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CONULLITY AND REPLACEABILITY OF MATRICES ON RATE SPACES

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In this paper conullity and replaceability of matrix methods, are presented.

Key Words: FK-Space; Conullity; AK-Space; Replaceability; Rate Space

1. INTRODUCTION

Let $\rho = (\rho_n)$, $\pi = (\pi_n)$ be sequences of positive numbers i.e , $\rho_n > 0$, $\pi_n > 0$, $\forall n \in \mathbb{N}$ ($\mathbb{N}$ positive integers) and X an FK space. Let w denote the space of all complex-valued sequences. We shall consider the sets of sequences $x = (x_n)$

$$X_\pi = \left\{ x \in w \, : \, \left(\frac{x_n}{\pi_n} \right) \in X \right\}.$$

The set X_π may be considered as FK-space. We shall call them as rate spaces[4]. If $X = c$ then we get the rate space $c_\pi = \left\{ x : \lim\limits_n \frac{x_n}{\pi_n} \text{ exists} \right\}$ which is a BK-space with $\| x \|_\pi := \sup\limits_n \left| \frac{x_n}{\pi_n} \right|$. Throughout the paper e denotes the sequences of ones, (1, 1, ..., 1, ...); δ^j, (j = 1, 2, ...) the sequence (0 0 ..., 0, 1, 0, ..) with the one in the jth position. By c, ℓ^∞, c_0 we denote the spaces of all convergent sequences bounded sequences and null sequences, respectively. These are FK-spaces under $\| x \| = \sup\limits_n | x_n |$. By ℓ we denote the space of absolute summable sequences. A sequence x in a locally convex sequence space X is said the property AK if $x^{(n)} \to x$ in X where

$$x^{(n)} = (x_1, x_2, ..., x_n, 0, ...) = \sum_{k=1}^{n} x_k \, \delta^k.$$

We shall be concerned with matrix transformations

$$y = Ax, \text{ where } x, y \in w, \; A = (a_{nk}), n, k = 1, 2, \ldots$$

and is an infinite matrix with complex coefficients, and $y_n = \sum_{k=1}^{\infty} a_{nk} x_k$ $(n = 1, 2, ...)$.

2. Known Theorems

Theorem $1^{2\&3}$ — *Let* $c_{\pi A} = \{x : Ax \in c_\pi\}$. *Domain* $c_{\pi A}$ *is an FK-space with seminorms*

$$p_0(x) = \sup_n \frac{1}{\pi_n} \left| \sum_{k=1}^{\infty} a_{nk} x_k \right|,$$

$$p_{2n}(x) = |x_n|, \quad (n = 1, 2, ...)$$

and

$$p_{2n-1}(x) = \sup_m \left| \sum_{k=1}^{m} a_{nk} x_k \right|, \quad (n = 1, 2, ...).$$

Theorem $2^{2\&3}$ — $f \in c^*_{\pi A}$ *if and only if*

$$f(x) = \sum_{k=1}^{\infty} \beta_k x_k + \sum_{n=1}^{\infty} t_n \sum_{k=1}^{\infty} a_{nk} x_k + \mu \lim_{\pi A} x, \qquad ... (1)$$

where $(t_n \pi_n) \in l$, $\mu \in \mathbb{C}$ ($\mathbb{C}$ *the set of all complex numbers),* $(\beta_k) \in c^\beta_{\pi A}$, *the* β*-dual of* $c_{\pi A}$, *and* $\lim_{\pi A} x = \lim_n \sum_{k=1}^{\infty} \frac{a_{nk}}{\pi_n} x_k$ *and* $\beta x := \sum_{k=1}^{\infty} \beta_k x_k$.

Theorem $3^{2\&3}$ — $A \in (c_\rho, c_\pi)$ *if and only if*

$$\lim_n \frac{a_{nk}}{\pi_n} = a_k^\pi \text{ exist for ali } k; \qquad ... (2)$$

$$\lim_n \frac{1}{\pi_n} \sum_{k=1}^{\infty} a_{nk} \rho_k = a^{\pi \rho} \text{ exists}; \qquad ... (3)$$

$$\sum_{k=1}^{\infty} |a_{nk}| \left(\frac{\rho_k}{\pi_n} \right) \le M \qquad ... (4)$$

where M is a constant independent of n.

Theorem 4^1 — *Let* $A \in (c, c_\pi)$. *Then we have* $\chi(f) = \mu \chi_\pi(A)$ *where*

$$\chi(f) = f(e) - \sum_{k=1}^{\infty} f(\delta^k);$$

$$\chi_\pi(A) = a^\pi - \sum_{k=1}^{\infty} a_k^\pi \quad \textit{with } a^\pi = \lim_n \sum_{k=1}^{\infty} \frac{a_{nk}}{\pi_n}$$

and

$$a_k^\pi = \lim_n \frac{a_{nk}}{\pi_n}.$$

Theorem 5[3] — *For each $f \in c^*_{\pi A}$ and for any rates π and ρ there exists a matrix B with $c_{\pi A} \subset c_{\rho B}$ and $\lim_{\rho B} x = f(x)$ $\forall x \in c_{\pi A}$. If f has a representation (1) with $\mu \neq 0$ then there exists a matrix B with $c_{\pi A} = c_{\rho B}$ and $\lim_{\rho B} x = f(x)$, $(\forall x \in c_{\pi A})$.*

Lemma 1[3] — If $A \in (c_\rho, c_\pi)$ then for any $f \in c^*_{\pi A}$ and for any $x \in \ell^\infty_\rho \cap c_{\pi A}$

$$f(x) = \sum_{k=1}^{\infty} s_k x_k + \mu \lim_{\pi A} x$$

where $\mu \in \mathbb{C}$, $s = (s_k) \in (\ell^\infty_\rho \cap c_{\pi A})^\beta$ and $\ell^\infty_\rho = \left\{ x \in w : \frac{x_n}{\rho_n} = O(1) \right\}$.

3. Main Results

Now we will get some results which are similar to those given in[6]. (Chapters 3 and 5).

Definition 1 — Let $A \in (c, c_\pi)$ and $\chi_\pi(A) = a^\pi - \sum_{k=1}^{\infty} a_k^\pi$. If $\chi_\pi(A) = 0$, then we say that the matrix A is π-conull. Otherwise it is π-coregular.

If we take $\pi = e$ in above, then we get the definition of ordinary conull matrix.

No we have

Theorem 6 — *Let $A \in (c, c_\pi)$. Then a matrix A i π-conull if and only if $c_{\pi A}$ is a conull space.*

Proof : By Definition 1, A is π-conull if and only if $\chi_\pi(\lim_{\pi A}) = 0$. By Theorem 4, $\chi_\pi(\lim_{\pi A}) = 0$ if and only if $\chi(f) = 0$ for all $f \in c^*_{\pi A}$. Conversely, suppose that $c_{\pi A}$ is a conull space. It follows from the Banach-Steinhaus theorem that $f = \lim_{\pi A} \in c^*_{\pi A}$. Therefore,

$$0 = \chi(\lim_{\pi A}) = \lim_{\pi A} e - \sum_{k=1}^{\infty} \lim_{\pi A} \delta^k = \chi_\pi(A),$$

this completes the proof of the theorem.

Theorem 7 — *Let $A \in (c, c_\pi)$, $B \in (c, c_\pi)$ with $c_{\pi A} \subset c_{\pi B}$. If A is π-conull, so is B.*

PROOF : Let A be π-conull. Then Theorem 6, implies that $c_{\pi A}$ is a conull space. By Theorem 4.6.3 of[6] $c_{\pi B}$ is a conull space. Hence B is π-conull.

Corollary 1 — Let $A \in (c, c_\pi)$. If c is closed in $c_{\pi A}$, then A is π-coregular.

PROOF : The proof is trivial by Theorem 6 and (4.6.4) of[6].

Corollary 2 — Let $A \in (c, c_\pi)$. If the matrix A is not π-conull, then $c_{\pi A}$ cannot have the property AK.

PROOF : Now asusme that $c_{\pi A}$ has the property AK, then for each $x \in c_{\pi A}$, we have

$$x = \sum_{k=1}^{\infty} x_k \delta^k.$$

In particular we have $e = \sum_{k=1}^{\infty} \delta^k$ since $e \in c_{\pi A}$. If we take $f = \lim_{\pi A} \in c^*_{\pi A}$, then we have

$$\lim_n \sum_{k=1}^{\infty} \frac{a_{nk}}{\pi_n} = \sum_{k=1}^{\infty} \lim_n \frac{a_{nk}}{\pi_n}$$

for each $f(e) = \sum_{k=1}^{\infty} f(\delta^k)$ which is a contradiction.

Theorem 8 — *Let $A \in (c, c_\pi)$. Then there exists a matrix $B \in (c, c_\pi)$, with $c_{\pi A} \cap \ell^\infty = c_{\pi B} \cap \ell^\infty$ and $\chi_\pi(A) = \chi_\pi(B)$ we have*

(i) if A is π-coregular then $\lim_n \sum_{k=1}^{\infty} \frac{b_{nk}}{\pi_n} = 1$,

(ii) if A is π-conull then $\lim_n \sum_{k=1}^{\infty} \frac{b_{nk}}{\pi_n} = 0$.

PROOF : To prove the theorem we use the technique given by Wilansky[6]. First let $\frac{b_{nk}}{\pi_n} = \frac{a_{nk}}{\pi_n} - a_k^\pi$. Then

$$\| B \|_\pi = \sup_n \sum_{k=1}^{\infty} \left| \frac{b_{nk}}{\pi_n} \right| \leq \| A \|_\pi + \sum_{k=1}^{\infty} \left| a_k^\pi \right| < \infty$$

by Theorem 3 and Proposition 4.4 (*iv*) of[2], $b_k^\pi = a_k^\pi - a_k^\pi = 0$, then B is multiplicative, that is;

$\lim_n \dfrac{b_{nk}}{\pi_n} = 0$. On the other hand, $\chi_\pi(B) = \lim_n \sum_{k=1}^{\infty} \dfrac{b_{nk}}{\pi_n} = \chi_\pi(A)$. Also, for $x \in \ell^\infty$

$$\sum_{k=1}^{\infty} \frac{b_{nk}}{\pi_n} x_k - \sum_{k=1}^{\infty} \frac{a_{nk}}{\pi_n} x_k = \sum_{k=1}^{\infty} a_k^\pi x_k$$

for all n, then $c_{\pi A} \cap \ell^\infty = c_{\pi B} \cap \ell^\infty$. Now we consider $\dfrac{b_{nk}}{\pi_n} = \dfrac{1}{\chi_\pi(A)}\left(\dfrac{a_{nk}}{\pi_n} - a_k^\pi\right)$. Then if A is π-coregular we get $\lim_n \sum_{k=1}^{\infty} \dfrac{b_{nk}}{\pi_n} = 1$ and if A is π-conull we have $\lim_n \sum_{k=1}^{\infty} \dfrac{b_{nk}}{\pi_n} = 0$. So the proof is completed.

By Wilansky[5,6], we define

$$W := W(c_{\pi A}) := \left\{ x \in c_{\pi A} : f(x) = \sum_{k=1}^{\infty} x_k f(\delta^k) \text{ for all } f \in c_{\pi A}^* \right\}.$$

We recall that

$$I = \left\{ x \in c_{\pi A} : \sum_{k=1}^{\infty} a_k^\pi x_k \text{ converges} \right\}.$$

Now consider the functional Λ_π defined by $\Lambda_\pi(x) = \lim_{\pi A} x - \sum_{k=1}^{\infty} a_k^\pi x_k$ for $x \in I$. In particular if $\Lambda_\pi(e) = \chi_\pi(A)$ then we get A is π-conull if and only if $e \in \Lambda_\pi^\perp$, where $\Lambda_\pi^\perp = \{x \in I : \Lambda_\pi(x) = 0\}$.

Lemma 2 — Let $A \in (c, c_\pi)$. Then, A is π-coregular if and only if $e \notin W$.

PROOF : If $x \in W$, take $f = \lim_{\pi A}$ in definition of W. This yields $\chi_\pi(A) = 0$. Conversely, if A is π-conull, Theorem 4 shows that $e \in W$.

Lemma 3 — Let $A \in (c_\rho, c_\pi)$ and $x \in c_{\pi A} \cap \ell_\rho^\infty$. Then $x \in W$ if and only if $\Lambda_\pi(x) = 0$.

PROOF : Let $x \in W$. If we take $f = \lim_{\pi A}$ in definition of W, then we have

$$\lim_{\pi A} x = \sum_{k=1}^{\infty} x_k \lim_{\pi A} \delta^k = \sum_{k=1}^{\infty} a_k^\pi x_k.$$

Conversely if $\Lambda_\pi(x) = 0$, let $f \in c_{\pi A}^*$. Then by Lemma 1, we get

$$f(x)=\mu \lim_{\pi A} x+sx=\mu \sum_{k=1}^{\infty} a_k^{\pi} x_k+sx$$

with hypothesis. Also,

$$\sum_{k=1}^{\infty} x_k f(\delta^k)= \sum_{k=1}^{\infty} x_k (\mu a_k^{\pi}+s_k)=f(x),$$

hence $x \in W$.

Theorem 9 — *Let $A \in (c, c_{\pi})$, $B \in (c, c_{\pi})$ and $\lim_n \frac{1}{\pi_n} = 1$, with $c_{\pi A} \cap \ell^{\infty} \subset c_{\pi B}$. If A is π-conull so is B.If B is π-coregular so is A. If $c_{\pi A} \cap \ell^{\infty} = c_{\pi B} \cap \ell^{\infty}$, then they are both π-coregular or both π-conull.*

PROOF : We proved only the first statement. Let $D=B-\chi_{\pi}(B) I$. Then D is π-conull, so is a conull space by Theorem 6. Hence $c_{\pi A} \cap c_{\pi A}$ is conull by (4.6.6) of[6] and so, by Theorem 6.1.1 of[6], contains a bounded divergent x. By hypothesis $x \in c_{\pi B}$, then $\chi_{\pi}(B)\, x=Bx-Dx$ and so $\chi_{\pi}(B) = 0$. This proves the result.

Let z be a sequence and A be an infinite matrix. Then we get immediately the following.

Lemma 4 — Let $A \in (c_{\rho}, c_{\pi})$ and $z \in c_{\pi A} \cap \ell^{\infty}_{\rho}$. Then $z \in W$ if and only if $A \cdot z=(a_{nk} z_k)$ is π-conull.

PROOF : This follows from the fact that $\chi_{\pi}(A \cdot z)=\Lambda_{\pi}(z)$, with Lemma 3.

Lemma 5 — Let $A \in (c, c_{\pi}), B \in (c, c_{\pi})$, with $c_{\pi A} \cap \ell^{\infty} \subset c_{\pi B}$. Then

$$W(c_{\pi A}) \cap \ell^{\infty} \subset W(c_{\pi B}).$$

PROOF : If $z \in W(c_{\pi A}) \cap \ell^{\infty}$, $A \cdot z$ is π-conull by Lemma 4 and so $B \cdot z$ is π-conull by Theorem 9. Hence $z \in W(c_{\pi B})$ by Lemma 4.

Now we prove a bounded consistency theorem.

Theorem 10 — *Let $A \in (c, c_{\pi})$ and $B \in (c, c_{\pi})$. Let A, B be π-coregular matrices with $c_{\pi A} \cap \ell^{\infty} \subset c_{\pi B}$ and $\lim_{\pi A} x=\lim_{\pi B}$ for $x \in c$. Then A, B are consistent for bounded sequences.*

PROOF : It is clear that $b_k^{\pi}= \lim_{\pi B} \delta^k= \lim_{\pi B} \delta^k=a_k^{\pi}$. Let $z \in c_{\pi A} \cap \ell^{\infty}$.

Step 1 : Supposed that $\Lambda_{\pi A}(z) = 0$. We have $\Lambda_{\pi B}(z) = 0$ by Lemma 5 and 3. Therefore, $\lim_{\pi B} z = \sum_{k=1}^{\infty} b_k^{\pi} z_k$ and $\lim_{\pi A} z = \sum_{k=1}^{\infty} a_k^{\pi} z_k$. These are equal as just mentioned.

Step 2 : Now assume that $\Lambda_{\pi A}(z) \neq 0$. Also, let $y = z - te$ with $t = \dfrac{\Lambda_{\pi A}(z)}{\Lambda_{\pi A}(e)}$. Thus $\Lambda_{\pi A}(y) = 0$ so $\lim_{\pi B} y = \lim_{\pi A} y$ by Step 1. Since $\lim_{\pi B} e = \lim_{\pi A} e$ it follows that $\lim_{\pi B} z = \lim_{\pi A} z$. Thus the theorem is proved.

4. REPLACEABILITY

Definition 2 — Let $\pi - (\pi_n)$ be sequence of positive numbers. A matrix A is c_{π}-replaceable if there is a matrix B with $c_{\pi A} = c_{\pi B}$ and $\lim_n \dfrac{b_{nk}}{\pi_n} = 0$, $(\forall k \in \mathbb{N})$.

If we take $\pi = e$ in definition 2, then we have the definition of ordinary c-replaceable.

If a matrix A is π-conull any multiplicative D with $c_{\pi A} = c_{\pi D}$ then it must be multiplicative-0. To see that, let $f \in \overset{*}{c}_{\pi A}$. If we take $f = \lim_{\pi D}$, then we have $\lim_{\pi D} \in \overset{*}{c}_{\pi A}$ by corollary 2.7. Also,

$$\chi_{\pi}\left(\lim_{\pi D}\right) = \chi_{\pi}(D) = \mu\chi_{\pi}(A)$$

by Theorem 4. Thus matrix D is π-conull by hypothesis. For a matrix D

$$\lim_{\pi D} x = \chi_{\pi}(D) \lim x + \sum_{k=1}^{\infty} d_k^{\pi} x_k$$

by Proposition 4.5. Since matrix D is multiplicative, we have $d_k^{\pi} = 0$ for $k = 1, 2, 3, \dots$. Therefore, we obtain $\lim_{\pi D} x = 0 \cdot \lim x$. On the other hand, a π-coregular matrix is c_{π}-replaceable if and only if there exists a regular matrix D with $c_{\pi A} = c_{\pi D}$.

We recall that

$$P = \{x \in c_{\pi A} : t(Ax) = (tA)x, \pi t \in \ell\}^1.$$

Theorem 11 — *Let A be π-coregular. Then the following are equivalent:*

(i) A is c_{π}-replaceable,

(ii) $e \notin \overline{c_0}$ (closure in $c_{\pi A}$),

(iii) $e \notin \overline{\phi}$ *(closure in* $c_{\pi A}$*),*

(iv) lim is continuous on c as a subspace of $c_{\pi A}$,

(v) $x \to \beta x$ *is continuous on c as a subspace of* $c_{\pi A}$.

PROOF : *(ii)* $\Leftrightarrow$ *(iii)* : It is enough to take $Y = c_{\pi A}, X = c_0$ and $E = \phi$ in Theorem 4.2.7 of[6]

(i) implies *(iv)* : Suppose that A is c_π-replaceable, then by the remarks preceding the theorem there exists a regular matrix D with $c_{\pi A} = c_{\pi D}$. Also, $\lim_{\pi D} \in \overset{*}{c}_{\pi A}$. We get $\lim_{\pi D} = \lim$ on c, so continuity is guaranteed by invariance of topology by Corollary 4.2.4 in[6].

(iv) implies *(i)* : Now let $f \in \overset{*}{c}_{\pi A}$, $f = \lim$ on c by the Hahn-Banach theorem.

Then $1 = \chi(f) = \mu \chi_\pi(A)$ by Theorem 4. Since A is π-coregular, $\mu \neq 0$. The result follows by Theorem 5.

(iv) implies *(ii)* : Let lim is continuous on c as a subspace of $c_{\pi A}$. Since $c_0 \subset c_{\pi A}$ and *lim* = 0 on c_0 and lim e = 1, by Hahn-Banach theorem $e \notin \overline{c_0}$.

(ii) implies *(iv)* : In c, with the $c_{\pi A}$ topology c_0 is a maximal subspace which is not dense. Therefore, by (5.0.1) of[6], it is closed. But $f^{\perp} = \lim^{\perp} = c_0$ so lim is continuous by (5.0.1) of[6].

(iv) $\Leftrightarrow$ *(v)* : By Proposition 4.5 of[2], for $x \in c$,

$$\lim_{\pi A} x = \chi_\pi(A) \lim x + \sum_{k=1}^{\infty} a_k^{\pi} x_k.$$

Since $\lim_{\pi A}$ is continuous and $\chi_\pi(A) \neq 0$ the result follows.

Theorem 12 — *If* c_0 *is not dense in P, then A is* c_π*-replaceable.*

PROOF : Suppose that there exists $f \in \overset{*}{c}_{\pi A}$ with $f = 0$ on c_0. Then since $f(\delta^k) = 0$, we have

$$f(x) = \mu \lim_{\pi A} x + t(Ax) + \sum_{k=1}^{\infty} \left[-\mu a_k^{\pi} - \sum_{k=1}^{\infty} t_n a_{nk} \right] x_k$$

by Theorem 2. If $\mu = 0$, then $f = 0$ on P. Therefore, we get $P \subset \overline{c_0}$ by Hahn-Banach theorem. On the other hand, by Theorem 12 of[1] $P = \overline{c_0}$, this contradict our present hypothesis. Hence $\mu \neq 0$. In that case there is a matrix D with $c_{\pi A} = c_{\pi D}$ and $\lim_{\pi D} f = 0$ by Theorem 5. If we take $x = \delta^k \in c_0$, $(k \in \mathbb{N})$, then

$$f(\delta^k) = \lim_{\pi D} \delta^k = \lim_n \frac{d_{nk}}{\pi_n} = 0$$

which proves the theorem.

Theorem 13 — *Let A be π-coregular. Then A is c_π-replaceable if and only if c_0 is not dense in P.*

PROOF : Since A is π-coregular, we have $P = \bar{c}$ by Theorem 12,[1]. Let A is c_π-replaceable, then $e \notin \overline{c_0}$ by Theorem 11. Thus, we get $P \neq \overline{c_0}$. Therefore, the result follows from Theorem 12.

A matrix is called $\pi\mu$-unique if all representations of each $f \in c^*_{\pi A}$ as in Theorem 2 have the same μ.

Theorem 14 — *A non-$\pi\mu$-unique matrix must be c_π-replaceable.*

PROOF : There is a representation of the function 0 with $\mu \neq 0$. So there exists D with $c_{\pi A} = c_{\pi D}$ and $\lim_{\pi D} = 0$ by Theorem 5. Hence A is c_π-replaceable.

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