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OPTIMAL EXISTENCE THEORY FOR SINGLE AND MULTIPLE POSITIVE SOLUTIONS TO SECOND ORDER NEUMANN BOUNDARY VALUE PROBLEMS*

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This paper deals with a new optimal existence theory for single and multiple positive solutions to second order Neumann boundary value problems by employing the fixed point theorem in cones. The paper improves some previous results and reports some new results.

Key Words: Neumann Boundary Value Problem; Positive Solution; The Fixed Point Theorem

1. INTRODUCTION

The purpose of the present paper is to deal with the optimal existence criteria for single and multiple positive solutions for Neumann boundary value problems (BVP)

$$\left.\begin{aligned} &-u''(t) + \rho_1^2 u(t) = f(t, u(t)),\ 0 < t < 1, \\ &u'(0) = 0,\ u'(1) = 0 \end{aligned}\right\} \qquad \ldots (1.1)_1$$

and

$$\left.\begin{aligned} &u''(t) + \rho_2^2 u(t) = f(t, u(t)),\ 0 < t < 1, \\ &u'(0) = 0,\ u'(1) = 0 \end{aligned}\right\} \qquad \ldots (1.1)_2$$

where $f(t, u) : [0, 1] \times [0, \infty)$ is nonnegative continuous, ρ_1 is a positive constant and $0 < \rho_2 < \frac{\pi}{2}$.

To the knowledge of the authors, Neumann boundary value problems have been studied by many different methods[2-6] and Krasnosel'skii fixed point theorem has been used to establish the existence of positive solutions to non-Neumann boundary value problems by many authors[7,8]. However, as far as we know that there are few works on the existence of positive solutions to Neumann boundary value problems.

Very recently, the authors[9] have dealt with the existence of one positive solution for the Neumann BVP$(1.1)_i$ ($i = 1, 2$). The main results of the paper[9] are as follows.

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***Theorem* A** — *BVP(1.1)$_i$ (i = 1, 2) has at least one positive solution, provided one of the following conditions holds :*

$$\lim_{u\downarrow 0}\ \min_{t\in[0,1]} \frac{f(t,u)}{u} = \infty \text{ and } \lim_{u\uparrow\infty}\ \max_{t\in[0,1]} \frac{f(t,u)}{u} = 0 \text{ (sublinear)},$$

or

$$\lim_{u\downarrow 0}\ \max_{t\in[0,1]} \frac{f(t,u)}{u} = 0 \text{ and } \lim_{u\uparrow\infty}\ \min_{t\in[0,1]} \frac{f(t,u)}{u} = \infty \text{ (superlinear)},$$

The proof of Theorem A is by employing a cone extension or compression theorem due to Krasnoselskii[1].

Motivated by the work above, in this paper we shall deal with a new optimal existence theory for single and multiple positive solutions of BVP(1.1)$_i$ (i = 1, 2). Our argument is based on the fixed point theorem in cones[1].

For convenience, we introduce the following notations

$$f_0 = \lim_{u\to 0^+}\inf\ \min_{t\in[0,1]} \frac{f(t,u)}{u},\quad f^0 = \lim_{u\to 0^+}\sup\ \max_{t\in[0,1]} \frac{f(t,u)}{u},$$

$$f_\infty = \lim_{u\to+\infty}\inf\ \min_{t\in[0,1]} \frac{f(t,u)}{u},\quad f^\infty = \lim_{u\to+\infty}\sup\ \max_{t\in[0,1]} \frac{f(t,u)}{u}.$$

Let

$$\sigma_i = \begin{cases} \dfrac{1}{ch^2\rho_i}, & i=1, \\ \cos^2\rho_i, & i=2 \end{cases} \qquad \ldots (1.2)$$

where

$$chx = \frac{e^x+e^{-x}}{2},\quad shx = \frac{e^x-e^{-z}}{2}.$$

In this paper, some of the following hypotheses are satisfied:

$(H1)_i$ $f_0 > \rho_i^2$, $f_\infty > \rho_i^2$

$(H2)_i$ $f^0 < \rho_i^2$, $f^\infty < \rho_i^2$.

$(H3)_i$ There is a $p_i > 0$ such that $\sigma_i p_i \le u \le p_i$ implies

$$f(t,u) < \rho_i^2 p_i,\qquad 0 \le t \le 1,$$

where σ_i is as in (1.2).

$(H4)_i$ There is a $p_i > 0$ such that $\sigma_i p_i \le u \le p_i$ implies

$$f(t,u) > \rho_i^2 u, \qquad 0 \le t \le 1.$$

We always let $i = 1, 2$.

Remark 1.1. : If there is a $p_i > 0$ such that $\sigma_i p_i \le u \le p_i$ implies

$$f(t,u) < \rho_i^2 u, \qquad 0 \le t \le 1,$$

then $(H3)_i$ holds.

Remark 1.2 : If there is a $p_i > 0$ such that $\sigma_i p_i \le u \le p_i$ implies

$$f(t,u) > \rho_i^2 \rho_i, \qquad 0 \le t \le 1,$$

then $(H4)_i$ holds.

2. Main Results

First of all, we point out that to find a solution of BVP$(1.1)_i$ is equivalent to find a solution of the integral equation

$$y_i(t) = \int_0^1 G_i(t,s) f(s, y_i(s))\, ds, \qquad \text{... (2.1)}$$

where

$$G_i(t,s) := \begin{cases} \begin{cases} \dfrac{ch\,\rho_i(1-t)\,ch\,\rho_i s}{\rho_i\, sh\,\rho_i}, & 0 \le s \le t \le 1, \\[2ex] \dfrac{ch\,\rho_i(1-s)\,ch\,\rho_i t}{\rho_i\, sh\,\rho_i}, & 0 \le t \le s \le 1, \end{cases} & i = 1, \\[6ex] \begin{cases} \dfrac{\cos\rho_i(1-t)\cos\rho_i s}{\rho_i \sin\rho_i}, & 0 \le s \le t \le 1, \\[2ex] \dfrac{\cos\rho_i(1-s)\cos\rho_i t}{\rho_i \sin\rho_i}, & 0 \le t \le s \le 1, \end{cases} & i = 2, \end{cases} \qquad \text{.. (2.2)}$$

One can find that, for $t, s \in [0, 1]$, we have

$$A_1 \ \underline{\underline{def}}\ \frac{1}{\rho_1\, sh\,\rho_1} \le G_1(t,s) \le \frac{ch^2\rho_1}{\rho_1\, sh\,\rho_1}\ \underline{\underline{def}}\ B_1,$$

$$A_2 \ \underline{\underline{def}}\ \frac{\cos^2\rho_2}{\rho_2 \sin\rho_2} \le G_2(t,s) \le \frac{1}{\rho_2 \sin\rho_2}\ \underline{\underline{def}}\ B_2. \qquad \text{... (2.3)}$$

Thus $\sigma_i = A_i/B_i$, where σ_i is as in (1.2).

Let

$$X = C[0, 1], \qquad \text{... (2.4)}$$

and define

$$\|y\| = \sup_{t \in [0, 1]} \{|y(t)| : y \in X\}.$$

Then X with the norm $\|\cdot\|$ is a Banach space.

By using (2.1)-(2.2), we know for every positive solution $y_i(t)$ of BVP$(1.1)_i$, one has

$$\|y_i\| \le B_i \int_0^1 f(s, y_i(s))\, ds,$$

and

$$y_i(t) \ge A_i \int_0^1 f(s, y_i(s))\, ds,$$

so we have

$$y_i(t) \ge \frac{A_i}{B_i} \|y_i\| = \sigma_i \|y_i\|. \qquad \text{... (2.5)}$$

The following theorems are our main results.

Theorem 1 — *Assume that* $(H1)_i$ *and* $(H3)_i$ *are satisfied, then* BVP$(1.1)_i$ $(i = 1, 2)$ *has at least two positive solutions* $\overset{1}{y_i}$ *and* $\overset{2}{y_i}$ *such that*

$$0 < \|\overset{1}{y_i}\| < p_i < \|\overset{2}{y_i}\|.$$

Corollary 1 — Using the following $(H1^*)$ instead of $(H1)_i$, the conclusion of Theorem 1 is true.

$$(H1^*) \quad f_0 = \infty, \quad f_\infty = \infty.$$

Theorem 2 — *Assume that* $(H2)_i$ *and* $(H4)_i$ *are satisfied, then* BVP$(1.1)_i$ $(i = 1, 2)$ *has at least two positive solutions* $\overset{1}{y_i}$ *and* $\overset{2}{y_i}$ *such that*

$$0 < \|\overset{1}{y_i}\| < p_i < \|\overset{2}{y_i}\|.$$

Corollary 2 — Using the following $(H2^*)$ instead of $(H2)_i$, the conclusion of Theorem 2 is true.

$(H2^*)$ $f^0 = 0$, $f^\infty = 0$.

Theorem 3 — *BVP(1.1)$_i$ (i = 1, 2) has at least one positive solution, provided one of the following conditions holds:*

(1)$_i$ $f_0 > \rho_1^2$ *and* $f^\infty < \rho_i^2$

(2)$_i$ $f_0 < \rho_1^2$ *and* $f^\infty > \rho_i^2$.

Corollary 3 — *BVP(1.1)$_i$ (i = 1, 2) has at least one positive solution, provided one of the following conditions holds:*

(1)$_i$: $f_0 = \infty$ *and* $f^\infty = 0$ *(sublinear);*

(2)$_i$: $f^0 = 0$ and $f_\infty = \infty$ (superlinear).

Remark 2.1 : Theorem 3 extends and improves Theorems A in[9].

Remark 2.2 : Noting that ρ_1^2 and ρ_2^2 are respectively the eigenvalue of the linear Neumann BVP

$$\begin{cases} -u''(t) + \rho_1^2 u(t) = \lambda u(t), \ 0 < t < 1 \\ u'(0) = 0, \ u'(1) = 0, \end{cases}$$

and

$$\begin{cases} u''(t) + \rho_2^2 u(t) = \lambda u(t), \ 0 < t < 1 \\ u'(0) = 0, \ u'(1) = 0, \end{cases}$$

the conditions in Theorems 1-3 are optimal.

3. Proof of Main Results

Let X be a real Banach space and K a closed, nonempty subset of X. K is a cone provided (i) $\alpha u + \beta \upsilon \in K$, for all $u, \upsilon \in K$ and all $\alpha, \beta \geq 0$. (ii) $u, -u \in K$ imply $u = 0$.

First, we state a fixed point theorem in cones which will be used later.

Lemma 1[1] — Let $X = (X, \|\cdot\|)$ be a Banach space and let K be a cone in X. Also, r, R are constants with $0 < r < R$. Suppose $\Phi : \overline{\Omega}_R \cap K \to K$ (here $\Omega_R = \{x \in X, \|x\| < R\}$) be a continuous and completely continuous operator such that

(i) $x \neq \lambda \Phi x$, for $\lambda \in [0, 1]$ and $x \in K \cap \partial \Omega_r$, and

(ii) there exists $\psi \in K \setminus \{0\}$ such that $x \neq \Phi x + \delta \psi$ for $x \in K \cap \partial \Omega_R$ and $\delta \geq 0$.

Then Φ has a fixed point in $K \cap \{ x \in X : r < \|x\| < R \}$.

Remark 3.1 : In Lemma 1, if (i) and (ii) are replaced by

$(i)^*$ $x \neq \lambda \Phi x$, for $\lambda \in [0, 1]$ and $x \in K \cap \partial \Omega_R$, and

$(ii)^*$ there exists $\psi \in K \setminus \{0\}$ such that $x \neq \Phi x + \delta \psi$ for $x \in K \cap \partial \Omega_r$ and $\delta \geq 0$. Then Φ has a fixed point in $K \cap \{ x \in X : r < \| x \| < R \}$.

Let X as in (2.4). Define an operator on X as following

$$y_i = \Phi_i y_i \quad \text{... (3.1)}$$

where Φ_i is defined by

$$(\Phi_i y_i)(t) = \int_0^1 G_i(t, s) f(s, y_i(s))\, ds, \quad \text{... (32.)}$$

for $y_i \in X$. Clearly, Φ_i is a completely continuous operator on X.

Let

$$K_i = \{y_i \in X : y_i(t) \geq 0 \quad \text{and} \quad y_i(t) \geq \sigma_i \| y_i \|\},$$

here σ_i is as in (1.2). It is not difficult to verify that K_i is a cone in X.

Lemma 2 — $\Phi_i(K_i) \subset K_i$.

PROOF : For any $y_i \in K_i$, we have

$$\| \Phi_i y_i \| \leq B_i \int_0^1 f(s, y_i(s))\, ds,$$

and

$$(\Phi_i y_i)(t) \geq A_i \int_0^1 f(s, y_i(s))\, ds.$$

So we have

$$(\Phi_i y_i)(t) \geq \frac{A_i}{B_i} \| \Phi_i y_i \| = \sigma_i \| \Phi_i y_i \|,$$

i.e. $\Phi_i y_i \in K_i$. This completes the proof of Lemma 2.

PROOF OF THEOREM 1 — Suppose that $(H1)_i$ and $(H3)_i$ hold. By using the first inequality of $(H1)_i$, i.e., $f_0 > \rho_i^2$, one can find $0 < r_i < p_i$ and $\varepsilon > 0$ such that

$$f(t, u) \geq \rho_i^2 (1 + \varepsilon) u, \quad \text{whenever} \quad 0 \leq u \leq r_i. \quad \text{... (3.3)}$$

Thus, if $y_i \in K_i$ with $\| y_i \| = r_i$, then $y_i(t) \geq \sigma_i r_i$.

Let $\psi \equiv 1$ and we prove that

$$y_i \neq \Phi_i y_i + \delta \psi \text{ for } y_i \in K_i \cap \partial \Omega_{r_i} \text{ and } \delta \geq 0. \qquad \text{... (3.4)}$$

If not, there exist $y_{i_0} \in K_i \cap \partial \Omega_{r_i}$ and $\delta_{i_0} \geq 0$ such that

$$y_{i_0} = \Phi_i y_{i_0} + \delta_{i_0} \psi.$$

Let $\mu_i = \min_{t \in [0,1]} y_{i_0}(t)$. Then for $t \in [0, 1]$, we have

$$\begin{aligned} y_{i_0}(t) &= (\Phi_i y_{i_0})(t) + \delta_{i_0} \\ &= \int_0^1 G_i(t,s) f(s, y_{i_0}(s))\, ds + \delta_{i_0} \\ &\geq \int_0^1 G_i(t,s)\, \rho_i^2 (1+\varepsilon)\, y_{i_0}(s)\, ds \\ &\geq \mu_i (1+\varepsilon) \int_0^1 G_i(t,s)\, \rho_i^2\, ds \\ &= \mu_i (1+\varepsilon), \end{aligned}$$

and this implies $\mu_i \geq \mu_i(1+\varepsilon)$, a contradiction.

On the other hands, by using the inequality of $(H3)_i$, we prove that

$$y_i \neq \lambda \Phi_i y_i \quad \text{for} \quad y_i \in K_i \cap \partial \Omega_{p_i} \quad \text{and} \quad 0 \leq \lambda \leq 1. \qquad \text{... (3.5)}$$

If not, there exist $y_{i_0} \in K_i \cap \partial \Omega_{p_i}$ and $0 \leq \lambda_{i_0} \leq 1$ such that

$$y_{i_0} = \lambda_{i_0} \Phi_i y_{i_0}.$$

Clearly, $\lambda_{i_0} > 0$. Thus, $\| y_{i_0} \| = p_i$ and $\sigma_i p_i \leq y_{i_0}(t) \leq p_i$ for $t \in [0, 1]$, so we have

$$f(t, y_{i_0}(t)) < \rho_i^2 p_i, \quad t \in [0, 1]. \qquad \text{... (3.6)}$$

Then we obtain

$$\begin{aligned} y_{i_0}(t) &= \lambda_{i_0} (\Phi_i y_{i_0})(t) \\ &= \lambda_{i_0} \int_0^1 G_i(t,s) f(s, y_{i_0}(s))\, ds \end{aligned}$$

$$< \int_0^1 G_i(t,s)\,\rho_i^2\,p_i\,ds$$

$$= p_i,$$

and this implies $\| y_{i_0} \| = p_i < p_i$, a contradiction.

In view of (3.4) and (3.5), by Lemma 1, we see that Φ_i has a fixed point $y_i^1 \in K_i$ and $r_i < \left\| y_i^1 \right\| < p_i$. Thus $y_i^1(t) \geq \sigma_i r_i > 0$, which means that $y_i^1(t)$ is a positive solution of $(1.1)_i$.

Next, by using the second inequality of $(H1)_i$, i.e., $f_\infty > \rho_i^2$, one can find $r_{i_1} > p_i$ and $\varepsilon > 0$ such that

$$f(t,u) \geq \rho_i^2 (1+\varepsilon)\,u, \quad \text{whenever} \quad u \geq r_{i_1}. \qquad \text{... (3.7)}$$

Let $R_i = \dfrac{r_{i_1}}{\sigma_i}$, so we have,

$$u(t) \geq \sigma_i \| u \| = \sigma_i R_i = r_{i_1} \quad \text{for} \quad u \in K_i \cap \partial\Omega_{R_i}. \qquad \text{... (3.8)}$$

Thus, if $y_i \in K_i$ with $\| y_i \| = R_i$, then $y_i(t) \geq \sigma_i R_i = r_{i_1}$.

Let $\psi \equiv 1$ and we prove that

$$y_i \neq \Phi_i y_i + \delta\psi \text{ for } y_i \in K_i \cap \partial\Omega_{R_i} \quad \text{and} \quad \delta \geq 0. \qquad \text{... (3.9)}$$

If not, there exist $y_{i_0} \in K_i \cap \partial\Omega_{R_i}$ and $\delta_{i_0} \geq 0$ such that

$$y_{i_0} = \Phi_i y_{i_0} + \delta_{i_0}\psi.$$

Let $\mu_i = \min_{t \in [0,1]} y_{i_0}(t)$. Then for $t \in [0,1]$, we have

$$y_{i_0}(t) = (\Phi_i y_{i_0})(t) + \delta_{i_0}$$

$$= \int_0^1 G_i(t,s) f(s, y_{i_0}(s))\,ds + \delta_{i_0}$$

$$\geq \int_0^1 G_i(t,s)\,\rho_i^2 (1+\varepsilon)\,y_{i_0}(s)\,ds$$

$$\geq \mu_i (1+\varepsilon) \int_0^1 G_i(t,s) \rho_i^2 \, ds$$

$$= \mu_i (1+\varepsilon),$$

and this implies $\mu_i \geq \mu_i (1+\varepsilon)$, a contradiction.

In view of (3.5) and (3.9), by Lemma 1, we see that Φ_i has a fixed point $y_i^2 \in K_i$ and $p_i < \left\| y_i^2 \right\| < R_i$. Thus $y_i^2(t) \geq \sigma_i p_i > 0$, which means that $y_i^2(t)$ is a positive solution of $(1.1)_i$.

This complete the proof of Theorem 1.

Remark 3.1 : Note to deduce the existence of y_i^1 in Theorem 1, we need only assume $(H3)_i$ and $f_0 > \rho_i^2$. A similar remark applies for y_i^2.

PROOF OF THEOREM 2 : Suppose Suppose that $(H2)_i$ and $(H4)_i$ hold. By using the first inequality of $(H2)_i$, i.e., $f^0 < \rho_i^2$, one can find $0 < r_i < p_i$ and $0 < \varepsilon < 1$ such that

$$f(t,u) \leq \rho_i^2 (1-\varepsilon)\, u, \quad \text{whenever} \quad 0 \leq u \leq r_i. \qquad \text{... (3.10)}$$

Thus, if $y_i \in K_i$ with $\| y_i \| = r_i$, then $y_i(t) \geq \sigma_i r_i$. We want to show that

$$\lambda \Phi_i y_i \ \text{for}\ y_i \in K_i \cap \partial \Omega_{r_i} \quad \text{and} \quad 0 \leq \lambda \leq 1. \qquad \text{... (3.11)}$$

If not, there exist $y_{i_0} \in K_i \cap \partial \Omega_{r_i}$ and $0 \leq \lambda_{i_0} \leq 1$ such that

$$y_{i_0} = \lambda_{i_0} \Phi_i y_{i_0}.$$

Clearly, $\lambda_{i_0} > 0$. Then we have

$$y_{i_0}(t) = \lambda_{i_0} (\Phi_i y_{i_0})(t)$$

$$= \lambda_{i_0} \int_0^1 G_i(t,s) f(s, y_{i_0}(s))\, ds$$

$$\leq \int_0^1 G_i(t,s) \rho_i^2 (1-\varepsilon)\, y_{i_0}(s)\, ds$$

$$\leq (1-\varepsilon) \| y_{i_0} \| \int_0^1 G_i(t,s) \rho_i^2 \, ds$$

$$= (1-\varepsilon) \| y_{i_0} \|,$$

and this implies $\| y_{i_0} \| \le (1-\varepsilon) \| y_{i_0} \|$, a contradiction.

On the other hand, by using inequality of $(H4)_i$, let $\psi \equiv 1$ and we prove that

$$y_i \ne \Phi_i y_i + \delta \psi \text{ for } y_i \in K_i \cap \partial \Omega_{p_i} \quad \text{and} \quad \delta \ge 0. \qquad \dots (3.12)$$

If not, there exist $y_{i_0} \in K_i \cap \partial \Omega_{p_i}$ and $\delta_{i_0} \ge 0$ such that

$$y_{i_0} = \Phi_i y_{i_0} + \delta_{i_0} \psi.$$

Thus, $\| y_{i_0} \| = p_i$ and $\sigma_i p_i \le y_{i_0}(t) \le p_i$ for $t \in [0, 1]$, so we have

$$f(t, y_{i_0}(t)) > \rho_i^2 \, y_{i_0}(t), \qquad t \in [0, 1]. \qquad \dots (3.13)$$

Let $\mu_i = \min_{t \in [0,1]} y_{i_0}(t)$. Then for $t \in [0, 1]$, we have

$$\begin{aligned} y_{i_0}(t) &= (\Phi_i y_{i_0})(t) + \delta_{i_0} \\ &= \int_0^1 G_i(t, s) f(s, y_{i_0}(s))\, ds + \delta_{i_0} \\ &> \int_0^1 G_i(t, s)\, \rho_i^2 \, y_{i_0}(s)\, ds \\ &\ge \mu_i \int_0^1 G_i(t, s)\, \rho_i^2 \, ds \\ &= \mu_i, \end{aligned}$$

and this implies $\mu_i > \mu_i$, a contradiction.

In view of (3.11) and (3.12), by Lemma 1, we see that Φ_i has a fixed point $y_i^1 \in K_i$ and $r_i < \| y_i^1 \| < p_i$. Thus $y_i^1(t) \ge \sigma_i r_i > 0$, which means that $y_i^1(t)$ is a positive solution of $(1.1)_i$.

Next, by using the second inequality of $(H2)_i$, i.e., $f^\infty < \rho_i^2$, one can find $r_{i_1} > p_i$ and $0 < \varepsilon > 1$ such that

$$f(t, u) \le \rho_i^2 (1-\varepsilon)\, u, \quad \text{whenever} \quad u \ge r_{i_1}. \qquad \dots (3.14)$$

Let $R_i = \frac{r_{i_1}}{\sigma_i}$, so we have,

$$u(t) \geq \sigma_i \| u \| = \sigma_i R_i = r_{i_1} \quad \text{for} \quad u \in K_i \cap \partial \Omega_{R_i}. \qquad \ldots (3.15)$$

Thus, if $y_i \in K_i$ with $\| y_i \| = R_i$, then $y_i(t) \geq \sigma_i R_i = r_{i_1}$.

We could also show that

$$y_i \neq \lambda \Phi_i y_i \quad \text{for} \quad y_i \in K_i \cap \partial \Omega_{R_i} \quad \text{and} \quad 0 \leq \lambda \leq 1. \qquad \ldots (3.16)$$

If not, there exist $y_{i_0} \in K_i \cap \partial \Omega_{R_i}$ and $0 \leq \lambda_{i_0} \leq 1$ such that

$$y_{i_0} = \lambda_{i_0} \Phi_i y_{i_0}.$$

Clearly, $\lambda_{i_0} > 0$. Then we have

$$\begin{aligned} y_{i_0}(t) &= \lambda_{i_0} (\Phi_i y_{i_0})(t) \\ &= \lambda_{i_0} \int_0^1 G_i(t, s) f(s, y_{i_0}(s))\, ds \\ &\leq \int_0^1 G_i(t, s)\, \rho_i^2 (1 - \varepsilon)\, y_{i_0}(s)\, ds \\ &\leq (1 - \varepsilon) \| y_{i_0} \| \int_0^1 G_i(t, s)\, \rho_i^2\, ds \\ &= (1 - \varepsilon) \| y_{i_0} \|, \end{aligned}$$

and this implies $\| y_{i_0} \| \leq (1 - \varepsilon) \| y_{i_0} \|$, a contradiction.

In view of (3.12) and (3.16), by Lemma 1, we see that Φ_i has a fixed point $y_i^2 \in K_i$ and $p_i < \left\| y_i^2 \right\| < R_i$. Thus $y_i^2(t) \geq \sigma_i p_i > 0$, which means that $y_i^2(t)$ is a positive solution of $(1.1)_i$.

This complete the proof of Theorem 2.

Remark 3.2 : Note to deduce the existence of y_i^1 in Theorem 2, we need only assume $(H4)_i$ and $f^0 < \rho_i^2$. A similar remark applies for y_i^2.

PROOF OF THEOREM 3. The proof follows the ideas in the proof of Theorems 1 and 2.

4. EXAMPLES

In this section, we apply the main result obtained in previous section to study some examples.

Example 4.1 — Consider the BVP

$$\begin{aligned} &-u''(t) + \rho^2 u(t) = \delta(u^a + u^b),\ 0 < t < 1 \\ &u'(0) = 0,\ u'(1) = 0 \end{aligned}, \qquad \text{... (4.1)}$$

where $0 < a < 1 < b$ and $0 < \delta$, $0 < \rho$.

Applying Theorem 1 (or Corollary 1), we can find that BVP(4.1) has two positive solutions provided

$$\frac{\delta}{\rho^2} < \sup_{x \in (0, \infty)} \frac{x}{x^a + x^b}. \qquad \text{... (4.2)}$$

Set $f(t, u) = \delta(u^a + u^b)$, then

$$\lim_{u \downarrow 0} \min_{t \in [0, 1]} \frac{f(t, u)}{u} = \infty \quad \text{and} \quad \lim_{u \uparrow \infty} \min_{t \in [0, 1]} \frac{f(t, u)}{u} = \infty,$$

so $(H1)_1$ (or $(H1^*)$) holds. Set

$$T(x) := \frac{x}{x^a + x^b},\ x > 0,$$

then $T(0+) = 0$, $T(\infty) = 0$, and

$$T(p_1) = \sup_{x \in (0, \infty)} T(x),\ p_1 = \left(\frac{1-a}{b-1}\right)^{\frac{1}{b-a}}$$

Then for $\sigma_1 p_1 \le u \le p_1$, we have

$$\begin{aligned} f(t, u) &\le \delta\left(p_1^a + p_1^b\right) \\ &= \rho^2\left(p_1^a + p_1^b\right)\frac{\delta}{\rho^2} \\ &< \rho^2\left(p_1^a + p_1^b\right)T(p_1) \\ &= \rho^2 p_1, \end{aligned}$$

so $(H3)_1$ holds. Then the result follows from Theorem 1 (or Corollary 1).

Example 4.2 — Consider the BVP

$$-u''(t) + \rho^2 u(t) = \delta e^u, \quad 0 < t < 1,$$
$$u'(0) = 0, \; u'(1) = 0 \qquad \text{... (4.3)}$$

where δ and ρ are positive constants.

Applying Theorem 1 (or Corollary 1), we can find that BVP(4.3) has two positive solutions provided

$$\frac{\delta}{\rho^2} < \frac{1}{e}. \qquad \text{... (4.4)}$$

Set $f(t, u) = \delta e^u$, then

$$\lim_{u \downarrow 0} \min_{t \in [0, 1]} \frac{f(t, u)}{u} = \infty \quad \text{and} \quad \lim_{u \uparrow \infty} \min_{t \in [0, 1]} \frac{f(t, u)}{u} = \infty,$$

so $(H1)_1$ (or $(H1^*)$) holds. Set

$$T(x) := \frac{x}{e^x}, \; x > 0,$$

then $T(0+) = 0$, $T(\infty) = 0$, and

$$T(p_1) = \sup_{x \in (0, \infty)} T(x), \; p_1 = 1.$$

Then for $\sigma_1 p_1 \leq u \leq p_1$ $(p_1 = 1)$, we have

$$\begin{aligned} f(t, u) &\leq \delta e^{p_1} \\ &= \rho^2 e^{p_1} \frac{\delta}{\rho^2} \\ &< \rho^2 e^{p_1} T(p_1) \\ &= \rho^2 p_1, \end{aligned}$$

so $(H3)_1$ holds. Then the result follows from Theorem 1 (or Corollary 1).

Example 4.3 — Consider the BVP

$$u''(t) + \rho^2 u(t) = \delta (u^a + u^b), \quad 0 < t < 1,$$
$$u'(0) = 0, \; u'(1) = 0 \qquad \text{... (4.5)}$$

where $0 < a < 1 < b$ and $0 < \delta$, $0 < \rho < \frac{\pi}{2}$.

Applying Theorem 1 (or Corollary 1), we can find that BVP(4.5) has two positive solutions provided

$$\frac{\delta}{\rho^2} < \sup_{x \in (0, \infty)} \frac{x}{x^a + x^b}. \quad \ldots (4.6)$$

The proof is similar as Example 4.1.

Example 4.4 — Consider the BVP

$$u''(t) + \rho^2 u(t) = \delta e^{u}, \quad 0 < t < 1,$$
$$u'(0) = 0, \ u'(1) = 0 \quad \ldots (4.7)$$

where δ is positive constant and $0 < \rho < \frac{\pi}{2}$.

Applying Theorem 1 (or Corollary 1), we can find that BVP(4.7) has two positive solutions provided

$$\frac{\delta}{\rho^2} < \frac{1}{e}. \quad \ldots (4.8)$$

The proof is similar as Example 4.2.

REFERENCES

1. K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.
2. A. Cabada and P. Habets, *J. Differential Equations*, **166** (2000), 385-401.
3. A. Cabada and R. R. L. Pouse, *Nonlinear Anal.*, **30** (1997), 1733-42.
4. I. Rachunkva and S. Stanke, *Nonlinear Anal.*, **27**(3) (1996), 271-85.
5. I. Rachunkva, *Ann. Polon. Math.*, **65** (1997), 235-44.
6. Hai Dang and F. Seth, *J. Math. Anal. Appl.*, **198** (1996), 35-48.
7. Ruyun Ma, *J. Math. Anal. Appl.*, **201** (1996), 375-86.
8. L. H. Erbe and H. Wang, *Proc. Amer. Math. Soc.*, **120** (1994), 743-48.
9. Jiang Daqing and Liu Huizhao, *J. Math. Res. Exposition*, **20** (2000), 360-64.