Indian J. pure appl. Math., 35(4) : 555-560, April 2004

ON RECURRENCE OR PSEUDO-SYMMETRY OF THE SASAKIAN METRIC ON THE TANGENT BUNDLE OF A RIEMANNIAN MANIFOLD

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(*Received* 21 *May* 2003; *accepted* 20 *January* 2004)

We consider the Riemannian space (M, g) and TM endowed with the Sasakian metric G associated with g. The aim of the paper is to show that if the curvature $\overline{R}$ of G is recurrent or Pseudo-symmetric (in the sense of Chaki), then both (M, g) and (TM, G) must be flat.

Key Words: Sasakian Metric; Recurrence; Pseudo-symmetry

1. INTRODUCTION

The notion of Pseudo-symmetric Riemannian manifold was introduced by Chaki[2] and studied intensively by De[3], Tarafdar[4] and others. Let (M, g) be a Riemannian manifold and (TM, G) its tangent bundle with the induced Sasakian metric. Kowalski[5] proved that the Riemannian manifold (TM, g) is locally symmetric if and only if (M, g) is flat. In this case also (TM, g) is flat. In the present paper we show that the theorem of Kowalski holds also in a more general case, namely we prove the

Theorem 1 — *If (TM, g) is recurrent or a Pseudo-symmetric Riemannian manifold, then (M, g) must be flat and thus (TM, g) must be-flat too. The converse is trivially true.*

2. PRELIMINARIES

Let (M, g) be a Riemannian manifold and let ∇ denote its Levi-Civita connection. If X is a vector field on M, its vertical lift on TM at $t \in T_x M$ is the tangent vector of the curve $c(s) = t + sX(x)$ at $c(0)$. Let $\alpha(s)$ be a smooth curve on M such that $\alpha(0) = x$, its tangent vector at x is $X(x)$ and let $T(s)$ be the parallel vector field along $\alpha(s)$ with $T(0) = t$. Then the horizontal lift of X at t is defined as the tangent vector $T(s)$ at t. The vertical and horizontal lifts of X are uniquely defined. Locally, let $\pi : TM \to M$ be the tangent bundle. If $(x^1, x^2, ..., x^n)$ are local coordinates on M, set $q^i = x^i \circ \pi$, then $(q^1, ..., q^n)$ together with the fibre coordinates $(v^1, ..., v^n)$ form the local coordinates

*Dedicated to Professor M. C. Chaki on the occassion of his 90th Birthday.

on TM. By direct calculation we obtain : if $X = X^i \frac{\partial}{\partial z^i}$, then the vertical and the horizontal lift of X at the point $t = (q^1, ..., q^n, v^1, ..., v^n)$ are:

$$X^V|_t = X^i \frac{\partial}{\partial v^i}|_t \quad \text{and} \quad X^H|_t = X^i \frac{\partial}{\partial q^i}|_t - X^i v^j \Gamma_{ij}^k \frac{\partial}{\partial v^k}|_t .$$

The set of horizontal lifts at t is called the horizontal subspace H_t of $T_t TM$. Respectively, the vertical subspace V_t of $T_t TM$ is the set of the vertical lifts at t. We have the direct sum $V_t \oplus H_t = T_t TM$. The connection map $K : TTM \to TM$ is defined by

$$KX^H = 0, \quad KX^V|_t = X_{\pi(t)}, \quad t \in TM. \qquad \text{... (1)}$$

Now the Sasakian metric G on TM, Sasaki[6], is defined by

$$G(\tilde{X}, \tilde{Y}) = (g(\pi_*(\tilde{X}), \pi_*(\tilde{Y})) + g(K\tilde{X}, K\tilde{Y})) \circ \pi, \qquad \text{... (2)}$$

where $\tilde{X}$ and $\tilde{Y}$ are vector fields on TM. Let R be the curvature tensor of the Levi-Civita connection ∇ of g. The Levi-Civita connection $\tilde{\nabla}$ of G and its curvature tensor $\tilde{R}$ were computed by Kowalski[5]. We can also refer to Blair[1], 9.1. From the results of Kowalski we recall the following:

$$(a)\ (\tilde{\nabla}_{X^H} Y^H)|_t = (\nabla_X Y)^H|_t - \frac{1}{2}(R(X, Y)t)^V|_t ,$$

$$(b)\ (\tilde{\nabla}_{X^V} Y^H)|_t = (\nabla_X Y)^V|_t - \frac{1}{2}(R(t, Y)X)^H|_t , \qquad \text{... (3)}$$

$$(c)\ (\tilde{\nabla}_{X^V} Y^H)|_t = \frac{1}{2}(R(t, X)Y)^H|_t ,$$

$$(d)\ (\nabla_{X^V} Y^V)|_t = 0.$$

From these formulas it follows

$$(a)\ \tilde{R}(X^V, Y^V)Z^V = 0, \qquad \text{... (4)}$$

$$(b)\ \tilde{R}(X^V, Y^V)Z^H|_t$$

$$= \left[R(X, Y)Z + \frac{1}{4}R(t, X)R(t, Y)Z - \frac{1}{4}R(t, Y)R(t, X)Z \right]^H |_t ,$$

$$(c)\ \tilde{R}(X^H, Y^V)Z^V|_t = -\left[\frac{1}{2}R(Y, Z)X + \frac{1}{4}R(t, Y)R(t, Z)X \right]^H |_t ,$$

$$(d)\ \tilde{R}(X^H, Y^V)Z^H|_t = \frac{1}{2}\left[(\nabla_X R)(t, Y, Z) \right]^H |_t$$

$$+\left[\frac{1}{2}R(X,Z)\,Y+\frac{1}{4}R(R(t,Y)\,Z,X)\,t\right]^{V}\Big|_{t},$$

$$(e)\ \tilde{R}(X^{H},Y^{H})Z^{V}|_{t}=\frac{1}{2}\left[(\nabla_{X}R)(t,Z,Y)-(\nabla_{Y}R)(t,Z,X)\right]^{H}\Big|_{t}$$

$$+\left[R(X,Y)Z+\frac{1}{4}R(R(t,Z)Y,X)\,t-\frac{1}{4}R(R(t,Z)\,X,Y)\,t\right]^{V}\Big|_{t},$$

$$(f)\ \tilde{R}(X^{H},Y^{H})Z^{H}|_{t}=\left[R(X,Y)Z+\frac{1}{4}R(t,R(Z,Y)\,t)\,X\right.$$

$$\left.+\frac{1}{4}R(t,R(X,Z)\,t)\,Y+\frac{1}{2}R(t,R(X,Y)\,t)\,Z\right]^{H}\Big|_{t}$$

$$+\frac{1}{2}\left[(\nabla_{Z}R)(X,Y,t)\right]^{V}\Big|_{t}.$$

3. Proof of the Theorem

We show the flatness of (M,g), first if $\tilde{R}$ is recurrent, and then if $\tilde{R}$ is Pseudo-symmetric.

1. If $\tilde{R}$ is recurrent then there exists a 1-form α on TM so that

$$(\tilde{\nabla}_{\tilde{W}}\tilde{R})(\tilde{X},\tilde{Y})\tilde{Z}=\alpha(\tilde{W})\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z}.$$

Then

$$\alpha(W^{H})(\tilde{R}(X^{H},Y^{V})Z^{V})|_{t}=[(\tilde{\nabla}_{W^{H}}\tilde{R})(X^{H},Y^{V})Z^{V}]|_{t}.$$

Using the formulas of covariant derivation and those of (3a, b) and (4c), we obtain

$$\alpha(W^{H})(\tilde{R}(X^{H},Y^{V})Z^{V})|_{t}$$

$$=\tilde{\nabla}_{W^{H}}\left[-\frac{1}{2}R(Y,Z)X-\frac{1}{4}R(t,Y)\,R(t,Z)\,X\right]^{H}\Big|_{t}$$

$$-\tilde{R}\left[(\nabla_{W}X)^{H}|_{t}-\frac{1}{2}(R(W,X)\,t)^{V}|_{t}),Y^{V}\right]Z^{V}$$

$$-\tilde{R}\left[X^{H},\frac{1}{2}(R(t,Y)\,W)^{H}|_{t}+(\nabla_{W}Y)^{V}|_{t}\right]Z^{V}$$

$$-\tilde{R}(X^{H},Y^{V})\left[\frac{1}{2}(R(t,Z)\,W)^{H}|_{t}+(\nabla_{W}Z)^{V}|_{t}\right].$$

Using the formulas (3a), (4a,c,d,e) and taking the vertical parts of the two sides one can conclude on the flatness of the Riemannian manifold (M,g). In order to see this we repeat the nice proof of Kowalski[5]. We obtain:

$$0=\frac{1}{2}R\left[W,\frac{1}{2}R(Y,Z)X+\frac{1}{4}R(t,Y)\,R(t,Z)\right]t-R\left[X,\frac{1}{2}R(t,Y)W\right]Z$$

$$+\frac{1}{4}R\left[R(t,Z)\frac{1}{2}R(t,Y)W,X\right]t-\frac{1}{4}R\left[R(t,Z)X,\frac{1}{2}R(t,Y)W\right]t$$

$$-\frac{1}{4}R\left[R(t,Y)\frac{1}{.2}R(t,Z)W,X\right]t+\frac{1}{2}R\left[X,\frac{1}{2}R(t,Z)W\right]Y.$$

Setting $Y = t$ and $Z = t$ respectively we get

$$R(W,R(t,Z)X)t-R(X,R(t,Z)W)t=0$$

and

$$R(W,R(Y,t)X)t-2R(X,R(t,Y)W)t=0.$$

Now replace Y by Z in the second of these equations and add it to the first. We obtain that $R(X,R(t,Z)W)t=0$. Setting $W = X$ and taking the inner product with Z yields $|R(t,Z)X| = 0$. Hence (M, g) is flat.

2. If $\tilde{R}$ is Pseudo-symmetric then there exists a 1-form α on TM so that (see Chaki[2]).

$$(\tilde{\nabla}_{\tilde{X}}\tilde{R})(\tilde{Y}\tilde{Z}\tilde{V})=2\alpha(\tilde{X})\tilde{R}(\tilde{Y},\tilde{Z})\tilde{V}+\alpha(\tilde{Y})\tilde{R}(\tilde{X},\tilde{Z})\tilde{V}$$

$$+\alpha(\tilde{Z})\tilde{R}(\tilde{Y},\tilde{X})\tilde{V}+\alpha(\tilde{V})\tilde{R}(\tilde{Y},\tilde{Z})\tilde{X}+G(\tilde{R}(\tilde{Y},\tilde{Z})\tilde{V},\tilde{X})\tilde{A}, \quad \ldots (5)$$

where $\tilde{A}$ is the vector field corresponding through G to the 1-form α by $G(X,\tilde{A})=\alpha(\tilde{X})$.

Now we consider (5) for W^H, X^H, Y^V and Z^V. From the formulas of the covariant derivative and from (3) and (4) we have

$$2\alpha(W^H)\tilde{R}(X^H,Y^V)Z^V+\alpha(X^H)\tilde{R}(W^H,Y^V)Z^V$$

$$+\alpha(Y^V)\tilde{R}(X^H,W^H)Z^V+\alpha(Z^V)\tilde{R}(X^H,Y^V)W^H$$

$$+G(\tilde{R}(X^H,Y^V)Z^V,W^H)\tilde{A}$$

$$=\tilde{\nabla}_{W^H}\left[-\frac{1}{2}R(Y,Z)X-\frac{1}{4}R(t,Y)R(t,Z)X\right]^H\Big|_t$$

$$-\tilde{R}\left[(\nabla_W X)^H|_t-\frac{1}{2}(R(W,X)t)^V|_t),Y^V\right]Z^V$$

$$-\tilde{R}(X^H,\frac{1}{2}(R(t,Y)W)^H|_t+(\nabla_W Y)^V|_t)Z^V$$

$$-\tilde{R}(X^H,Y^V)\left[\frac{1}{2}(R(t,Z)W)^H|_t+(\nabla_W Z)^V|_t\right].$$

Using the formulas (4) for the curvature tensor $\tilde{R}$, taking the vertical parts of both sides and taking into account (1) and (2) we have

$$\alpha(Y^V)\left[R(X,W)Z+\frac{1}{4}R(R(t,Z)W,X)t-\frac{1}{4}R(R(t,Z)X,W)t\right]$$

$$+\alpha(Z^V)\left[\frac{1}{2}R(X,W)Y+\frac{1}{4}(R(t,Y)W,X)t\right]$$

$$-g\left(\frac{1}{2}R(Y,Z)X+\frac{1}{4}R(t,Y)R(t,Z)X,W\right)\tilde{A}^V$$

$$=\frac{1}{2}R\left[W,\frac{1}{2}R(Y,Z)X+\frac{1}{4}R(t,Y)R(t,Z)X\right]t$$

$$-R\left(X,\frac{1}{2}R(t,Y)W\right)Z+\frac{1}{4}R\left[R(t,Z)\frac{1}{2}R(t,Y)W,X\right]t$$

$$-\frac{1}{4}R\left[R(t,Z)X,\frac{1}{2}R(t,Y)W\right]t-\frac{1}{4}R\left[R(t,Y)\frac{1}{2}R(t,Z)W,X\right]t$$

$$+\frac{1}{2}R\left[X,\frac{1}{2}R(t,Z)W\right]Y.$$

By setting $Y = t$, respectively $Z = t$ we get

$$\alpha(t^V)\left[R(X,W)Z+\frac{1}{4}R(R(t,Z)W,X)t-\frac{1}{4}R(R(t,Z)X,W)t\right]$$

$$+\alpha(Z^V)\frac{1}{2}R(X,W)t-\frac{1}{2}g(R(t,Z)X,W)\tilde{A}^V=R[W,R(t,Z)X]t$$

$$-R[X,R(t,Z)W]t,$$

and

$$\alpha(Y^V)R(X,W)t+\alpha(t^V)\left[\frac{1}{2}R(X,W)Y+\frac{1}{4}R(R(t,Y)W,X)t\right]$$

$$-\frac{1}{2}g(R(Y,t)X,W)\tilde{A}^V=R[W,R(Y,t)X]t-2R[X,R(t,Y)W]t.$$

Replace Y by Z in the second of these equations and add it to the first. We get

$$\alpha(Z^V)\left[R(X,W)t+\frac{1}{2}R(X,W)t\right]+\alpha(t^V)$$

$$\left[R(X,W)Z+\frac{1}{2}R(X,W)Z+\frac{1}{2}R(R(t,Z)W,X)t\right. \qquad \ldots (6)$$

$$\left.-\frac{1}{4}R(R(t,Z)X,W)t\right]-2R[X,R(t,Z)W]t$$

$$=-3R(X,R(t,Z)W)t.$$

By setting $Z = t$ we obtain $3\alpha(t^V)\ R(X,W)t = 0$. If $\alpha(t^V)$ is no 0 then we have $R(X,W)t = 0$. If $\alpha(t^V) = 0$, then from (6) we obtain

$$\alpha(Z^V)\left(\frac{1}{2}R(X,W)t+R(X,W)t\right)=-3R(X,R(t,Z)W)t.$$

By setting now $W = X$ we get $R(X, R(t, Z)X)t = 0$. Taking the inner product with Z, it follows that $g(R(t, Z)X, R(t, Z)X) = 0$, and thus $R(t, Z)X = 0$. Taking now the inner product with an arbitrary vectorfield Y and using an elementary property of the curvature tensor we have $g(R(X, Y)t, Z) = 0$ for any vector field Z. Thus $R(X, Y)t = 0$ for any vector fields X, Y, t on M. So we have $R = 0$, and hence (M, g) is flat.

Conversely, if $R = 0$ then from the formula (4) we immediately have $\tilde{R} = 0$. This means that also (TM, G) is flat.

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