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L-FUZZY QUASI-UNIFORMIZABLE SPACES

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We introduce the notion of *L*-fuzzy quasi-uniform spaces. We investigate the relationship between *L*-fuzzy topological spaces and *L*-fuzzy quasi-uniform spaces. In particular, we show that every [0, 1]-fuzzy topological space is [0, 1]-fuzzy quasi-uniformizable.

Key Words: ***L*-fuzzy topological spaces; *L*-fuzzy quasi-uniform spaces; *L*-fuzzy quasi-uniformizable spaces; *L*-fuzzy continuous, *L*-fuzzy quasi-uniformly continuous.**

1. INTRODUCTION

Extending Höhle's notion of openness as a predicate[4], Kubiak[12] and Šostak[16] independently introduced the notion of *L*-fuzzy topological spaces as a generalizationof Chang [0, 1]-topological spaces[2]. Höhle and Šostak[6] analyzed the fixed-basis foundations for *L*-fuzzy topological spaces having fixed base *L*. Moreover, Samanta[15] introduced the concept of [0, 1]-fuzzy uniform spaces as an expansion of Hutton *L*-uniform spaces[7].

In this paper, we introduce the notion of *L*-fuzzy quasi-uniform spaces in a sense of Samanta [0, 1]-fuzzy uniform spaces[15]. We investigate the relationship between *L*-fuzzy topological spaces and *L*-fuzzy quasi-uniform spaces. The family $\Sigma(\mathcal{T})$ of all [0, 1]-fuzzy quasi-uniformities $\mathcal{U}$ compatible with an [0, 1]-fuzzy topological $\mathcal{T}$ on X is never empty and it contains an [0, 1]-fuzzy quasi-uniformity $\mathcal{U}_{\mathcal{T}}$ which is the coarsest member of $\Sigma(\mathcal{T})$. From this fact, we show that every [0, 1]-fuzzy topological space is [0, 1]-fuzzy quasi-uniformizable.

In this paper, all the notations and the defintions are referred to Höhle and Rodabaugh[5].

2. PRELIMINARIES

Let X be a nonempty set. Let a complete lattice $L = (L, \leq, \vee, \wedge, ')$ be a completely distributive lattice with an order-reversing involution $'$, 0 and 1 denote the least and the greatest element in L (ref.[13]). For $\alpha \in L, \overline{\alpha}(x) = \alpha$ for each $x \in X$ and $L^1 = L - \{1\}$. L is called a chain if for each $a, b \in L, a \leq b$ or $a \geq b$. L is called order dense if for each $a, b \in L$ such that $a < b$, there exists $c \in L$ such that $a < c < b$.

Definition 2.1[12,16] — A function $\mathcal{T}: L^X \to L$ is called an *L*-fuzzy topology on X if it satisfies the following conditions :

(01) $\mathcal{T}(\overline{0}) = \mathcal{T}(\overline{1}) = 1$.

(02) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge (\mu_2)$ for $\mu_1, \mu_2 \in L^X$.

(O3) $\mathcal{T}(\vee_{i\in\Gamma}\mu_i) \geq \wedge_{i\in\Gamma}\mathcal{T}(\mu_i)$ for any $\{\mu_i\}_{i\in\Gamma} \subset L^X$.

The pair $(X, \mathcal{T})$ is called an L-fuzzy topological space.

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be L-fuzzy topological spaces. A function ψ: $(X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is called an L-fuzzy continuous map if $\mathcal{T}_2(\mu) \leq \mathcal{T}_1(\psi^{\leftarrow}(\mu))$ for all $\mu \in L^Y$.

Theorem 2.2 — *Let $(X, \mathcal{T})$ be an L-fuzzy topological space. Define, for each $r \in L^1$, $\lambda \in L^X$,*

$$I_{\mathcal{T}}(\lambda, r) = \vee\left\{\mu \in L^X \mid \mu \leq \lambda,\ \mathcal{T}(\mu) > r\right\}.$$

Then it satisfies the following properties : for $\lambda, \mu \in L^X$ and $r, s \in L^1$,

(1) $I_{\mathcal{T}}(\bar{1}\, r) = \bar{1}$,

(2) $I_{\mathcal{T}}(\lambda, r) \leq \lambda$,

(3) if $\lambda \leq \mu$ *and* $r \leq s$, *then* $I_{\mathcal{T}}(\lambda, s) \leq I_{\mathcal{T}}(\mu, r)$,

(4) $I_{\mathcal{T}}(\lambda \wedge \mu, r) \geq I_{\mathcal{T}}(\lambda, r) \wedge I_{\mathcal{T}}(\mu, r)$,

(5) $I_{\mathcal{T}}(I_{\mathcal{T}}(\lambda,, r), r) \geq I_{\mathcal{T}}(\lambda, r)$.

PROOF : (1), (2) and (3) are trivial from the definition of $I_{\mathcal{T}}$.

(4) Suppose $I_{\mathcal{T}}(\lambda_1 \wedge \lambda_2, r) \not\geq I_{\mathcal{T}}(\lambda_1, r) \wedge I_{\mathcal{T}}(\lambda_2, r)$. Since L is a completely distributive lattice, by the definitions of $I_{\mathcal{T}}(\lambda_i, r)$, for each $i \in \{1, 2\}$, there exist $\mu_i \in L^X$ with $\mu_i \leq \lambda_i$ and $\mathcal{T}(\mu_i) > r$ such that $I_{\mathcal{T}}(\lambda_1 \wedge \lambda_2, r) \not\geq \mu_1 \wedge \mu_2$. Since $\mathcal{T}(\mu_1 \wedge \mu_2) > r$ and $\mu_1 \wedge \mu_2 \leq \lambda_1 \wedge \lambda_2$, we have $I_{\mathcal{T}}(\lambda_1 \wedge \lambda_2, r) \geq \mu_1 \wedge \mu_2$. It is a contradiction.

(5) Suppose $I_{\mathcal{T}}(I_{\mathcal{T}}(\lambda,, r), r) \not\geq I_{\mathcal{T}}(\lambda, r)$. From the definitions of $I_{\mathcal{T}}(\lambda, r)$, there exist $\mu \in L^X$ with $\mu \leq \lambda$ and $\mathcal{T}(\mu) > r$ such that $I_{\mathcal{T}}(I_{\mathcal{T}}(\lambda,, r), r) \not\geq \mu$. Since $\mu = I_{\mathcal{T}}(\mu, r) \leq I_{\mathcal{T}}(\lambda, r)$, $I_{\mathcal{T}}(I_{\mathcal{T}}(\lambda,, r), r) \geq \mu$. It is a contradiction. ■

Remark 2.3 : The above theorem introduces an L-fuzzy interior operator different from Höhle and Šostak L-fuzzy interior operator[6] in order for it to be well-behaved to L-fuzzy uniformities.

Theorem 2.4 — *Let $(X, \mathcal{T})$ be an L-fuzzy topological space. The function $\mathcal{T}_{I_{\mathcal{T}}}: L^X \to L$ is defined by*

$$\mathcal{T}_{I_{\mathcal{T}}}(\lambda) = \vee\left\{r \in L \mid I_{\mathcal{T}}(\lambda, r) \geq \lambda\right\}.$$

Then we have the following properties:

(1) $\mathcal{T}_{I_\mathcal{T}}$ is an L-fuzzy topology on X.

(2) *If* L *is an order dense chain, then* $\mathcal{T}_{I_\mathcal{T}} = \mathcal{T}$

PROOF : (1) We show that $\mathcal{T}_{I_\mathcal{T}}$ is an L-fuzzy topology on X.

(O1) It is trivial.

(O2) Let $I_\mathcal{T}(\lambda_i, r_i) \geq \lambda_i$ for each $i \in \{1, 2\}$, by Theorem 2.2 (3.4),

$$I_\mathcal{T}(\lambda_1 \wedge \lambda_2, r_1 \wedge r_2) \geq I_\mathcal{T}(\lambda_1, r_1) \wedge I_\mathcal{T}(\lambda_2, r_2) \geq \lambda_1 \wedge \lambda_2.$$

Hence $\mathcal{T}_{I_\mathcal{T}}(\lambda_1 \wedge \lambda_2) \geq r_1 \wedge r_2$. Since L is a completely distributive lattice, $\mathcal{T}_{I_\mathcal{T}}(\lambda_1 \wedge \lambda_2) \geq \mathcal{T}_{I_\mathcal{T}}(\lambda_1) \wedge \mathcal{T}_{I_\mathcal{T}}(\lambda_2)$.

(O3) Let $I_\mathcal{T}(\lambda_i, r_i) \geq \lambda_i$ for each $i \in \Gamma$, by Theorem 2.2 (3.4),

$$I_\mathcal{T}\left(\bigvee_{i \in \Gamma} \lambda_i, \wedge\ r_i \right) \geq I_\mathcal{T}(\lambda_i, r_i) \geq \lambda_i.$$

Hence $\mathcal{T}_{I_\mathcal{T}}\left(\bigvee_{i \in \Gamma} \lambda_i \right) \geq \bigwedge_{i \in \Gamma} r_i$. Since L is a completely distributive lattice, $\mathcal{T}_{I_\mathcal{T}}\left(\bigvee_{i \in \Gamma} \lambda_i \right) \geq \bigwedge_{i \in \Gamma} \mathcal{T}_{I_\mathcal{T}}(\lambda_i)$.

(2) We will show that $\mathcal{T}_{I_\mathcal{T}} = \mathcal{T}$. Suppose $\mathcal{T}_{I_\mathcal{T}} \not\leq \mathcal{T}$. Since L is an order dense chain, by the definition of $\mathcal{T}_{I_\mathcal{T}}$, there exists $r_0 \in L^1$ with $I_\mathcal{T}(\lambda, r_0) \geq \lambda$ such that

$$\mathcal{T}_{I_\mathcal{T}}(\lambda) \geq r_0 > \mathcal{T}(\lambda).$$

On the other hand, since $I_\mathcal{T}(\lambda, r_0) \geq \lambda$, we have $\mathcal{T}(\lambda) \geq r_0$ from the definition of $I_\mathcal{T}(\lambda, r_0)$. It is a contradiction. Hence $\mathcal{T}_{I_\mathcal{T}} \leq \mathcal{T}$.

Suppose $\mathcal{T}_{I_\mathcal{T}} \not\geq \mathcal{T}$. Since L is an order dense chain, there exist $\rho \in L^X$ and $r_1 \in L^1$ such that

$$\mathcal{T}_{I_\mathcal{T}}(\rho) < r_1 < \mathcal{T}(\rho).$$

On the other hand, since $\mathcal{T}(\rho) > r_1$, by the definition of $I_\mathcal{T}(\rho, r_1) \geq \rho$. Hence $\mathcal{T}_{I_\mathcal{T}}(\rho) \geq r_1$. It is a contradiction. Hence $\mathcal{T}_{I_\mathcal{T}} \geq \mathcal{T}$.

Example 2.5 — Let $L = \{0, 1, a, b\}$ be the diamond-type lattice, that is, $a \vee b = 1$, $a \wedge b = 0$ and $\mu \neq \rho \in L^X - \{\bar{0}, \bar{1}\}$. Let $\mathcal{T}$ be an L-fuzzy topology on X defined as follow:

$$\mathcal{T}(\lambda) = \begin{cases} 1 \text{ if } \lambda \in \{\bar{0}, \bar{1}, \mu \vee \rho\} \\ a \text{ if } \lambda = \rho, \\ b \text{ if } \lambda \in \{\mu, \mu \wedge \rho\}, \\ 0 \text{ otherwise.} \end{cases}$$

From Theorem 2.2, we have

$$I_{\mathcal{T}}(\lambda, a) = \begin{cases} \bar{1} \text{ if } \lambda = \bar{1}, \\ \mu \vee \rho \text{ if } \mu \vee \rho \leq \lambda \neq \bar{1}, \\ \bar{0} \text{ otherwise.} \end{cases}$$

$$I_{\mathcal{T}}(\lambda, b) = \begin{cases} \bar{1} \text{ if } \lambda = \bar{1}, \\ \mu \vee \rho \text{ if } \mu \vee \rho \leq \lambda \neq \bar{1}, \\ \bar{0} \text{ otherwise.} \end{cases}$$

From Theorem 2.4, we can obtain $\mathcal{T}_{I_{\mathcal{T}}}: L^X \to L$ as follows:

$$\mathcal{T}_{I_{\mathcal{T}}}(\lambda) = \begin{cases} 1 \text{ if } \lambda \in \{\bar{0}, \bar{1}, \mu \vee \rho\} \\ 0 \text{ otherwise.} \end{cases}$$

Hence, in general, $\mathcal{T}_{I_{\mathcal{T}}} \neq \mathcal{T}$.

Lemma 2.6[7,11,13] — Let Ω_X be the family of all functions $f: L^X \to L^X$ such that:

(*a*) $\mu \leq f(\mu)$ for every $\mu \in L^X$;

(*b*) $f\left(\bigvee_{i \in \Gamma} \mu_i\right) = \bigvee_{i \in \Gamma} f(\mu_i)$ for $\{\mu_i\}_{i \in \Gamma} \subset L^X$.

For every $f, g, h, f_1, g_1 \in \Omega_X$, we have the following properties.

(1) Define, for all $\mu \in L^X, f^{-1}(\mu) = \wedge \{\rho \in L^X \mid f(\rho') \leq \mu'\}$. Then $f^{-1} \in \Omega_X$.

(2) Difine, for all $\mu \in L^X, (f \wedge g)(\mu) = \wedge \{f(\mu_1) \vee g(\mu_2) \mid \mu_1 \vee \mu_2 = \mu\}$. Then $f \vee g \in \Omega_X$.

(3) Define, for all $\mu \in L^X, f \circ g(\mu) = f(g(\mu))$. Then $f \circ g \in \Omega_X$.

(4) If $f \leq f_1, g \leq g_1$, then $f \wedge g \leq f_1 \wedge g_1$.

(5) $f \wedge g \leq f$, $f \wedge g \leq g$ and $f \wedge f = f$.

(6) $(f^{-1})^{-1} = f$.

(7) $f \leq g$ iff $f^{-1} \leq g^{-1}$.

(8) $f(\mu) \leq \rho$ iff $f^{-1}(\rho') \leq \mu'$.

(9) Let a function $f_{\overline{1}} : L^X \to L^X$ be defined by

$$f_{\overline{1}}(\mu) = \begin{cases} \overline{1} \text{ if } \mu \neq \overline{0}, \\ \overline{0} \text{ if } \mu \neq \overline{0}. \end{cases}$$

Then $f_{\overline{1}} = f_{\overline{1}}^{-1} \in \Omega_X$ and $f \wedge f_{\overline{1}} = f$.

(10) $(f \circ g)^{-1} = f^{-1} \circ g^{-1}$.

(11) $(f \wedge g)^{-1} = f^{-1} \wedge g^{-1}$.

(12) $(f \wedge g) \wedge h = f \wedge (g \wedge h)$.

3. Some Properties of L-Fuzzy Quasi-Uniform Spaces

Definition 3.1[15] — A function $\mathcal{U} : \Omega_X \to L$ is said to be an *L*-fuzzy quasi-uniformity on *X* if it satisfies the following conditions:

(FQU1) If $f \leq g$, then $\mathcal{U}(f) \leq \mathcal{U}(g)$.

(FQU2) $\mathcal{U}(f \wedge g) \geq \mathcal{U}(f) \wedge \mathcal{U}(g)$, for each $f, g \in \Omega_X$.

(FQU3) For each $f \in \Omega_X$, $\bigvee \{\mathcal{U}(g) \mid g \circ g \leq f\} \geq \mathcal{U}(f)$.

(FQU4) There exists $f \in \Omega_X$ such that $\mathcal{U}(f) = 1$.

The pair $(X, \mathcal{U})$ is said to be an *L*-fuzzy quasi-uniform space.

An *L*-fuzzy quasi-uniform space $(X, \mathcal{U})$ is called an *L*-fuzzy uniform space if it satisfies

(FU) For each $f \in \Omega_X$, $\bigvee \{\mathcal{U}(g) \mid g \leq f^{-1}\} \geq \mathcal{U}(f)$.

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be *L*-fuzzy (quasi-)uniformities on *X*. $\mathcal{U}_1$ is finder than $\mathcal{U}_2$ (or $\mathcal{U}_2$ is coarser than $\mathcal{U}_1$), denoted by $\mathcal{U}_2 \leq \mathcal{U}_1$, iff for any $f \in \Omega_X$, $\mathcal{U}_2(f) \leq \mathcal{U}_1(f)$.

Remark 3.2 : Let $(X, \mathcal{U})$ be an *L*-fuzzy quasi-uniform space.

(1) Put $\mathcal{U}_r = \{f \in \Omega_X \mid \mathcal{U}(f) > r\}$ for each $r \in L^1$. If *L* is an order dense chain, then $\mathcal{U}_r$ is a Hutton *L*-quasi-uniformity on *X*[7].

(2) We define for $f \in \Omega_X$, $\mathcal{U}^{-1}(f) = \mathcal{U}(f^{-1})$. From Lemma $2.6^{(6,7,10,11)}$, $\mathcal{U}^{-1}$ is an L-fuzzy quasi-uniformity on X.

(3) From (FQU1), (FQU2) and Lemma 2.6(5), we have $\mathcal{U}(f \wedge g) = \mathcal{U}(f) \wedge \mathcal{U}(g)$.

(4) From Lemma 2.6^9 and (FQU4), since $f \leq f_T$ for all $f \in \Omega_X$, we have $\mathcal{U}(f_T) = 1$.

(5) If $(X, \mathcal{U})$ is an L-fuzzy uniform space, then, by (FU), (FQU1) and Lemma 2.6^6, we have $\mathcal{U}(f) = \mathcal{U}(f^{-1})$.

***Theorem* 3.3** — *Let $(X, \mathcal{U})$ be L-fuzzy quasi-uniform space. We define, for each $f \in \Omega_X$,*

$$\mathcal{U}^*(f) = \bigvee \left\{ \mathcal{U}(f_1) \wedge \mathcal{U}^{-1}(f_2) \mid f_1 \wedge f_2 \leq f \right\}.$$

Then the structure $\mathcal{U}^*$ is the coarsest L-fuzzy uniformity on X finer than $\mathcal{U}$ and $\mathcal{U}^{-1}$.

PROOF : From Remark 3.2^2, $(X, \mathcal{U}^{-1})$ is L-fuzzy quasi-uniform space. First, we will show $\mathcal{U}^*$ is an L-fuzzy uniformity on X.

(FQU1) It is trivial

(FQU2) Suppose there exist $f, g \in \Omega_X$ such that

$$\mathcal{U}^*(f \wedge g) \not\geq \mathcal{U}^*(f) \wedge \mathcal{U}^*(g).$$

Since L is a completely distributive lattive, by the definition of $\mathcal{U}^*(f)$, there exist $f_1, f_2 \in \Omega_X$ with $f_1 \wedge f_2 \leq f$ such that

$$\mathcal{U}^*(f \wedge g) \not\geq (\mathcal{U}(f_1) \wedge \mathcal{U}^{-1}(f_2)) \wedge \mathcal{U}^*(g).$$

Again, by the definitions of $\mathcal{U}^*(g)$, there exist $g_1, g_2 \in \Omega_X$ with $g_1 \wedge g_2 \leq g$ such that

$$\mathcal{U}^*(f \wedge g) \not\geq (\mathcal{U}(f_1) \wedge \mathcal{U}^{-1}(f_2)) \wedge (\mathcal{U}(g_1) \wedge \mathcal{U}^{-1}(g_2)).$$

On the other hand, since $(f_1 \wedge f_2) \wedge (g_1 \wedge g_2) = (f_1 \wedge g_1) \wedge (f_2 \wedge g_2) \leq f \wedge g$ from Lemma 2.6,

$$\begin{aligned} \mathcal{U}^*(f \wedge g) &\geq \mathcal{U}(f_1 \wedge g_1) \wedge \mathcal{U}^{-1}(f_2 \wedge g_2) \\ &\geq (\mathcal{U}(f_1) \wedge \mathcal{U}(g_1)) \wedge (\mathcal{U}^{-1}(f_2) \wedge \mathcal{U}^{-1}(g_2)). \end{aligned}$$

It is a contradiction. Thus, $\mathcal{U}^*(f \wedge g) \geq \mathcal{U}^*(f) \wedge \mathcal{U}^*(g)$, for each $f, g \in \Omega_X$.

(FQU3) Suppose there exists $f \in \Omega_X$ such that

$$\bigvee \left\{ \mathcal{U}^*(g) \mid g \circ g \leq f \right\} \not\geq \mathcal{U}^*(f).$$

By the definition of $\mathcal{U}^*(f)$, there exist $f_1, f_2 \in \Omega_X$ with $f_1 \wedge f_2 \leq f$ such that

$$\vee \left\{ \mathcal{U}^*(g) \mid g \circ g \leq f \right\} \not\geq \mathcal{U}(f_1) \wedge \mathcal{U}^{-1}(f_2).$$

Since $\vee \left\{ \mathcal{U}(g_1) \mid g_1 \circ g_1 \leq f \right\} \geq \mathcal{U}(f_1)$ and $\vee \left\{ \mathcal{U}(g_2) \mid g_2 \circ g_2 \leq f_2 \right\} \geq \mathcal{U}^{-1}(f_2)$, we have

$$\vee \left\{ \mathcal{U}^*(g) \mid g \circ g \leq f \right\} \not\geq \vee \left\{ \mathcal{U}^*(g_1) \mid g_1 \circ g_1 \leq f_1 \right\} \wedge \left\{ \mathcal{U}^{-1}(g_2) \mid g_2 \circ g_2 \leq f_2 \right\}.$$

Since L is a completely distributive lattice, there exist $g_1, g_2 \in \Omega_X$ with $g_1 \circ g_1 \leq f_1$ and $g_2 \circ g_2 \leq f_2$ such that

$$\vee \left\{ \mathcal{U}^*(g) \mid g \circ g \leq f \right\} \not\leq \mathcal{U}(g_1) \wedge \mathcal{U}^{-1}(g_2).$$

On the other hand, since $(g_1 \wedge g_2) \circ (g_1 \wedge g_2) \leq (g_1 \circ g_1) \wedge (g_2 \circ g_2) \leq f_1 \wedge f_2 \leq f$,

$$\mathcal{U}^*(g_1 \wedge g_2) \geq \mathcal{U}(g_1) \wedge \mathcal{U}^{-1}(g_2).$$

It is a contradiction.

(FQU4) There exist $f, g \in \Omega_X$ such that $\mathcal{U}(f) = 1$ and $\mathcal{U}^{-1}(g) = 1$. Hence $\mathcal{U}^*(f \wedge g) = 1$.

(FU) Suppose there exist $f \in \Omega_X$ such that

$$\vee \left\{ \mathcal{U}^*(g) \mid g \leq f^{-1} \right\} \not\geq \mathcal{U}^*(f).$$

By the definition of $\mathcal{U}^*(f)$, there exist $f_1, f_2 \in \Omega_X$ with $f_1 \wedge f_2 \leq f$ such that

$$\vee \left\{ \mathcal{U}^*(g) \mid g \leq f^{-1} \right\} \not\geq \mathcal{U}^*(f_1) \wedge \mathcal{U}^{-1}(f_2).$$

On the other hand, since $f_1 \wedge f_2 \leq f$ iff $f_1^{-1} \wedge f_2^{-1} \leq f^{-1}$ from Lemma 2.6[6,7,10],

$$\mathcal{U}^*(f_1^{-1} \wedge f_2^{-1}) \geq \mathcal{U}(f_2^{-1}) \wedge \mathcal{U}^{-1}(f_1^{-1}) = \mathcal{U}(f_1) \wedge \mathcal{U}^{-1}(f_2).$$

It is contradiction.

We will show that the structure $\mathcal{U}^*$ is the coarsest L-fuzzy uniformity on X finer than $\mathcal{U}$ and $\mathcal{U}^{-1}$. Since $f \wedge f_1 = f$ from Lemma 2.6[9], by Remark 3.2[4], $\mathcal{U}^*(f) \geq \mathcal{U}(f)$ and $\mathcal{U}^*(f) \geq \mathcal{U}^{-1}(f)$. Thus, $\mathcal{U}^*$ is finer than $\mathcal{U}$ and $\mathcal{U}^{-1}$. If $\mathcal{V}$ is finer than $\mathcal{U}$ $\mathcal{U}^{-1}$,
we have $\mathcal{U}^* \leq \mathcal{V}$ from the following:

$$\mathcal{U}^*(f) = \vee \left\{ \mathcal{U}(f_1) \wedge \mathcal{U}^{-1}(f_2) \mid f_1 \wedge f_2 \leq f \right\}$$

$$\leq \vee \{ \mathcal{V}(f_1) \wedge \mathcal{V}(f_2) \mid f_1 \wedge f_2 \leq f \}$$

$$\leq \vee \{ \mathcal{V}(f_1 \wedge f_2) \mid f_1 \wedge f_2 \leq f \}$$

$$\leq \mathcal{V}(f).$$

■

Theorem 3.4 — *Let $\mathcal{U}$ be an L-fuzzy quasi-uniformity on X. For each $r \in L^1, \lambda \in L^X$, we define*

$$C_{\mathcal{U}}(\lambda, r) = \wedge \left\{ f^{-1}(\lambda) \mid \mathcal{U}(f) > r \right\}.$$

Then it satisfies the following:

(1) $C_{\mathcal{U}}(\bar{0}, r) = \bar{0}$.

(2) $C_{\mathcal{U}}(\lambda, r) \geq \lambda$.

(3) *If* $\lambda_1 \leq \lambda_2$, *then* $C_{\mathcal{U}}(\lambda_1, r) \leq C_{\mathcal{U}}(\lambda_2, r)$.

(4) *If* $r \leq s$, *then* $C_{\mathcal{U}}(\lambda, r) \leq C_{\mathcal{U}}(\lambda, s)$.

(5) $C_{\mathcal{U}}(\lambda_1 \vee \lambda_2, r) \leq C_{\mathcal{U}}(\lambda_1, r) \vee C_{\mathcal{U}}(\lambda_2, r)$.

(6) *If L is a chain* $C_{\mathcal{U}}(C_{\mathcal{U}}(\lambda, r), r) \leq C_{\mathcal{U}}(\lambda, r)$.

PROOF : (1), (2), (3) and (4) are easily proved from the definition of $C_{\mathcal{U}}$.

(5) Suppose $C_{\mathcal{U}}(\lambda_1 \vee \lambda_2, r)(x) \not\leq C_{\mathcal{U}}(\lambda_1, r)(x) \vee C_{\mathcal{U}}(\lambda_2, r)(x)$.

Since L is a completely distributive lattice, by the definitions of $C_{\mathcal{U}}(\lambda_i, r)$, for each $i \in \{1, 2\}$, there exist $f_i \in \Omega_X$ with $\mathcal{U}(f_i) > r$ such that

$$C_{\mathcal{U}}(\lambda_1 \vee \lambda_2, r)(x) \not\leq f_1^{-1}(\lambda_1)(x) \vee f_2^{-1}(\lambda_2)(x).$$

Since $\mathcal{U}(f_1 \wedge f_2) > r$, we have

$$C_{\mathcal{U}}(\lambda_1 \vee \lambda_2, r)(x) \leq (f_1 \wedge f_2)^{-1}(\lambda_1 \vee \lambda_2)(x)$$

$$= (f_1^{-1} \wedge f_2^{-1})(\lambda_1 \vee \lambda_2)(x) \text{ (by Lemma 2.6(11))}$$

$$\leq f_1^{-1}(\lambda_1)(x) \vee f_2^{-1}(\lambda_2)(x).$$

It is a contradiction.

(6) Suppose $C_{\mathcal{U}}(C_{\mathcal{U}}(\lambda, r), r) \not\leq C_{\mathcal{U}}(\lambda, r)$. By the definition $C_{\mathcal{U}}(\lambda, r)$, there exist $f \in \Omega_X$ and

$x \in X$ such that $\mathcal{U}(f) > r$ and

$$C_{\mathcal{U}}(C_{\mathcal{U}}(\lambda, r), r)(x) \not\leq f^{-1}(\lambda)(x).$$

On the other hand, by (FQU3), since $\vee \{\mathcal{U}(g) \mid g \circ g \leq f\} \geq \mathcal{U}(f) > r$ and L is a chain, there exists $g_1 \in \Omega_X$ with $g_1 \circ g_1 \leq f$ and $\mathcal{U}(g_1) > r$. It follows that

$$C_{\mathcal{U}}(C_{\mathcal{U}}(\lambda, r), r) \leq g_1^{-1}(C_{\mathcal{U}}(\lambda, r)) \leq g_1^{-1}(g_1^{-1}(\lambda)) \leq f^{-1}(\lambda).$$

It is a contradiction. ■

Theorem 3.5 — *Let $(X\ \mathcal{U})$ be an L-fuzzy quasi-uniform space. Define, for each $r \in L^1$, $\lambda \in L^X$,*

$$I_{\mathcal{U}}(\lambda, r) = \vee \{\mu \in L^X \mid f(\mu) \leq \lambda,\ \mathcal{U}(f) > r\}.$$

Then we have

$$(I_{\mathcal{U}}(\lambda, r))' = C_{\mathcal{U}}(\lambda', r).$$

PROOF : It is proved from:

$$\begin{aligned}(I_{\mathcal{U}}(\lambda, r))' &= \left(\vee \{\mu \in L^X \mid f(\mu) \leq \lambda,\ \mathcal{U}(f) > r\}\right)' \\ &= \wedge \{\mu' \in L^X \mid f(\mu) \leq \lambda,\ \mathcal{U}(f) > r\} \\ &= \wedge \{\mu' \in L^X \mid f^{-1}(\lambda') \leq \mu',\ \mathcal{U}(f) > r\} \text{ (by Lemma 2.6(8))} \\ &= \wedge \{f^{-1}(\lambda') \mid \mathcal{U}(f) > r\} \\ &= C_{\mathcal{U}}(\lambda', r).\end{aligned}$$

■

Similarly, we can define $I_{\mathcal{U}^{-1}}(\lambda, r)$ and $I_{\mathcal{U}^*}(\lambda, r)$.

Theorem 3.6 — *Let $(X\ \mathcal{U})$ be an L-fuzzy quasi-uniform space. The function $\mathcal{T}_{\mathcal{U}} : L^X \to L$ is defined by, for each $\lambda \in L^X$,*

$$\begin{aligned}\mathcal{T}_{\mathcal{U}}(\lambda) &= \vee \{r \in L \mid I_{\mathcal{U}}(\lambda, r) \geq \lambda\} \\ &= \vee \{r \in L \mid C_{\mathcal{U}}(\lambda', r) \leq \lambda'\}.\end{aligned}$$

Then $\mathcal{T}_{\mathcal{U}}$ is an L-fuzzy topology on X.

PROOF : The operator $I_{\mathcal{U}}$ satisfies the condition (1-4) of Theorem 2.2. Thus, we can obtain and L-fuzzy topology $\mathcal{T}_{\mathcal{U}}$ as a similar method as in Theorem 2.4. ∎

Similarly, since $(X, \mathcal{U}^{-1})$ be an L-fuzzy quasi-uniform space, $\mathcal{T}_{\mathcal{U}^{-1}}$ is an L-fuzzy topology on X. The space $(X, \mathcal{T}_{\mathcal{U}}, \mathcal{T}_{\mathcal{U}^{-1}})$ induced by $(X, \mathcal{U})$ is called an L-fuzzy bitopological space.

Lemma 3.7 — Let $\psi: X \to Y$ be a function. For each $f \in \Omega_Y$, a function $\psi^{-1}(f): L^X \to L^X$ is defined by, for all $\mu \in L^X$,

$$\psi^{-1}(f)(\mu) = (\psi^{\leftarrow} \circ f \circ \psi^{\rightarrow})(\mu) = \psi^{\leftarrow}(f(\psi^{\rightarrow}(\mu))).$$

For $f, f_1, f_2 \in \Omega_Y$, we have the following properties.

(1) $\psi^{-1}(f) \in \Omega_X$.

(2) If $f_1 \leq f_2$, then $\psi^{-1}(f_1) \leq \psi^{-1}(f_2)$.

(3) $\psi^{-1}(f_1) \circ \psi^{-1}(f_2) \leq \psi^{-1}(f_1 \circ f_2)$ with equality if ψ is onto.

(4) $(\psi^{-1}(f))^{-1} = \psi^{-1}(f^{-1})$.

(5) $\psi^{-1}(f_1) \wedge \psi^{-1}(f_2) = \psi^{-1}(f_1 \wedge f_2)$.

(6) $\psi^{\rightarrow}((\psi^{-1}(f))^{-1}(\lambda)) \leq f^{-1}(\psi^{\rightarrow}(\lambda))$, for all $\lambda \in L^X$.

PROOF : (1), (2), (3) and (4) follow from Proposition 5.5 of[10].

(5) Using (2) and Lemma 2.6[5], we get $\psi^{-1}(f_1) \wedge \psi^{-1}(f_2) \geq \psi^{-1}(f_1 \wedge f_2)$.

Suppose that $\psi^{-1}(f_1) \wedge \psi^{-1}(f_2) \not\leq \psi^{-1}(f_1 \wedge f_2)(\mu)$.

Since $\psi^{-1}(f_1 \wedge f_2)(\mu) = \psi^{\leftarrow}(f_1 \wedge f_2)\, \psi^{\rightarrow}(\mu_2))$, by the definition of $f_1 \wedge f_2$, there exist $\mu_1, \mu_2 \in I^Y$ such that $\mu_1 \vee \mu_2 = \psi^{\rightarrow}(\mu)$ and

$$\psi^{-1}(f_1) \wedge \psi^{-1}(f_2)(\mu) \not\leq \psi^{\leftarrow}(f_1(\mu_1)) \vee \psi^{\leftarrow}(f_2(\mu_2)).$$

On the other hand, since $\mu_1 \vee \mu_2 = \psi^{\rightarrow}(\mu)$, we have

$$\mu \leq \psi^{\leftarrow}(\psi^{\rightarrow}(\mu)) = \psi^{\leftarrow}(\mu_1) \vee \psi^{\leftarrow}(\mu_2).$$

It implies $(\psi^{\leftarrow}(\mu_1) \wedge \mu)(\psi^{\leftarrow}(\mu_2) \wedge \mu) = \mu$. By the definition of $\psi^{-1}(f_1) \wedge \psi^{-1}(f_2)$,

$$(\psi^{-1}(f_1) \wedge \psi^{-1}(f_2))(\mu) \leq \psi^{-1}(f_1)(\psi^{\leftarrow}(\mu_1) \wedge \mu) \vee \psi^{-1}(f_2)(\psi^{\leftarrow}(\mu_2) \wedge \mu)$$

$$= \psi^{\leftarrow}(f_1(\psi^{\rightarrow}(\psi^{\leftarrow}(\mu_1)\wedge\mu)) \vee \psi^{\leftarrow}(f_2(\psi^{\rightarrow}(\psi^{\leftarrow}(\mu_2)\wedge\mu)))$$

$$\leq \psi^{\leftarrow}(f_1(\mu_1)) \vee \psi^{\leftarrow}(f_2(\mu_2)).$$

It is a contradiction.

(6) From (4), we have for all $\lambda \in L^X$,

$$\psi^{\rightarrow}((\psi^{-1}((f))^{-1}(\lambda)) = \psi^{\rightarrow}(\psi^{-1}(f^{-1})(\lambda))$$

$$= \psi^{\rightarrow}(\psi^{\leftarrow}(f^{-1}(\psi^{\rightarrow}(\lambda)))) \text{ (by 4))}$$

$$\leq f^{-1}(\psi^{\rightarrow}(\lambda)).$$

Definition 3.8 — Let $(X, \mathcal{U})$, $(X, \mathcal{V})$ be L-fuzzy (quasi)-uniform spaces. A function $\psi : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is L-fuzzy (quasi)-uniformly continuous if $\mathcal{V}(f) \leq \mathcal{U}(\psi^{-1}(f))$, for every $f \in \Omega_Y$.

The following theorem is immediate from Definition 3.8.

Theorem 3.9 — *Let $(X, \mathcal{U})$, $(X, \mathcal{V})$ and $(X, \mathcal{W})$ be L-fuzzy (quasi)-uniform spaces. If $\psi : (X, \mathcal{U}) \to (Y, \mathcal{V})$ and $\phi : (Y, \mathcal{V}) \to (Z, \mathcal{W})$ are L-fuzzy (quasi)-uniformly continuous, then $\phi \circ \psi : (X, \mathcal{U}) \to (Z, \mathcal{W})$ is L-fuzzy (quasi)-uniformly continuous.*

Theorem 3.10 — *Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be L-fuzzy quasi-uniform spaces. If $\psi : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is L-fuzzy quasi-uniformly continuous, then:*

(1) *$\psi : (X, \mathcal{U}^{-1}) \to (Y, \mathcal{V}^{-1})$ is L-fuzzy quasi-uniformly continuous,*

(2) *$\psi : (X, \mathcal{U}^{*}) \to (Y, \mathcal{V}^{*})$ is L-fuzzy uniformly continuous.*

PROOF : (1) For all $f \in \Omega_Y$, we have

$$\mathcal{V}^{-1}(f) = \mathcal{V}(f^{-1}) \leq \mathcal{U}(\psi^{-1}(f^{-1}))$$

$$= \mathcal{U}((\psi^{-1}(f))^{-1} \text{ (by Lemma 3.7}^4).$$

$$= \mathcal{U}^{-1}(\psi^{-1}(f)).$$

(2) Suppose that $\psi : (X, \mathcal{U}^{*}) \to (Y, \mathcal{V}^{*})$ is not L-fuzzy uniformly continuous. There exists $f \in \Omega_Y$, such that $\mathcal{V}^{*}(f) \not\leq \mathcal{U}^{*}(\psi^{-1}(f))$. By the definition of $\mathcal{V}^{*}$ from Theorem 3.3. Let there exist $f_1, f_2 \in \Omega_Y$ such that

$$\mathcal{V}(f_1) \wedge \mathcal{V}^{-1}(f_2) \not\leq \mathcal{U}^{*}(\psi^{-1}(f)), \quad f_1 \wedge f_2 \leq f.$$

On the other hand, by Lemma $3.5^{2,5}$ we have

$$\psi^{-1}(f_1 \wedge f_2) = \psi^{-1}(f_1) \wedge \psi^{-1}(f_2) \leq \psi^{-1}(f).$$

Since $\psi: (X, \mathcal{U}) \to (Y, \mathcal{V})$ and $\psi: (X, \mathcal{U}^{-1}) \to (Y, \mathcal{V}^{-1})$ are L-fuzzy quasi-uniformly continuous, then we have

$$\mathcal{V}(f_1) \wedge \mathcal{V}^{-1}(f_2) \leq \mathcal{U}(\psi^{-1}(f_1)) \wedge \mathcal{U}^{-1}(\psi^{-1}(f_2)) \leq \mathcal{U}^*(\psi^{-1}(f)).$$

It is a contradiction. ■

***Theorem* 3.11** *Let $(X, \mathcal{U})$ and $Y, \mathcal{V})$ be L-fuzzy quasi-uniform spaces. Let $\psi: (X, \mathcal{U}) \to (Y, \mathcal{V})$ be L-fuzzy quasi-uniformly continuous. For each $\lambda \in L^X$, $\nu \in L^Y$ and $r \in L^1$, we have the following properties.*

(1) $C_{\mathcal{V}}(\psi^{\to}(\lambda), r) \geq \psi^{\to}(C_{\mathcal{V}}(\lambda, r))$,

$C_{\mathcal{V}^{-1}}(\psi^{\to}(\lambda), r) \geq \psi^{\to}(C_{\mathcal{U}^{-1}}(\lambda, r))$,

(2) $C_{\mathcal{U}}(\psi^{\leftarrow}(\nu), r) \leq \psi^{\leftarrow}(C_{\mathcal{V}}(\nu, r))$.

PROOF : (1) For $\lambda \in L^X, r \in L^1$, we have

$$C_{\mathcal{V}}(\psi^{\to}(\lambda), r) = \wedge \left\{f^{-1}(\psi^{\to}(\lambda)) \mid \mathcal{V}(f) > r\right\}$$

$$\geq \wedge \left\{\psi^{\to}((\psi^{-1}(f))^{-1}(\lambda)) \mid \mathcal{U}(\psi^{-1}(f)) \geq \mathcal{V}(f) > r\right\}$$

(by Lemma 3.7[6] and ψ is L-quasi-uniformly continuous),

$$\geq \psi^{\to}\left(\wedge \left\{(\psi^{-1}(f))^{-1}(\lambda)) \mid \mathcal{U}(\psi^{-1}(f)) > r\right\}\right)$$

$$\geq \psi^{\to}\left(\wedge \left\{f^{-1}(\lambda) \mid \mathcal{U}(f) > r\right\}\right)$$

$$= \psi^{\to}(C_{\mathcal{U}}(\lambda, r)).$$

Similarly, we have $C_{\mathcal{V}^{-1}}(\psi^{\to}(\lambda), r) \geq \psi^{\to}(C_{\mathcal{U}^{-1}}(\lambda, r))$.

(2) For all $r \in L^1$ and $\nu \in L^Y$, we have

$$C_{\mathcal{U}}(\psi^{\leftarrow}(\nu), r) \leq \psi^{\leftarrow}(\psi^{\to}(C_{\mathcal{U}}(\psi^{\leftarrow}(\nu), r)))$$

$$\leq \psi^{\leftarrow}(C_{\mathcal{V}}(\psi^{\to}(\psi^{\leftarrow}(\nu)), r)) \text{ (by (1))}$$

$$\leq \psi^{\leftarrow}(C_{\mathcal{V}}(\nu, r)).$$ ■

***Theorem* 3.12** — *Let $(X, \mathcal{U})$, $(X, \mathcal{V})$ be L-fuzzy (quasi)-uniform spaces. If $\psi: (X, \mathcal{U}) \to (Y, \mathcal{V})$ is L-fuzzy quasi-uniformly continuous, then:*

(1) $\psi : (X, \mathcal{T}_{\mathcal{U}}) \to (Y, \mathcal{T}_{\mathcal{V}})$ *is L-fuzzy continuous,*

(2) $\psi : (X, \mathcal{T}_{\mathcal{U}^{-1}}) \to (Y, \mathcal{T}_{\mathcal{V}^{-1}})$ *is L-fuzzy continuous,*

(3) $\psi : (X, \mathcal{T}_{\mathcal{U}^*}) \to (Y, \mathcal{T}_{\mathcal{V}^*})$ *is L-fuzzy continuous.*

PROOF : (1) Suppose that $\psi : (X, \mathcal{T}_{\mathcal{U}}) \to (Y, \mathcal{T}_{\mathcal{V}})$ is not L-fuzzy continuous. Then there exists $\rho \in L^Y$ such that $\mathcal{T}_{\mathcal{V}}(\rho) \not\leq \mathcal{T}_{\mathcal{U}}(\psi^{\leftarrow}(\rho))$. Hence there exists $r \in L^1$ such that

$$r \not\leq \mathcal{T}_{\mathcal{U}}(\psi^{\leftarrow}(\rho)), C_{\mathcal{V}}(\rho', r) \leq \rho'. \qquad \ldots (A)$$

It implies

$$\begin{aligned} \psi^{\leftarrow}(\rho)' &= \psi^{\leftarrow}(\rho') \\ &\geq \psi^{\leftarrow} C_{\mathcal{V}}(\rho', r)) \quad \text{(by } (A)) \\ &\geq C_{\mathcal{U}}(\psi^{\leftarrow}(\rho'), r) \text{ (by Theorem 3.11 (2))} \\ &= C_{\mathcal{U}}(\psi^{\leftarrow}(\rho)' r). \end{aligned}$$

Hence $C_{\mathcal{U}}(\psi^{\leftarrow}(\rho'), r) \leq \psi^{\leftarrow}(\rho)'$. Therefore, $\mathcal{T}_{\mathcal{U}}(\psi^{\leftarrow}(\rho)) \geq r$ from Theorem 3.6. It is a contradiction. Similarly, we can prove (2) and (3). ■

4. L-FUZZY QUASI-UNIFORMIZABLE SPACES

Lemma 4.1. — Let $(X, \mathcal{T})$ be an L-fuzzy topological space and

$$\mathcal{T}_0 = \left\{ \bar{0} \neq \rho \in L^X \mid \mathcal{T}(\rho) \neq 0 \right\}.$$

For every $\rho \in \mathcal{T}_0$, we define $f_\rho : L^X \to L^X$ as follows :

$$f_\rho(\lambda) = \begin{cases} \bar{0} \text{ if } \lambda = \bar{0}, \\ \rho \text{ if } \bar{0} \neq \lambda \leq \rho, \\ \bar{1} \text{ otherwise.} \end{cases}$$

Then we have the following properties:

(1) $f_\rho \in \Omega_X$ and $f_\rho^{-1} = \rho_{\rho'}$.

(2) $f_\rho \circ f_\rho = f_\rho$ and $f \leq f_{\bar{1}}$ for all $f \in \Omega_X$.

(3) If $f_{\rho i}$ for $i = 1, \ldots, n$, $\Gamma = \left\{ J \subset \{1, \ldots, n\} \lambda \leq \bigvee_{j \in J} \rho_j \right\}$

and $\rho_J = \bigvee_{j \in J} \rho_j$, then

$$\wedge_{i=1}^{n} f_{\rho i}(\lambda) = \begin{cases} \overline{0} & \text{if } \lambda = \overline{0}, \\ \bigwedge_{j \in \Gamma} \rho_j & \text{if } \Gamma \neq \phi, \\ \overline{1} & \text{if } \Gamma = \phi. \end{cases}$$

PROOF : (1) By the definition of f_ρ, we have $\lambda \leq f_\rho(\lambda)$.

We show that $f_\rho\left(\bigvee_{i \in \Gamma} v_i\right) = \bigvee_{i \in \Gamma} f_\rho(v_i)$ from the following conditions:

(a) $\bigwedge_{j \in \Gamma} v_i \leq \rho$ iff for all $i \in \Gamma$, $v_i \leq \rho$,

(b) $\bigwedge_{j \in \Gamma} v_i \not\leq \rho$ iff for some $i \in \Gamma$, $v_i \not\leq \rho$.

We prove that $f_\rho^{-1}(\lambda) = f_{\rho'}(\lambda)$, for each $\lambda \in L^X$, from the followings:
If $\lambda = \overline{0}$, it is trivial. If $\overline{1} \neq \lambda \leq \rho'$,

$$\begin{aligned} f_\rho^{-1}(\lambda) &= \wedge\left\{ v \in L^X \mid f_\rho(v') \leq \lambda' \right\} \\ &= \wedge\left\{ v \in L^X \mid f_\rho(v') = \rho \right\} \\ &= \wedge\left\{ v \in L^X \mid v \geq \rho' \right\} \\ &= \rho' = f_{\rho'}(\lambda). \end{aligned}$$

If $\lambda \not\leq \rho'$,

$$\begin{aligned} f_\rho^{-1}(\lambda) &= \wedge\left\{ v \in L^X \mid f_\rho(v') \leq \lambda' \right\} \\ &= \wedge\left\{ v \in L^X \mid f_\rho(v') = \overline{0} \right\} \\ &= \overline{1} = f_{\rho'}(\lambda). \end{aligned}$$

(2) It is easily proved from the definition of f_ρ.

(3) If $\lambda = \overline{0}$ and $\Gamma = \phi$,

$$\wedge_{i=1}^{n} f_{\rho i}(\lambda) = \bigwedge_{J \in \Gamma} \rho_J$$

Suppose $\wedge_{i=1}^{n} f_{\rho i}(\lambda) \not\leq \bigwedge_{J \in \Gamma} \rho_j$ There exists $J \in \Gamma$ such that

$$\wedge_{i=1}^{n} f_{\rho i}(\lambda) \not\leq \rho_J, \quad \lambda \leq \bigwedge_{J \in \Gamma} \rho_j$$

Put for $i \in \{1, ..., n\}$,

$$\lambda_i = \begin{cases} \lambda \wedge \rho_i & \text{if } i \in J, \\ \overline{0} & \text{otherwise.} \end{cases}$$

Since $\lambda = \bigvee_{i \in J} \lambda_i$ and $\lambda_i \leq \rho_i$ for all $i \in J$, we obtain

$$\wedge_{i=1}^{n} f_{\rho i}(\lambda) \leq \bigvee_{i=1}^{n} f_{\rho i}(\lambda_i) \leq \bigvee_{i \in J} \rho_i = \rho_J.$$

It is a contradiction. Hence $\wedge_{i=1}^{n} f_{\rho i}(\lambda) \leq \bigwedge_{J \in \Gamma} \rho_j$

Suppose $\wedge_{i=1}^{n} f_{\rho i}(\lambda) \not\geq \bigwedge_{J \in \Gamma} \rho_j$ There exist $\lambda_i \in L^X$ with $\lambda = \bigvee_{i=1}^{n} \lambda_i$ such that

$$\bigvee_{i=1}^{n} f_{\rho i}(\lambda_i) \not\geq \bigwedge_{J \in \Gamma} \rho_J.$$

Put $\rho = \bigvee_{i=1}^{n} f_{\rho i}(\lambda_i)$ and $K = \{k \in \{1, ..., n\} \mid \rho_k \leq \rho\}$. We obtain $\rho_K \leq \rho$. If $i \notin K$, then $\rho_i \not\leq \rho$. Hence $f_{\rho i}(\lambda_i) = \overline{0}$, which implies $\lambda_i = \overline{0}$.

If $k \in K$, then $\lambda_k \leq \rho_k$ because $f_{\rho k}(\lambda_k) \neq \overline{1}$.

It implies that

$$\lambda = \bigvee_{i=1}^{n} \lambda_i = \bigvee_{k \in K} \lambda_k \leq \bigvee_{k \in K} \rho_k.$$

Then there exists $K \in \Gamma$ such that

$$\bigvee_{i=1}^{n} f_{\rho i}(\lambda_i) = \rho \geq \rho_K \geq \bigvee_{k \in \Gamma} \rho_K.$$

It is a contradiction. Hence.

$$\wedge_{i=1}^{n} f_{\rho i}(\lambda) \geq \bigwedge_{J \in \Gamma} \rho_J .$$ ■

Theorem 4.2 — *Let* $(X, \mathcal{T})$ *be an L-fuzzy topological space. Define a function* $\mathcal{U}_{\mathcal{T}} : \Omega_X \to L$ *by*

$$\mathcal{U}_{\mathcal{T}}(f) = \bigvee \left\{ \bigwedge_{i=1}^{n} \mathcal{T}(\rho_i) \mid \bigwedge_{i=1}^{n} f_{\rho i} \leq f \right\}$$

where the first $\vee$ *is taken over every finite family* $\{f_{\rho i} \mid i = 1, ..., n\}$. *Then* $\mathcal{U}_{\mathcal{T}}$ *is an L-fuzzy quasi-uniformity on X.*

PROOF : (FQU1). It is trivial

(FQU2) Suppose there exist $f, g \in \Omega_X$ such that

$$\mathcal{U}_{\mathcal{T}}(f \wedge g) \not\geq \mathcal{U}_{\mathcal{T}}(f) \wedge \mathcal{U}_{\mathcal{T}}(g).$$

There exist two finite families $\left\{ \rho_i \in \mathcal{T}_0 \mid \bigwedge_{i=1}^{m} f_{\rho i} \leq f \right\}$ and $\left\{ v_i \in \mathcal{T}_0 \mid \bigwedge_{i=1}^{m} f_{v_j} \leq g \right\}$ such that

$$\mathcal{U}_{\mathcal{T}}(f \wedge g) \not\geq \left(\bigwedge_{i=1}^{m} \mathcal{T}(\rho_i) \right) \left(\bigwedge_{j=1}^{n} \mathcal{T}(v_j) \right)$$

On the other hand, since $f \wedge g \geq \left(\bigwedge_{i=1}^{m} f_{\rho_i} \right) \wedge \left(\bigwedge_{j=1}^{n} f(v_j) \right)$, we have

$$\mathcal{U}_{\mathcal{T}}(f \wedge g) \geq \left(\bigwedge_{i=1}^{m} \mathcal{T}(\rho_i) \right) \left(\bigwedge_{j=1}^{n} \mathcal{T}(v_j) \right).$$

It is a contradiction.

(FQU3) Suppose there exists $f \in \Omega_X$ such that

$$\bigvee \left\{ \mathcal{U}_{\mathcal{T}}(g) \mid g \circ g \leq f \right\} \not\geq \mathcal{U}_{\mathcal{T}}(f).$$

There exists a finite family $\left\{ \rho_i \in \mathcal{T}_0 \mid \bigwedge_{i=1}^{m} f_{\rho i} \leq f \right\}$ such that

$$\bigvee \left\{ \mathcal{U}_{\mathcal{T}}(g) \mid g \circ g \leq f \right\} \not\geq \bigwedge_{i=1}^{m} \mathcal{T}(\rho_i). \qquad \dots (B)$$

On the other hand, since $f_{\rho i} \circ f_{\rho i} = f_{\rho i}$ for each $i \in \{1, ..., m\}$ from Lemma 4.1[2], we have

$$\left(\bigwedge_{i=1}^{m} f_{\rho i} \right) \circ \left(\bigwedge_{i=1}^{m} f_{\rho i} \right) \leq f_{\rho i} \circ f_{\rho i} = f_{\rho i}.$$

From Lemma 2.6[4,5], it implies

$$\left(\bigwedge_{i=1}^{m} f_{\rho i} \right) \circ \left(\bigwedge_{i=1}^{m} f_{\rho i} \right) \leq \left(\bigwedge_{i=1}^{m} f_{\rho i} \right) \leq f. \text{ Put } g = \bigwedge_{i=1}^{m} f_{\rho i}.$$

Then $g \circ g \leq f$ and

$$\bigvee \left\{ \mathcal{U}_{\mathcal{T}}(g) \mid g \circ g \leq f \right\} \geq \bigwedge_{i=1}^{m} \mathcal{T}(\rho_i).$$

It is a contradiction for the eq. (B).

(FQU4) Since $\mathcal{T}(\overline{1}) = 1$, there exists $f_{\overline{1}} \in \Omega_X$ such that $\mathcal{U}_{\mathcal{T}}(f_{\overline{1}}) = 1$. Hence $\mathcal{U}_{\mathcal{T}}$ is an L-fuzzy quasi-uniformity on X. ■

Definition 4.3 — An L-fuzzy quasi-uniform space $(X, \mathcal{U})$ is said to be compatible with an L-fuzzy topological space $(X, \mathcal{T})$ if $\mathcal{T} = \mathcal{T}_{\mathcal{U}}$. The class $\Sigma(\mathcal{T})$ denotes the family of all L-fuzzy quasi-uniformities which are compatible with an L-fuzzy topology $\mathcal{T}$.

Theorem 4.4 — *Let L be an order dense chain. Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $\mathcal{U}_{\mathcal{T}}$ denoted by the L-fuzzy quasi-uniformity induced by $\mathcal{T}$. Then we have*

(1) $\mathcal{T}_{\mathcal{U}_{\mathcal{T}}} = \mathcal{T}$.

(2) *$\mathcal{U}_{\mathcal{T}}$ is the coarsest number of $\Sigma(\mathcal{T})$.*

PROOF : (1) Since L is an order dense chain, by Theorem 2.4, $\mathcal{T}_{I_{\mathcal{T}}} = \mathcal{T}$. Thus, we only show that each $\lambda \in L^X$ and $r \in L^1$, $I_{\mathcal{U}_{\mathcal{T}}}(\lambda, r) = I_{\mathcal{T}}(\lambda, r)$ defined as follows:

$$I_{\mathcal{U}_{\mathcal{T}}}(\lambda, r) = \bigvee \left\{ \mu \in L^X \mid f(\mu) \leq \lambda, \ \mathcal{U}_{\mathcal{T}}(f) > r \right\},$$

$$I_{\mathcal{T}}(\lambda, r) = \bigvee \left\{ \rho \in L^X \mid \rho \le \lambda,\ \mathcal{T}(\rho) > r \right\}.$$

If $\lambda = \bar{0}$ or $\bar{1}$, it is trivial. If $\mathcal{T}(\rho) > r$ with $\rho \le \lambda$, then there exists $f_\rho \in \Omega_X$ such that

$$\rho = f_\rho(\rho) \le \lambda,\ \mathcal{U}_{\mathcal{T}}(f_\rho) \ge \mathcal{T}(\rho) > r.$$

Hence $I_{\mathcal{T}}(\lambda, r) \le I_{\mathcal{U}}(\lambda, r)$.

Suppose that there exist $\lambda \in L^X$ and $r \in L^1$ such that

$$I_{\mathcal{T}}(\lambda, r) \not\ge I_{\mathcal{U}_{\mathcal{T}}}(\lambda, r).$$

From the definition of $I_{\mathcal{U}_{\mathcal{T}}}(\lambda, r)$, there exists $f \in \Omega_X$ with $\mathcal{U}_{\mathcal{T}}(f) > r$ such that

$$f(\mu) \le \lambda,\ I_{\mathcal{T}}(\lambda, r) \not\ge \mu.$$

On the other hand, since $\mathcal{U}_{\mathcal{T}}(f) > r$ and L is an order dense chain, there exist $r_1 \in L^1$ and a family

$$\left\{ f_{\rho 1}, \ldots, f_{\rho n} \mid \bigwedge_{i=1}^{n} f_{\rho i} \le f \right\}$$

such that

$$\bigwedge_{i=1}^{n} \mathcal{T}(\rho_i) \ge r_1 > r,\quad \bigwedge_{i=1}^{n} f_{\rho i}(\mu) \le f(\mu).$$

Since $\bigwedge_{i=1}^{n} f_{\rho i}(\mu) \le \lambda$ and $\lambda \ne \bar{1}$, by Lemma 4.1, there exists

$$\Gamma = \left\{ J \subset \{1, \ldots, n\} \mid \mu \le \rho_J = \bigvee_{j \in J} \rho_j \right\}$$

such that

$$\bigwedge_{i=1}^{n} f_{\rho i}(\mu) = \bigwedge_{j \in J} \rho_j \le \lambda$$

and

$$\mathcal{T}(\rho_J) = \mathcal{T}\left(\bigvee_{j \in J} \rho_j \right) \ge \bigwedge_{j \in J} \mathcal{T}(\rho_j) \ge r_1 > r.$$

Since $\mathcal{T}\left(\bigvee_{j\in\Gamma}\rho_j\right) \geq \bigwedge_{J\in\Gamma}\mathcal{T}(\rho_J) \geq r_1 > r$, we have

$$I_{\mathcal{T}}(\lambda, r) \geq \bigvee_{J\in\Gamma}\rho_J \geq \mu.$$

It is a contradiction. Hence, $I_{\mathcal{T}}(\lambda, r) \geq I_{\mathcal{T}_{\mathcal{U}}}(\lambda, r)$.

(2) By (1), we have that $\mathcal{U}_{\mathcal{T}}$ is compatible with $\mathcal{T}$. Let $\mathcal{U}$ be an arbitrary member of $\Sigma(\mathcal{T})$. We will show that $\mathcal{U}_{\mathcal{T}}(f) \leq \mathcal{U}(f)$, for all $f \in \Omega_X$.

Suppose that there exists $f \in \Omega_X$ such that $\mathcal{U}_{\mathcal{T}}(f) \not\leq \mathcal{U}(f)$. Then, by the definition of $\mathcal{U}_{\mathcal{T}}$, there exists a family $\left\{ f_{\rho i} \mid \bigwedge_{i=1}^{n} f_{\rho i} \leq f \right\}$ such that

$$\bigwedge_{i=1}^{n} \mathcal{T}(\rho_i) \not\leq \mathcal{U}(f).$$

Put $\mathcal{U}(f) = r$. Since $\mathcal{T}_{\mathcal{U}} = \mathcal{T}$ and L is a chain, $\mathcal{T}_{\mathcal{U}}(\rho_i) > r$ for all $i = 1, \ldots, n$, that is, $I_{\mathcal{T}_{\mathcal{U}}}(\rho_i, r) = \rho_i$. From the definition of $I_{\mathcal{T}_{\mathcal{U}}}(\rho_i, r)$, for each $i = 1, \ldots, n$ there exist $g_i \in \Omega_X$ with $\mathcal{U}(g_i) > r$ and $\mu_i \in L^X$ with $g_i(\mu_i) \leq \rho_i$ such that $\mu_i \leq I_{\mathcal{T}_{\mathcal{U}}}(\rho_i, r) = \rho_i$. Thus, by the definition of $f_{\rho i}$, $g_i \leq f_{\rho i}$ for all $i = 1, \ldots, n$. Put $g = \bigwedge_{i=1}^{n} g_i$. We have $\mathcal{U}(g) > r$. Hence $f \geq g$ and $\mathcal{U}(f) \geq \mathcal{U}(g) > r$. Thus $\mathcal{U}(f) > r$. It is a contradiction. Therefore, $\mathcal{U}_{\mathcal{T}}(f) \leq \mathcal{U}(f)$, for all $f \in \Omega_X$. ■

Example 4.5 — Let $\mathcal{T}$ be an [0, 1]-fuzzy topology on X defined as follows:

$$\mathcal{T}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{1} \text{ or } \overline{0}, \\ \frac{1}{2} & \text{if } \lambda = \rho, \\ 0 & \text{otherwise.} \end{cases}$$

From Theorem 4.2, we have

$$\mathcal{U}_{\mathcal{T}}(f) = \begin{cases} 1 & \text{if } f = f_{\overline{1}} \\ \frac{1}{2} & \text{if } f_{\rho} \leq f < f_{\overline{1}}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $I_{\mathcal{U}_{\mathcal{T}}}(\rho, r) = \rho$, for each $0 \le r < \frac{1}{2}$. Hence $\mathcal{T}_{\mathcal{U}_{\mathcal{T}}} = \mathcal{T}$.

Definition 4.6 — An L-fuzzy topological space $(X, \mathcal{T})$ is said to be L-fuzzy quasi-uniformizable if there exists an L-fuzzy quasi-uniformity $\mathcal{U}$ on X such that $\mathcal{T} = \mathcal{T}_{\mathcal{U}}$.

Since $L = [0, 1]$ is a completely distributive and order dense chain, the following corollory is immediate from Theorem 4.2 and Theorem 4.4[1].

Corollary 4.7 — Every [0, 1]-fuzzy topological space is [0, 1]-fuzzy quasi-uniformizable.

Lemma 4.8 — Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be L-fuzzy topological spaces. If $\psi : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is L-fuzzy continuous and $\mathcal{T}_2(\rho) > 0$, then $\psi^{-1}(f_\rho) = f_{\psi^{\leftarrow}(\rho)}$.

PROOF : Since $\mathcal{T}_1(\psi^{\leftarrow}(\rho)) \ge \mathcal{T}_2(\rho) > 0$, we have $\psi^{\leftarrow}(\rho) \in (\mathcal{T}_1)_0$. Since $\psi^{-1}(f_\rho)(\lambda) = \psi^{\leftarrow}(f_\rho(\psi^{\rightarrow}(\lambda)))$, for all $\lambda \in L^X$, and $\psi^{\rightarrow}(\lambda) \le \rho$ iff $\lambda \le \psi^{\leftarrow}(\rho)$, we have

$$\psi^{-1}(f_\rho)(\lambda) = \begin{cases} \bar{0} & \text{if } \lambda = \bar{0}, \\ \psi^{\leftarrow}(\rho) & \text{if } \bar{0} \ne \lambda \le \psi^{\leftarrow}(\rho), \\ \bar{1} & \text{otherwise.} \end{cases}$$

Thus, $\psi^{-1}(f_\rho) = f_{\psi^{\leftarrow}(\rho)}$. ■

Theorem 4.9 — *Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be L-fuzzy topological spaces.*

(1) *If $\psi : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is* L-fuzzy continuous, then $\psi : (X, \mathcal{U}_{\mathcal{T}_1}) \to (Y, \mathcal{U}_{\mathcal{T}_2})$ is L-fuzzy quasi-uniformly continuous.

(2) *If L is an order dense chain, the converse of (1) is true.*

PROOF : (1) Suppose that ψ is not L-fuzzy uniformly continuous. There exists $f \in \Omega_Y$ such that

$$\mathcal{U}_{\mathcal{T}_1}(\psi^{-1}(f)) \not\ge \mathcal{U}_{\mathcal{T}_2}(f).$$

By the definition of $\mathcal{U}_{\mathcal{T}_2}$, there exists a family $\left\{ f_{\rho i}, \mid \bigwedge_{i=1}^{n} f_{\rho i} \le f \right\}$ such that

$$\mathcal{U}_{\mathcal{T}_1}(\psi^{-1}(f)) \not\ge \bigwedge_{i=1}^{n} \mathcal{T}_2(\rho_i). \qquad \text{... } (C)$$

On the other hand, since f is L-fuzzy continuous, by Lemma 4.8, we have for each $i = 1, \ldots, n$, $\psi^{-1}(f_{\rho i}) = f_{\psi^{\leftarrow}(\rho_i)}$. Hence, by Lemma 3.7(5), we have

$$\bigwedge_{i=1}^{n} f_{\psi^{\leftarrow}(\rho_i)} = \bigwedge_{i=1}^{n} \psi^{-1}(f_{\rho i}) = \psi^{-1}\left(\bigwedge_{i=1}^{n} f_{\rho i}\right) \leq \psi^{-1}(f)$$

and, by the definiton of $\mathcal{U}_{\mathcal{T}_1}$,

$$\mathcal{U}_{\mathcal{T}_1}(\psi^{-1}(f)) \geq \bigwedge_{i=1}^{n} \mathcal{T}_1(\psi^{\leftarrow}(\rho_i))$$

$$\geq \bigwedge_{i=1}^{n} \mathcal{T}_2(\rho_i).$$

It is a contradiction for the eq. (*C*).

(2) It is easily proved from Theorem 3.12[1] and Theorem 4.4[1].

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