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ANALYTIC CONTINUATION OF SERIES SOLUTION REPRESENTING FLOW BETWEEN PLATES

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The flow of an incompressible fluid between two rectangular and circular plates is reinvestigated. The problem admits similarity solutions there by reducing unsteady Navier-Stokes equations to non-linear ordinary differential equation of order four involving a small parameter $\alpha = \frac{h(t)\,\dot{h}(t)}{\nu}$. The proposed new semi-analytic semi-numerical scheme is convenient in obtaining analytic continuation of series solution. The series so generated yields expression for pressure gradient and it is analyzed using Pade' approximants and other useful techniques (Van Dyke[10]) of series analysis.

Key Words: Analyic Continuation; Computer Extended Unconstrained Optimization (Powell's method); Padé Approximants; Series Solution

1. INTRODUCTION

The flow of a viscous incompressible fluid between two parallel infinitely long rectangular (two dimensional) and two parallel circular plates (axisymmetric) is of interest in Fluid Mechanics. Such flow situations are frequently encountered in lubrication and are termed as squeezed films. The problem of unsteady squeezing of viscous fluid between two plates occurs in unsteady loading. Attempts have been made by several investigators e.g. Chandrashekharan and Ramnaih[5] and Jackson and Symmons[6] to study the effect of inertia on the bearing characteristics. Large class of nonlinear ordinary differential equations arising from mathematical modelling are listed systematically by Sachdev[8,9] and valuable hints for their analysis are given by him.

In the present study, we investigate the problem for small and moderately large Reynolds number α and present interesting results based on a new type of series analysis. Van Dyke[10,11,12] and his associates have clearly shown the potential application of these methods in fluid dynamics. Recently, Bujurke, Pai and Jayaraman[3] have successfully used series analysis in their study associated with flows in pipes. For simple geometries the method proposed here provide accurate results and have advantages over pure numerical methods. In numerical methods a separate scheme is to be developed for calculating derived quantities. If the computation of derivatives is required, the numerical scheme to be used will be very sensitive to the grid or step size. This itself will be an elaborate scheme. However, this difficulty does not arise in the case of a series method. A single computer run yields the solution for a range of the expansion quantity. In addition, the method reveals the analytic structure of the solution which is silent in numerical solution. The precise nature of polynomial solutions obtained during initial approximation enables us to propose series expansion with polynomial coefficients to calculate sufficient terms (universal coefficients) of the low Reynolds number (α) perturbation series by computer. Using the Domb-Sykes plot the nature and location of the nearest singularity which restricts the convergence of the series are obtained. For different values of α the series is analysed using Pade' approximants for summing it. The results are also analysed by Powell's method of unconstrained optimization (Press, W.H. *et al.*[7]) and Brown's method (Byrne and Hall[4]).

2. Mathematical Formulation

Analysis of two dimensional flow:

Fig. 1 shows the geometry of viscous incompressible fluid between two parallel infinitely long rectangular plates of with $2a$ separated by a small distance $h(t)$. The upper plate is assumed to move downwards while the lower plate remains fixed.

For the above two dimensional flow the fluid has velocity vector $(u, 0, w)$ and the equations of continuity and of motion of the fluid referred to Cartesian coordinates (Batchelor[1]) are :

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial x}+\dfrac{\partial w}{\partial z}=0 \\[2ex] \dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+w\dfrac{\partial u}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+\upsilon\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial z^2}\right) \\[2ex] \dfrac{\partial w}{\partial t}+u\dfrac{\partial w}{\partial x}+w\dfrac{\partial w}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}+\upsilon\left(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial z^2}\right) \\[2ex] \text{and the boundary conditions are} \\[1ex] u=w=0 \text{ at } z=0 \\[1ex] u=0 \text{ and } w=\dot{h} \text{ at } z=h \end{array}\right\} \quad \text{... (2.1)}$$

For two-dimensional flow equations, we introduce the following transformations (Chandrashekharan and Ramanaih[5]).

Let

$$w=-\frac{\upsilon}{h}F(\eta)$$

where

$$\eta=\frac{z}{h(t)}, \text{ so that } u=\frac{x\upsilon F'}{h^2} \quad \text{... (2.2)}$$

where a prime denotes differentiation with respect to η and $\dot{h}=\dfrac{dh}{dt}$.

Substituting these into eq. (2.1) and eliminating pressure we obtain a non-linear ordinary differential equation.

$$F''''+FF'''-\alpha\eta F'''-F'F''-3\alpha F''=0 \quad \text{... (2.3)}$$

with the boundary conditions

$$\left\{\begin{array}{lll} F=0, & F'=0 & \text{at } \eta=0 \\ F=\alpha, & F'=0, & \text{at } \eta=1 \end{array}\right\} \quad \text{... (2.4)}$$

Analysis of axisymmetric flow:

The cylindrical co-ordinates system used for the description of the problem of axisymmetric flow between two parallel circular plates of radius a separated by a small distance $h(t)$ is shown in Fig. 1. In this case the upper plate moves towards the lower plate which is fixed. u and w represent the velocity components in the radial and axial directions respectively. The axisymmetric unsteady low is governed by the following equations of continuity and of motion

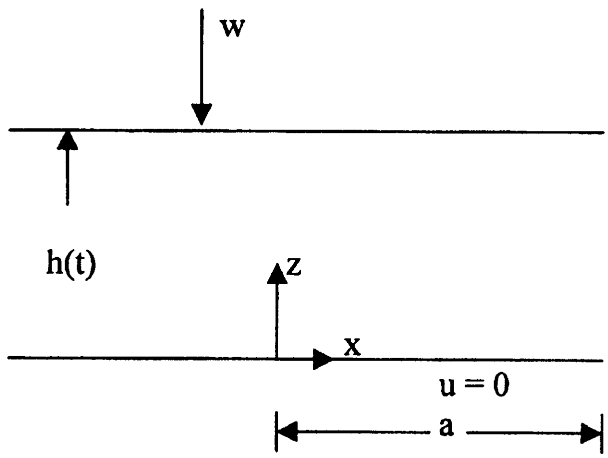

FIG. 1. Geometry of flow between parallel rectangular plates and circular plates

$$\left\{\begin{aligned} &\frac{\partial}{\partial x}(xu) + \frac{\partial}{\partial z}(xw) = 0 \\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial z^2} - \frac{u}{x^2}\right) \\ &\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{1}{x}\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial z^2}\right) \\ &\text{The relevant boundary conditions are} \\ &u = w = 0 \ \text{at} \ z = 0 \\ &u = 0 \ \text{and} \ w = \dot{h} \ \text{at} \ z = h \end{aligned}\right\} \quad \ldots (2.7)$$

For axisymmetric flow equations, we introduce the following transformations

$$w = -\frac{2\upsilon}{h} F(\eta)$$

$$u = \frac{x\upsilon F'}{h^2} \quad \ldots (2.8)$$

where

$$\eta = \frac{z}{h(t)}.$$

Substituting (2.8) into eqs. (2.7), we get

$$F'''' + 2FF''' - \alpha\eta F''' - 3\alpha F'' = 0 \qquad \text{... (2.9)}$$

with the boundary conditions

$$\left\{\begin{array}{lll} F=0, & F'=0 & \text{at } \eta=0 \\ F=\frac{\alpha}{2}, & F'=0, & \text{at } \eta=1 \end{array}\right\} \qquad \text{... (2.10)}$$

These equations and boundary conditions for both two dimensional and axisymmetric flows can be put in the general form as

$$\left\{\begin{array}{l} F'''' + AFF''' - \alpha\eta F''' - BF'F'' - 3\alpha F'' = 0 \\ \text{with the boundary conditions} \\ F=0, \quad F'=0 \quad \text{at } \eta=0 \\ F=C\alpha, \quad F'=0, \quad \text{at } \eta=1 \end{array}\right\} \qquad \text{... (2.11)}$$

For two dimensional flow, $A = 1$, $B = -1$ and $C = 1$ and for axisymmetric flow, $A = 2$, $B = 0$ and $C=\frac{1}{2}$.

3. Method of Solution

We seek solution of eqs. (2.3)-(2.4) in a power series of α in the form

$$F = \sum_{n=1}^{\infty} \alpha^n F_n(\eta). \qquad \text{... (3.1)}$$

Substituting (3.1) in (2.3) and equating like powers of α on both sides, we obtain

$$F_1'''' = 0; F_n'''' = F_{n-1}'' + 3F_{n-1}''$$

$$- \sum_{r=1}^{n-1} \left(F_r F_{n-r}''' - F_r' F_{n-r}'' \right), n = 2, 3, 4 \ldots. \qquad \text{... (3.2)}$$

The relavant boundary conditions are

$$\left\{\begin{array}{l} F_1(0) = F_n(0) = 0, \; F_1'(0) = F_n'(0) = 0, \\ F_1(1) = 1, F_n(1) = 0, F_1'(1) = F_n'(1) = 0. \quad n = 2, 3, 4 \ldots \end{array}\right\} \qquad \text{... (3.3)}$$

The solutions of the above equations up to $O(\alpha^3)$ are

$$F_1(\eta) = -2\eta^3 + 3\eta^2$$

$$F_2(\eta) = \frac{2}{35}\eta^7 - \frac{1}{5}\eta^6 - \frac{1}{10}\eta^5 + \frac{3}{4}\eta^4 - \frac{24}{35}\eta^3 + \frac{5}{28}\eta^2$$

$$F_3(\eta) = \frac{4}{5775}\eta^{11} - \frac{2}{525}\eta^{10} + \frac{11}{630}\eta^9 - \frac{1}{20}\eta^8$$

$$+ \frac{227}{4900}\eta^7 + \frac{187}{4200}\eta^6 - \frac{71}{700}\eta^6 - \frac{71}{700}\eta^5 + \frac{5}{112}\eta^4$$

$$+ \frac{283}{38808}\eta^3 - \frac{1229}{215600}\eta^2. \quad \text{... (3.4)}$$

Chandrashekharan and Ramanaih[5] also have obtained above solution but in $F_3(\eta)$ there is minor error.

It is not sufficient to analyze the problem with just these three approximations. It is essential to get higher approximations (large number of universal polynomials) in the series if it is to reveal the true nature of the function represented by the series. As we proceed to higher approximations, the algebra becomes cumbersome and it is difficult to calculate the terms manually. The nature of solution (3.4) enables us to propose a systematic series expansion with polynomial coefficients which is quite useful and efficient in the calculation of higher approximations of the series. Towards this goal, we consider $F_n(\eta)$ to be of the form

$$F_n(\eta) = \sum_{k=2}^{4n-3} A_{n,k}\,(1-\eta)^2\,\eta^k, \quad n = 2, 3, 4 \ldots \quad \text{... (3.5)}$$

in (3.1). This expression yields exactly the earlier calculated terms $F_n(n = 2, 3)$ and also enables us to find F_n for $n \geq 3$. We substitute eq. (3.5) into eq. (3.2) and equate various powers of η on both sides to get a recurrence relation for unknowns $A_{n,k}$ in the form

$$A_{n,(2N_2-(J+5))} = 2A_{n,(2N_2-(J+4))} - A_{n,(2N_2-(J+3))}$$

$$+ \frac{1}{(2N_2-(J+3))\,(2N_2-(J+4))\,(2N_2-(J+5))\,(2N_2-(J+6))}$$

$$\times \left\{ \begin{array}{l} \displaystyle\sum_{i=1}^{5} A_{(n-1),(2N_2-i-(J+4))}\,P_i\,(2N_2-i-(J+4)) \\ \displaystyle + \sum_{L=1}^{n-2}\left[\sum_{r=-2}^{2}\left[\sum_{k=4L-J-r-3}^{4L-3} A_{L,k}A_{m,(2N_2-k-(J+r+6))}\,P_{8+r}\,(k, 2N_2-k-(J+r+6))\right]\right] \end{array} \right\}$$

... (3.6)

where $N_2 = 2n$, $m = n - L$ and J varies from $-2, -1, 0, \ldots, (4n-7)$.

$$P_1(k) = k(k-1)(k-2) + 3k(k-1)$$

$$P_2(k) = -2(k+1)\,k(k-1) - 6(k+1)\,k - 3k(k-1)(k-2) + 6k(k-1) + 6k$$

$$P_3(k) = (k+2)(k+1)(k) + 3(k+2)(k+1) + 2k(k-1)(k-2) + 6(k+1)k(k-1)$$
$$- 6k(k-1) - 12(k+1)k - 12(k+1) - 12k - 12$$
$$P_4(k) = -4(k+1)k(k-1) - 3(k+2)(k+1)k + 12(k+1)k + 6(k+2)(k+1)$$
$$+ 24(k+1) + 6(k+2) - 24$$
$$P_5(k) = 2(k+2)(k+1)k - 6(k+2)(K+1) - 12(k+2) + 12$$
$$P_6(k, k_1) = -k_1(k_1-1)(k_1-2) + k(k_1)(k_1-1)$$
$$P_7(k, k_1) = 2(k_1+1)k_1(k_1-1) + 2k_1(k_1-1)(k_1-2) - 2k(k_1+1)$$
$$k_1 - 2(k+1)k_1(k_1-1)$$
$$P_8(k, k_1) = -(k_1+2)(k_1+1)k_1 - 4(k_1+1)k_1(k_1-1) - k_1(k_1-1)$$
$$(k_1-2) + k(k_1+2)(k_1+1) + 4(k+1)(k_1+1)k_1 + (k+2)k_1(k_1-1)$$
$$P_9(k, k_1) = 2(k_1+2)(k_1+1)k_1 + 2(k_1+1)k_1(k_1-1) - 2(k+1)(k_1+2)$$
$$(k_1+1) - 2(k+2)(k_1+1)k_1$$
$$P_{10}(k, k_1) = -(k_1+2)(k_1+1)k_1 + (k+2)(k_1+2)(k_1+1)$$

and

$$A_{2,2} = \frac{5}{28}, \quad A_{2,3} = \frac{-23}{70}, \quad A_{2,4} = \frac{-3}{35}, \quad A_{2,5} = \frac{2}{35}.$$

The pressure gradient is given by

$$-F'''(0) = \sum_{n=1}^{\infty} \alpha^n C_n \qquad \text{... (3.7)}$$

where

$$C_n = 12A_{n,2} - 6A_{n,3}.$$

The calculated coefficients C_n of the series (3.7) for pressure gradient are listed in Table I. These coefficients are also checked by solving eqs. (3.2)-(3.3) using MATHEMATICA. They are decreasing in magnitude but have no regular pattern of sign. Domb-Sykes plot (Fig. 2&3), after extrapolation, confirms the radius of convergence of the series (3.7) to be α = 14.105. The region of validity of the series (3.7) is increased by considering Pade' sum which are given in Table 2.

4. Power Series Method

To confirm the values of the estimated pressure gradient it is proposed to analyze the same using power series in conjunction of optimization method and Brown's method in the following section. We assume power series solution

TABLE I

Coefficients C_n and C_n' of the expansion of $-F'''(0)$ for two Dimensional and axisymmetric flow

n	C_n	C_n'
1.	12.00000000000000	6.00000000000000
2.	4.11428571428570	1.67142857142860
3.	-4.3753865182436E-002	-1.2043908472480E-002
4.	2.5676160574120E-003	5.0731751241962E-004
5.	-1.6026994758868E-004	-2.2462589721959E-005
6.	1.0167292467709E-005	9.9906407285410E-007
7.	-6.4976467523038E-007	-4.4235214674815E-008
8.	4.1761479296334E-008	1.9461566338816E-009
9.	-2.6985749876974E-009	-8.5039954427862E-011
10.	1.7529227384235E-010	3.6893478492769E-012
11.	-1.1443512059849E-011	-1.5882735181502E-013
12.	7.5057377226235E-013	6.7798927821901E-015
13.	-4.9444835339735E-014	-2.8661768120748E-016
14.	3.2703718294549E-015	1.2018242793752E-017
15.	-2.17111785 82079E-016	-4.93 64294422174E-019
16.	1.4462759564151E-017	1.5265943345157E-020
17.	-9.6646477637456E-019	-4.2091783938780E-022
18.	6.4763510976882E-020	1.9915681444730E-021
19.	-4.3520462129283E-021	5.3830802841229E-022
20.	2.9339381638165E-022	2.3937544922232E-022
21.	-1.9769042231352E-023	1.6765027619978E-022
22.	1.33368158046521E-024	-8.7409489245384E-023
23.	-9.3573103583742E-026	-3.7823049110118E-023
24.	6.1241031633457E-027	1.7794838573503E-023
25.	-2.1574943479302E-028	-1.5976841018627E-024
26.	-1.1079704533580E-029	1.1492904577595E-024
27.	-9.1151588130900E-030	-1.12587913 89993E-024
28.	-1.8080280214670E-030	-5.4345970574018E-025
29.	1.5296344561707E-031	-1.5565140564149E-025
30.	2.3933361867281E-032	-4.6672504691794E-027
31.	-3.7218932780011E-032	-3.7664386228623E-026
32.	-3.3954140441470E-033	-3.5926884859144E-026
33.	-1.9428038810491E-033	-2.2768388172050E-027

34.	-4.3167947139329E-034	1.5331096954647E-027
35.	-2.4494620659510E-034	-2.5120083397015E-028
36.	-2.7911359564110E-035	-4.2438741344138E-029
37.	2.4601194591974E-036	-1.0889994395844E-029
38.	-1.6241803758251E-036	1.9267364920318E-029
39.	5.6234914074745E-037	2.2990777898348E-029
40.	6.3060500221868E-038	-3.5384522079074E-030

$$F(\eta) = \sum_{n=1}^{\infty} a_n \eta^{n+1} \qquad \dots (4.1)$$

for the solution of (2.3). Boundary conditions (2.4) takes the forms

$$\left\{ \begin{array}{l} \sum_{n=1}^{\infty} a_n = \alpha \\ \sum_{n=1}^{\infty} (n+1) a_n = 0 \end{array} \right\}. \qquad \dots (4.2)$$

Substituting (4.1) into (2.3), we get

$$a_3 = \frac{6a_1 \alpha}{24}$$

$$a_{n+3} = \frac{1}{(n+4)(n+3)(n+2)(n+1)}$$

$$\left\{ \left[\alpha a_{n+1} (n+1)(n+2)(n+3) \right] + \sum_{m=1}^{n} a_{n-m+1} a_m [m(m+1)(n-2m+3)] \right\} \qquad \dots (4.3)$$

If a_1 and a_2 are known then the rest of $\{a_n\}$ can be found from the recurrence relation (4.3). Effectively we have transformed a two point boundary value problem into a system of non-linear equations. We wish to find a_1 and a_2 such that (4.2) are satisfied. To solve this system of non-linear algebraic equations, we have used both Brown's method (Byrne and Hall[4]) and Powell's method of unconstrained optimization (Press, W. H. *et al.*[7]). Powell's method requires optimization (minimization) of

$$\left(\sum_{n=1}^{\infty} a_n - \alpha \right)^2 + \left(\sum_{n=1}^{\infty} (n+1) a_n \right)^2 \qquad \dots (4.4)$$

in conjugation of (4.3). In this process eq. (4.2) is satisfied automatically. a_n's so gnerated are listed in Table II for different α.

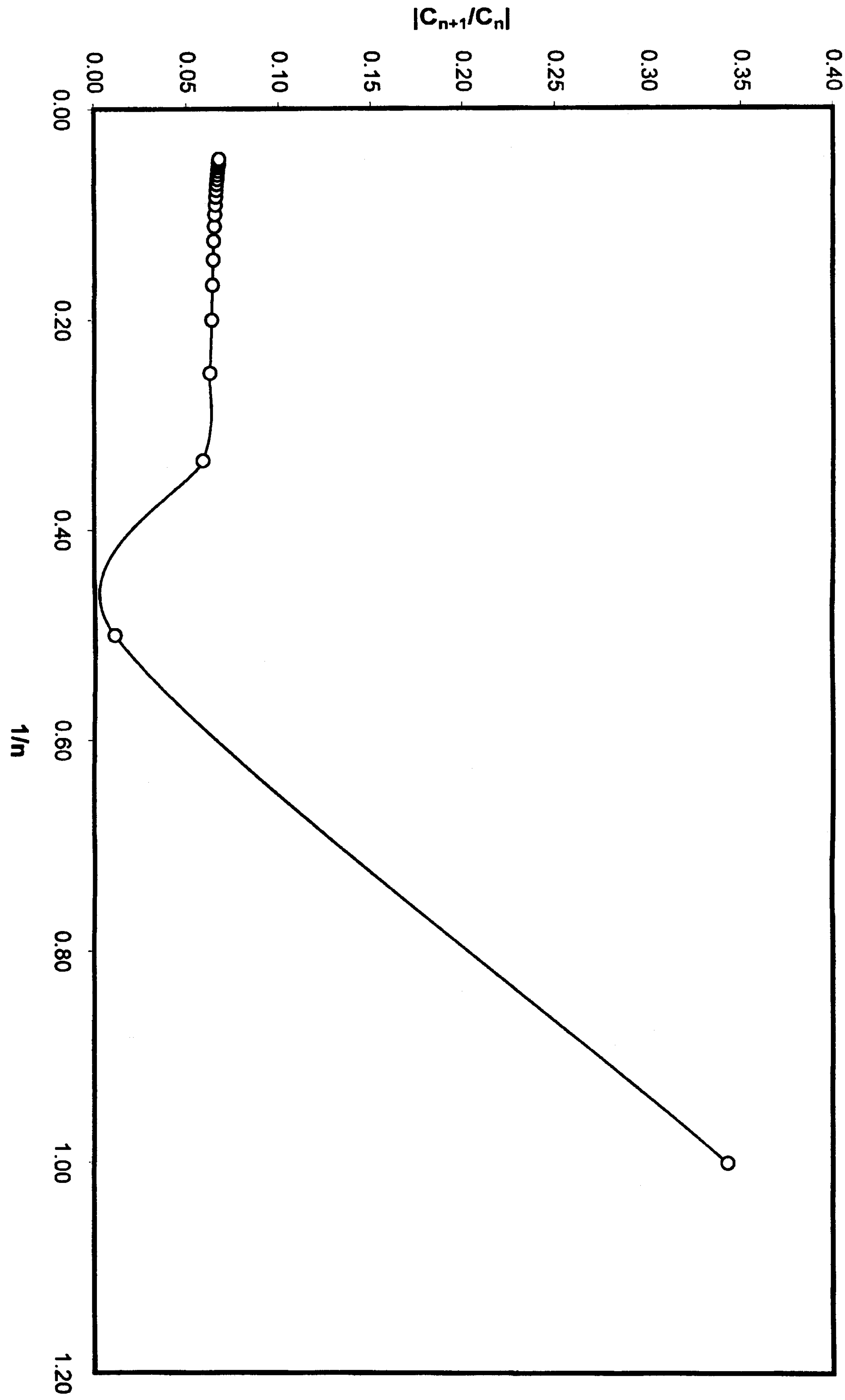

Fig. 2 : Domb-Sykes Plot for the coefficents C_n of the series
(Two Dimensional Flow)

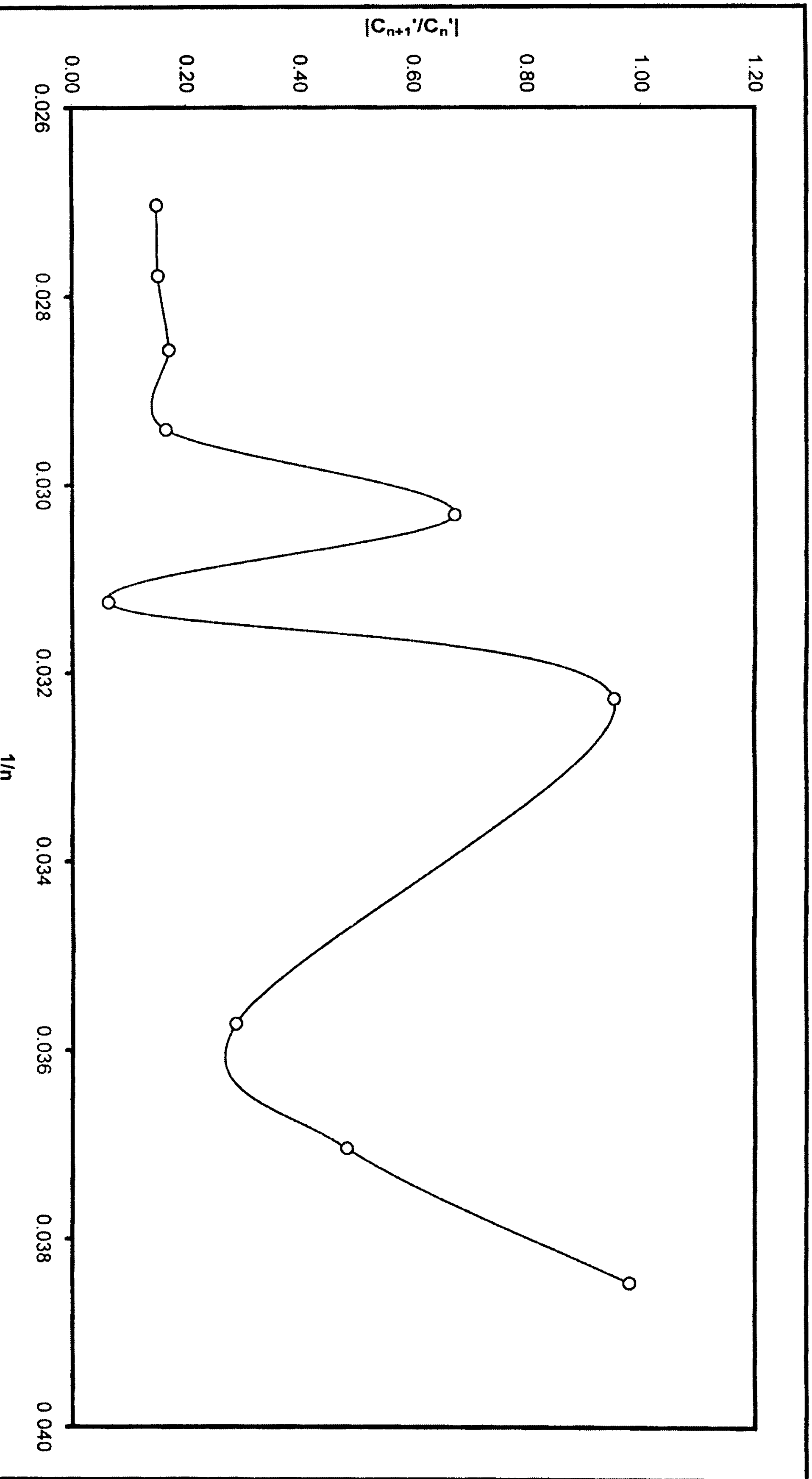

FIG 3 : Domb-Sykes Plot for the coefficents Cn' of the series (Axisynnetric Flow)

TABLE II

Pressure Gradient (represented by $-F'''(0)$) calculated using different methods for various α (Two dimensional flow)

α	$-F'''(0)$ Computer Extended Series		$-F'''(0)$ Power Series		$-F'''(0)$ Numerical [5]
	Using Recurrence Relation	Using MATHEMATICA	Optimization Method	Brown's Method	
0.	0.0	0.0	0.0	0.0	0.0
1	16.07	16.07	16.07	16.07	16.15
3	72.02	72.02	72.02	72.02	72.04
4	111.55	111.55	102.63	102.63	111.43
5	158.61	158.61	-------	-------	-------
10	503.53	503.53	-------	-------	-------
20	1719.80	1719.80	-------	-------	-------
30	3614.14	3614.15	-------	-------	-------
40	6171.77	6171.79	-------	-------	-------
50	9384.11	9384.15	-------	-------	-------
60	13245.07	13245.41	-------	-------	-------

TABLE III

Pressure Gradient (represented by $-F'''(0)$) calculated using different methods for various α (Axisymmetric flow)

α	$-F'''(0)$ Computer Extended Series		$-F'''(0)$ Power Series		$-F'''(0)$ Numerical [5]
	Using Recurrence Relation	Using MATHEMATICA	Optimization Method	Brown's Method	
0	0.0	0.0	0.0	0.0	0.0
1	7.659	7.659	7.659	7.659	7.72
3	32.75	32.75	32.75	32.75	32.95
4	50.08	50.08	50.08	50.08	50.69
5	70.53	70.53	-------	-------	-------
10	218.61	218.61	-------	-------	-------
20	735.37	735.37	-------	-------	-------
30	1536.38	1536.38	-------	-------	-------
40	2615.23	2615.24	-------	-------	-------
50	3968.12	3968.13	-------	-------	-------
60	5592.47	5592.52	-------	-------	-------

5. Results and Discussion

The problem of unsteady flow between plates is studied using computer extended series analysis. The motion of the fluid for two dimensional flow is governed by non-linear ordinary differential eq. (2.3) with boundary conditions (2.4). The series expansion scheme with polynomial coefficients (3.5) proposed enables in obtaining recurrence relations (3.6). Using this relation, we generate large number ($n = 40$) of universal coefficients ($A_{n,k}$, $k = 2, 3, \ldots, 4n - 3$; $n = 1, 2, \ldots 40$). A careful FORTRAN program consisting of number of Do loops makes it possible in performing complex algebra involved. The coefficients C_n's of the series (3.7) representing the pressure gradient $[-F'''(0)]$ are descending in magnitude, but have no regular sign pattern (Table I). Domb-Syke plot (Fig. 2) after extrapolation, confirms radius of convergence of the series to be $\alpha = 14.105$ (with an error of 10^{-6}). The direct sum of series is valid only for very small values of α. We use Pade' Approximants (Bender and Orszag[2]) for summing the series. Pressure gradient calculated using Pade' Approximants give converging results upto $\alpha = 60$ whereas pure numerical results gives pressure gradient only upto $\alpha = 4$. Pressure gradient increases considerably with increase in α in both the cases. This is an important contribution of this investigation. The analysis of the problem is also done using power series in conjugate with unconstrained optimization (Press, W. H. *et al.*[7]) and also the solution of resulting nonlinear algebraic equations is obtained by Brown's method (Byrne and Hall[4]). But these have limited validity compared with perturbation analysis because of slow convergence of power series for higher values of α. All these results are shown in Table II. Similar analysis is used in the study of axisymmetric flows and the results are given in Table III.

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