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ON PRIMES DUAL TO ASSOCIATED PRIMES FOR COTORSION MODULES

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Using the theory of flat covers, we introduce a set of primes dual to the set of associated primes. It turns out that this set has properties similar (or rather dual) to those of the set of associated primes. With the aid of this set of primes, we obtain a necessary and sufficient condition for a cotorsion module M, over a ring whose injective hull is flat, to have a nonzero injective cover.

Key Words: Flat Cover; Injective Cover; Cotorsion Module

1. INTRODUCTION

In the study of modules over a commutative Noetherian ring R, the set of associated prime ideals have proved to be an important tool. For a module M, if $E(M)$ stands for its injective envelope, then $\mathrm{Ass}_R(M)$, the set of associated primes of M, can be defined as the set of primes p such that $E(M)$ contains a copy of $E(R/p)$ and it can be computed as the set of prime ideals p such that $\mathrm{Hom}_{R_p}(k(p), (M_p)) \neq 0$ (here $k(p) = (R/p)_p$ is the residue field of p).

On the other hand, the theory of flat covers of modules, which is a suitable dual to the theory of injective envelopes, was introduced by Enochs in[6]. Recently it was proved by Bican, El Bashir and Enochs that every module admits a flat cover[2]. In[7], Enochs showed that over a commutative Noetherian ring R, a flat cover F of a cotorsion module M, can be uniquely written in the form $F = \prod T_p$, where T_p is a completion of a free R_p-module with respect to the p-adic topology[7] (an R-module C is called cotorsion if for any flat R-module F, $\mathrm{Ext}^1_R(F, C) = 1$).

These facts provide motivation for our definition for the dual of associated primes, for cotorsion modules.

In fact our strategy is to replace injective envelopes by flat covers and replace the structure of injective modules by the structure of flat cotorsion modules. So we define the set of dual associated primes of a cotorsion module M to be the set of all primes p such that $F(M)$, the flat cover of M, contains a copy of $\hat{R}_p$ as a direct summand. We will use Coass(M), to denote this set of primes of R. In fact Coass(M) can be characterized as the set of all primes p, for which $k(p) \otimes_{R_p} \mathrm{Hom}_R(R_p, M) \neq 0$. Comparing this with the computation of the primes of Ass (M), we see that Hom is replaced by tensor product and the localization M_p is replaced by $\mathrm{Hom}_R(R_p, M)$, which was called the colocalization of M at the prime ideal p, by Melkersson and Schenzel[12]. It turns out that this set of primes has properties similar or rather dual to those of associated primes. We show that $\mathrm{MinCos}_R(M) = \mathrm{Min\,Coass}(M)$, where $\mathrm{Cos}_R(M)$ denotes the co-support of M, which is the set of all primes p such that $\mathrm{Hom}_R(R_p, M) \neq 0$[12]. It also will be shown that this definition is equivalent with Macdonald's definition when M is Artinian or injective. We shall also establish some properties of Coass(M). For example, we show that for an R-module M and injective R-module E, $\mathrm{Coass}(\mathrm{Hom}_R(M, E)) = \{p \in \mathrm{Ass}_R(M) : p \subseteq q$ for some $q \in \mathrm{Ass}_R(E)\}$. In particular, in case M is finitely generated, $\mathrm{Coass}(\mathrm{Hom}_R(M, E) = \mathrm{Att}(\mathrm{Hom}_R(M, E))$.

It is shown by Golan and Teply[9], that every module admits an injective (pre)cover. Of course in many cases it zero. So finding the conditions on module for which there exists a nonzero injective cover is quite interesting. Towards the end of the paper we consider the questionof when every nonzero module M, has a nonzero injective cover, or more generally, when such an M has a nonzero linear map $E \to M$ with E injective. Using the theory of dual associated primes we will obtain a necessary and sufficient condition for a cotorsion module M, over a ring R whose injective hull is flat, to have such a property. In fact, for a cotorsion module M, over a ring R with $E(R)$ flat, we prove that, $\mathrm{Hom}_R(E, M) \neq 0$ for some injective R-module E, if and only if $\mathrm{Coass}(M) \cap \mathrm{Ass}_R(R) \neq \phi$. Note that the class of rings whose injective hulls are flat, contains all integral domains and also all Gorenstein rings.

Throughout R is a commutative Noetherian ring with identity. For an R-module M, $E(M)$ stands for its injective envelope and $F(M)$ stands for its flat cover. All other notations are standard.

2. Properties of Dual Associated Primes

For completeness, we begin with the formal definition.

Definition 2.1 — Let M be a cotorsion R-module. We define the set of dual associated primes, Coass(M), of M by

$$\mathrm{Coass}(M) = \{p \in \mathrm{Sepc}(R) : \hat{R}_p \text{ is a direct summand of } F(M)\}$$

Note that, in the light of[7], if $M \neq 0$,Coass(M) $\neq \phi$.

The following proposition provides a description of Coass(M). This formula turns out to be useful when we study properties of dual associated primes. The proof is motivated by Enochs and Xu' proof of Theorem 2.2 of[8]. For the convenience of the reader we will write the modified proof for our case.

Proposition 2.2 — Let M be a cotorsion R-module. Then for a prime ideal p of R, $p \in \text{Coass}(M)$ if and only if $k(p) \otimes_R \text{Hom}_R(R_p, M) \neq 0$.

PROOF : Let $\cdots \to F_1 \to F_0 M \to 0$ be a minimal flat resolution of M. Using[13], by taking colocalization at a prime p, we have a minimal flat resolution of $\text{Hom}_R(R_p, M)$ as an R_p-module,

$$\cdots \to \text{Hom}_R(R_p, F_1) \xrightarrow{\partial} \text{Hom}_R(R_p, F_0) \to \text{Hom}\,(R_p, M) \to 0.$$

Let $\bar{\partial}$ stands for $k(p) \otimes_{R_p} \partial$. Then, by the proof of Theorem 2.2 of[8], $\bar{\partial} = 0$. Hence

$$k(p) \otimes_{R_p} \text{Hom}_R\,(R_p, F_0) \cong k(p) \otimes_{R_p} \text{Hom}_R\,(R_p, M).$$

Now it is easy to see that

$k(p) \otimes_{R_p} \text{Hom}_R\,(R_p, F_0) \neq 0$ if and only if $\hat{R}_p$ appears in F_0. Therefore, $\text{Coass}(M) = \{p \in \text{Spec}(R) : k(p) \otimes_{R_p} \text{Hom}_R\,(R_p, M) \neq 0\}$. □

Remark 2.3 — (*a*) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of cotorsion R-modules. Apply the functor $- \otimes_{R_p} k(p)$ to the exact sequence

$$0 \to \text{Hom}_R(R_p, M') \to \text{Hom}_R\,(R_p, M) \to \text{Hom}_R\,(R_p, M'') \to 0,$$

and use the last three terms of the long exact sequence of 'Tor' modules to deduce, in view of the previous proposition, that

$$\text{Coass}(M'') \subseteq \text{Coass}(M) \subseteq \text{Coass}(M') \cup \text{Coass}(M'').$$

(*b*) Is $S \subseteq R$ is a multiplicatively closed subset of R, then it follows from[13] that

$$\text{Coass}\,(\text{Hom}_R\,(S^{-1} R, M)) = \{p \in \text{Spec}\,(R) : p \in \text{Coass}(M) \text{ and } p \cap S = \phi\}.$$

In[12], the concept of co-support of a module M, is defined as the set of primes p for which $\text{Hom}_R(R_p, M) \neq 0$ and is denoted by $\text{Cos}_R\,(M)$. By the Proposition 2.2, it is clear that $\text{Coass}(M) \subseteq \text{Cos}_R\,(M)$. In the following, we show that the set of minimal elements of these two sets coincide.

Theorem 2.4 — *For a cotorsion module M, the set of minimal elements of Coass(M) and coass* $\text{Cos}_R\,(M)$ *are coincide.*

PROOF : First, we show that for all prime ideals p of R, $\text{Hom}_R(R_p, M)$ is cotorsion. To this end, let I be an injective resolution of M. Since M is cotorsion, $\text{Hom}_R(R_p, I\ )$ is an injective resolution of $\text{Hom}_R(R_p, M)$. Now let F be an arbitrary flat R-module.

Using the natural equivalence

$$\text{Hom}_R(F, \text{Hom}_R(R_p)) \cong \text{Hom}_R(F \otimes_R R_p)$$

it follows that

$$\text{Ext}^i_R(F, \text{Hom}_R(R_p, M) \cong \text{Ext}^i_R(F \otimes_R R_p, M),$$

which is zero for all $i \geq 1$, because $F \otimes R_p$ is flat. So the claim follows.

To prove the result, it is enough to show that if p is a minimal element of $\text{Cos}_R(M)$ then $p \in \text{Coass}(M)$. So let $p \in \text{MinCos}_R(M)$. Since $\text{Hom}_R(R_p, M) \neq 0$, $\text{Coass}(\text{Hom}_R(R_p, M)) \neq \phi$. Let $q \in \text{Coass}(\text{Hom}_R(R_p, M))$. So by Remark 2.3($b$), $q \subseteq p$. On the other hand, the following isomorphism

$$k(q) \otimes_{R_q} \text{Hom}_R(R_q, \text{Hom}_R(R_p, M)) \cong k(q) \otimes_{R_q} \text{Hom}_R(R_q, M) \neq 0$$

shows that $q \in \text{Cos}_R(M)$, which implies that $q = p$. This completes the proof. □

Corollary 2.5 — Let M be a cotorsion module. Then for any $q \in \text{Cos}_R(M)$, there exists $p \in \text{Coass}(M)$ such that $p \subseteq q$.

PROOF : This follows easily from 2.4. □

There have been several attempts to dualize the theory of associated primes (see[11,3,15,5&14]). In[14], it is shown that, over a Neotherian ring, all these definitions are equivalent. According to them, the set Coass(N) of co-associated prime ideals of an arbitrary R-module N, is defined to be the set of prime ideals p such that there exists an Artinian homomorphic image L of N such that $p = 0 :_R L$. In view of the next theorem one can see that the theory of dual associated prime is different from the theory of co-associated primes.

In preparation to the following theorem we remind the reader that for an R-module M and injective R-module E, the module $\text{Hom}_R(M, E)$ is also cotorsion.

This follows from the natural equivalence

$$\text{Ext}^1_R(\,, \text{Hom}_R(M, E)) \cong \text{Hom}_R(\text{Tor}^R_1(\,, M), E).$$

***Theorem* 2.6** — *Let $p \in$ Spec(R). Then $p \in \text{Ass}_R(M)$ if and only if there exists $m \in \text{Max}(R) \cap V(p)$ such that $p \in$ Coass($D_m(M)$), where ($D_m(M)$) $= \text{Hom}_R(M, E(R/m))$.*

PROOF : It is well known that $p \in \text{Ass}_R(M)$ if and only if $\text{Hom}_{R_p}(k(p), M_p) \neq 0$ and this is equivalent to saying that $\text{Hom}_R(\text{Hom}_{R_p}(k(p), M_p), E(R/m)) \neq 0$ for some $m \in \text{Max}(R) \cap V(p)$. Now the result follows from the following isomorphisms,

$$k(p) \otimes_{R_p} \mathrm{Hom}_R(R_p, D_m(M)) \cong k(p) \otimes_{R_p} \mathrm{Hom}_R(M_p, E(R/m))$$

$$\cong \mathrm{Hom}_R(\mathrm{Hom}_R(k(p), M_p), E(R/m)). \qquad \square$$

In[11] Macdonald has developed the theory of attached prime ideals and secondary representation of modules, which is (in a certain sense) a dual to the theory of assiciated prime ideals and primary decompositions. Now we are in a position to show that, for Artinian modules, our definition is equivalent with Macdonald's. Note that every Artinian module is cotorsion.

Theorem 2.7 — *Let M be an Artinian R-module. Then Coass (M) = Att(M).*

PROOF : Let $p \in \mathrm{Coass}(M)$. So $k(p) \otimes_{R_p} \mathrm{Hom}_R(R_p, M) \neq 0$. On the other hand $k(p) \otimes R_p$ $\mathrm{Hom}_R(R_p, M)$ is a homomorphic image of $\mathrm{Hom}_R(R_p, M)$ with pR_p as its annihilator. So by[12], $pR_p \in \mathrm{Att}(\mathrm{Hom}_R(R_p, M))$ and hence in view of[12], $p \in \mathrm{Att}(M)$. Conversely, let $p \in \mathrm{Att}(M)$. By[14], there exists $m \in \mathrm{Max}(R) \cap V(p)$ and a homomorphic image L of M, such that $L \cong D_m(R/p)$. Now by 2.6, since $p \in \mathrm{Ass}_R(R/p)$, $p \in \mathrm{Coass}(D_m(R/p))$. So $p \in \mathrm{Coass}(M)$, by 2.3(*a*). $\square$

The following theorem provides a slight modification of[1], [2.1]. Note that by the reminder before 2.6, $\mathrm{Hom}_R(M, E)$ is cotorsion.

Theorem 2.8 — *Let M be a (not necessarily local) R-module. Then for an injective R-module E,*

$$\mathrm{Coass}(\mathrm{Hom}_R(M, E)) = \{p \in \mathrm{Ass}_R(M) : p \subseteq q \text{ for some } q \in \mathrm{Ass}_R(E)\}.$$

PROOF : Let $P \in \mathrm{Coass}(\mathrm{Hom}_R(M, E))$. Then by 2.2,

$$k(p) \otimes_{R_p} \mathrm{Hom}_R(R_p, \mathrm{Hom}_R(M, E)) \neq 0.$$

Now the following isomorphism

$$k(p) \otimes_{R_p} \mathrm{Hom}_R(M_p, E) \cong \mathrm{Hom}_R(\mathrm{Hom}_{R_p}(k(p), M_p), E)$$

implies that $\mathrm{Hom}_R(\mathrm{Hom}_{R_p}(k(p), M_p), E) \neq 0$. In particular, we can conclude that $\mathrm{Hom}_{R_p}(k(p), M_p) \neq 0$ and $\mathrm{Hom}_R(k(p), E) \neq 0$. The first one is equivalent to $p \in \mathrm{Ass}_R(M)$ and the second one implies that there exists $q \in \mathrm{Ass}_R(E)$ such that $p \subseteq q$.

For the converse inclusion let $p \in \mathrm{Ass}_R(M)$ and $p \subset q$ for some $q \in \mathrm{Ass}_R(E)$.

Therefore,

$$\mathrm{Hom}\, R_p(k(p), M_p) \cong \mathrm{Hom}_{R_p}(k(p), E(M)_p \neq 0$$

and

$$k(p) \otimes_{R_p} \mathrm{Hom}_R(E(R/p), E(R/q)) \cong k(p) \otimes_{R_p} T_p \neq 0.$$

Set $E(M)_p = E(R/p) \oplus E'$ and $E = E(R/q \oplus E'')$. Hence $\mathrm{Hom}_R(\mathrm{Hom}_{R_p}(k(p), E(M)_p), E)$ contains a copy of $k(p) \otimes_{R_p} T_p$, so it is non-zero Now the result follows from the following isomorphisms

$$\mathrm{Hom}_R(\mathrm{Hom}_{R_p}(k(p), E(M)_p), E) \cong \mathrm{Hom}_R(\mathrm{Hom}_{R_p}(k(p), M_p), E)$$

$$\cong k(p) \otimes_{R_p} \mathrm{Hom}_R(M_p, E).$$ □

Corollary 2.9 — Let M be a finitely generated module. Then for any injective R-module E,

$$\mathrm{Coass}(\mathrm{Hom}_R(M, E)) = \mathrm{Att}(\mathrm{Hom}_R(M, E)).$$

In particular Coass (E) = Att (E).

PROOF : This follows from 2.8 and[1], [2.1]. □

So far, it is not known, if for every module M, even over a Gorenstein ring, there exists a non-zero linear map $E \to M$, with E an injective R-module[13]. In[10], it is shown that if an R-module M has non-zero injective cover then $\mathrm{Ass}\, R \cap \mathrm{Coass}(M) \neq \phi$ and converse is true if M is Artinian. Now using the theory of dual associated primes, we are able to obtain a necessary and sufficient condition for a cotorsion module M to have a non-zero injective cover.

Theorem 2.10 — *Let R be a ring such that $E(R)$ if flat. Then for a cotorsion R-module M, the following are equivalent:*

(1) There exists a non-zero linear map $E \to M$, with E injective.

(2) $\mathrm{Coass}(M) \cap \mathrm{Ass}_R(R) \neq \phi$.

PROOF : (1) $\Rightarrow$ (2). Let E be an injective module such that $(\mathrm{Hom}_R(M, E)) \neq 0$. By[4] [Theorem 3] $F(E)$, the flat cover of E, is injective. Hence it can be written uniquely as a direct sum of some $E(R/p)$. Since $F(E)$ is flat, for all p that $E(R/p)$ appears in $F(E)$, $\mathrm{ht}(p) = 0$. This implies that $p \in \mathrm{Ass}_R(R)$ and $E(R/p) \cong R_p$. Therefore, we can write $F(E) = \oplus_{p \in \mathrm{Ass}_R(R)} R_p^{(X_p)}$. Since $F(E)$ is injective, it is a direct summand of $\prod_{p \in \mathrm{Ass}_R(R)} R_p^{(X_p)}$. Now, since $\mathrm{Hom}_R(E, M)) \neq 0$, it concludes that $\mathrm{Hom}_R(F(E), M) \neq 0$, which in turn implies $\mathrm{Hom}_R\left(\prod_{p \in \mathrm{Ass}_R(R)} R_p^{(X_p)}, M\right) \neq 0$. So there exists $p \in \mathrm{Ass}_R(R)$ such that $\mathrm{Hom}_R(R_p, M) \neq 0$. Therefore, it follows from the fact $\mathrm{ht}(p) = 0$, that $p \in \mathrm{MinCos}_R(M)$, which implies the result by 2.4.

(2) $\Rightarrow$ (1) : Let $p \in \mathrm{Coass}(M) \cap \mathrm{Ass}_R(R)$. Since $p \in \mathrm{Coass}(M)$, $\mathrm{Hom}_R(R_p, M) \neq 0$. But since $p \in \mathrm{Ass}(R)$, $R_p \cong E(R/p)$ is injective. This completes the proof. □

Corollary 2.11 — Let R be a ring such that $E(R)$ is flat. Then every cotorsion R-module M has a non-zero injective cover if and only if $\mathrm{Coass}\,(M) \cap \mathrm{Ass}_R\,(R) \neq \phi$.

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