known formula $h_i' = nV'/S_i'$ from the inequality (3.2), we can get inequality (3.1) □

Taking Ω' for the regular simplex and P the circumcenter of Ω in (3.1), we can derive the following well-known inequality.

Corollary 1[7,13,14] — For an n-dimensional simplex Ω, we have

$$V \leq \frac{(n+1)^{(n+1)/2}}{n!\, n^{n/2}} R^n, \qquad \ldots (3.3)$$

and the equality is valid if and only if Ω is regular.

Taking P for the centroid G of Ω in (3.1), applying the well-known geometric fact

$$|A_i G| = \frac{n}{n+1} m_i,$$

we get

Corollary 2 — Let Ω and Ω' be two n-dimensional simplices. Then

$$\sum_{i=0}^{n} S_i' m_i^2 \geq (n!)^{1/n} n^{1/2} (n+1)^{(3n-1)/2n} V^{2/n} V'^{(n-1)/n}, \qquad \ldots (3.4)$$

and equality is true if and Ω and Ω' are regular.

Taking Ω' for the regular simplex in (3.4), we obtain

Corollary 3 — For an n-dimensional simplex Ω, we have

$$\sum_{i=0}^{n} m_i^2 \geq \frac{(n!)^{2/n}(n+1)^{(2n-1)/n}}{n} V^{2/n}, \quad \text{... (3.5)}$$

and equality is true if and only if Ω is regular.

From (3.5), we can infer the following sharpening of inequality (3.3).

Corollary 4 — Let O and G be the circumcenter and centroid of Ω, respectively. Then

$$V^{2/n} \leq \frac{(n+1)^{(n+1)/n}}{n\,(n!)^{2/n}} (R^2 - |\overrightarrow{OG}|^2), \quad \text{... (3.6)}$$

and equality is true if and only if Ω is regular.

PROOF : Since

$$\sum_{i=0}^{n} \overrightarrow{GA_i} = \vec{0}, \quad |\overrightarrow{GA_i}| = \frac{n}{n+1} m_i,$$

we have

$$(n+1) R^2 = \sum_{i=0}^{n} \overrightarrow{OA_i}^2 = \sum_{i=0}^{n} (\overrightarrow{OG} + \overrightarrow{GA_i})^2$$

$$= (n+1)|\overrightarrow{OG}|^2 + 2\overrightarrow{OG} \cdot \sum_{i=0}^{n} \overrightarrow{GA_i} + \sum_{i=0}^{n} \overrightarrow{GA_i}^2$$

$$= (n+1)|\overrightarrow{OG}|^2 + \left(\frac{n}{n+1}\right)^2 \sum_{i=0}^{n} m_i^2.$$

Namely

$$R^2 - |\overrightarrow{OG}|^2 = \frac{n^2}{(n+1)^3} \sum_{i=0}^{n} m_i^2. \quad \text{... (3.7)}$$

The inequality (3.6) follows from (3.5) and (3.7). □

***Theorem* 3** — *For an n-dimensional simples Ω, we have*

$$\frac{R^2}{r} \geq n^{3/2} \left[\frac{n!}{(n+1)^{(n+1)/2}}\right]^{1/n} \cdot V^{1/n}, \quad \text{... (3.8)}$$

and equality is valid if and only if Ω is regular.

PROOF : Taking $\Omega' = \Omega, P$ the circumcenter of and Q the incenter of Ω in (1.1), we can get (3.8). □

Remark : Noting the known fact:

$$V^{1/n} \geq \left(\frac{1}{n!}\right)^{1/n} n^{1/2} (n+1)^{(n+1)/2n} r,$$

(see[8&14]), we know that inequality (3.7) is a sharpening of the well-known Euler inequality $R \geq nr$[14].

4. A CONJECTURE

We keep the notation of Section 1. We conjecture inequality (1.1) can be generalized, namely, we have

Conjecture

$$\sum_{i=0}^{n} \frac{D_i}{d_i'} \geq C(n) \left(\frac{V}{V'}\right)^{1/n},$$

where $C(n)$ is a constant only related to dimension n.

REFERENCES

1. L. Yang and J. Z. Zhang, *Bull. Austral. Math. Soc.*, **27** (1983), 203-14.
2. G. S. Leng, *Geom. Deddicata*, **66** (1997), 89-98.
3. G. s. Leng and L. H. Tang, *Acta Math. Sinica*, **40** (1997), 14-21.
4. G. S. Leng, *Geom. Deddicata*, **68** (1997), 43-48.
5. G. S. Leng and X. Z. Qian, *Disrete Math.*, **202** (1999), 163-72.
6. G. S. Leng, Z. Shen and L. H. Tang, *J. Math. Anal. Appl.*, **248** (2000), 429-37.
7. L. Gerber, *Pacific J. Math.*, **56** (1975), 97-111.
8. J. Z. Zhang and L. Yang, *J. China Univ. Sci. Tech.*, (in Chinese), **11** (1981), 1-8.
9. P. Bartoš, *Cas. Pest. Mat.*, **93** (1968), 273-77.
10. H. Q. Li and G. S. Leng, *Linear Algebra and Its Applications*, **185** (1993), 273-78.
11. J. Chen, *Bull. Math.*, (Fujian) **5** (1992), 8-9.
12. D. Mitrinovic, J. Pecaric and V. Volence, *Recent Advances in Geometric Inequalities*, Kluwer Academic, Dordrecht. 1989.
13. M. Ali, *Pacific J. Math.*, **33** (1970), 1-14.
14. L. Fejes, *Toth, Regular Figures*, Oxford-London-Edinburgh, 1964.