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STABILITY ON MULTIOBJECTIVE QUADRATIC PROGRAMMING PROBLEMS WITH FUZZY PARAMETERS

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This paper deals with multiobjective quadratic programming problems with fuzzy parametes (FMOQP^s). For such problems, some stability notions such as the solvability set and the stability set of the first kind (SSK 1) are defined and characterized. An algorithm is proposed for the determination of the set (SSK 1) for the fuzzy multiobjective quadratic problem after introducing the so called α-level set of the fuzzy numbers. A numerical example is included to clarify the developed theory and the proposed algorithm.

Key Words: Nonlinear Programming; Fuzzy Multiobjective; Quadratic Programming; α-Pareto Optimality; Stability

1. INTRODUCTION

In literature there are many papers that deal with the solution of fuzzy multiobjective optimization problems. Among the many algorithms suggested for these problems are those due to[4,6,8,9,10,11,12]. Sakawa and Yano[9] presented an interactive decision making method for multiobjective nonlinear programming problems with fuzzy parameters. These Fuzzy parameters were in the objective functions and in the constraints. In that work, the fuzzy parameters have been characterized by fuzzy numbers and the concept of α-Pareto optimality has been introduced.

In this paper, we present an algorithm for solving multiobjective quadratic programming problems with fuzzy parameters. The fuzzy parameters are in the objective functions and in the right-hand side of the constraints.

Following the methodology and concepts developed in[9], we shall discuss stability of the α-Pareto optimal solutions in the decision space for fuzzy multiobjective quadratic programming problems (FMOQP^S). The paper consists of six main sections : In Section 2, the multiobjective quadratic programming problem with fuzzy parameters (FMOQP) is formulated. Section 3 presents some basic stability notions for the (FMOQP). These notions are the solvability set and the stability set of the first kind (SSK 1). In Section 4, an algorithm is proposed for the determination of the set (SSK 1) to the fuzzy multiobjective quadratic programming problem after introducing the α-level set of the fuzzy numbers. In Section 5, an illustrative example is given to clarify the theory developed in this paper. Finally, Section 6 contains the concluding remarks.

2. PROBLEM FORMULATION AND SOLUTION CONCEPT

Consider the following multiobjective quadratic programming problem involving fuzzy parameters in the objective functions and in the right-hand side of the constraints. This problem will denoted by (FMOQP) and can be formulated mathematically as follows:

$$\text{(FMOQP):}\min f(x, \tilde{\lambda}, \tilde{\eta}) = \left\{ f_1(x, \tilde{\lambda}^1, \tilde{\eta}^1), f_2(x, \tilde{\lambda}^2, \tilde{\eta}^2), \dots, f_k(x, \tilde{\lambda}^K, \tilde{\eta}^K) \right\}$$

subject to

$$M(\tilde{\upsilon}) = \left\{ x \in R^n / g_r(x, \tilde{\upsilon}_r) = \sum_{j=1}^{n} a_{rj} x_j \le b_r + \tilde{\upsilon}_r, r = 1, 2, ..., m, \;\; x_j \ge 0, j = 1, 2, ..., n \right\}$$

where the kth objective function takes the form:

$$f_k(x, \tilde{\lambda}^k, \tilde{\eta}^k) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(C_{ij}^k + \tilde{\lambda}_{ij}^k \hat{C}_{ij}^k \right) x_i x_j$$

$$+ \sum_{j=1}^{n} \left(p_j^k + \hat{p}_j^k \tilde{\eta}_j^k \right) x_j, \; k = 1, 2, ..., K.$$

and the feasible region $M(\tilde{\upsilon})$ is assumed to be compact set i.e. closed and bounded for all fuzzy parameters $\tilde{\upsilon}_r$ whose membership function is $\mu_{\tilde{\upsilon}_r}$ $(r = 1, 2, ..., m)$ and will be defined later.

In the above problem (FMOQP), x is an n-dimensional vector of decision variables, $\tilde{\lambda}^k$ and $\tilde{\eta}^k$, $(k = 1, 2, ..., K)$ are $n \times n$ Symmetric matrices and n-dimensional vector of fuzzy parameters involved in the objective function $f_k(x, \tilde{\lambda}^k, \tilde{\eta}^k)$, respectively and $a_{rj}, b_r, C_{ij}^k, \hat{C}_{ij}^k\; p_j^k, \hat{p}_j^k$, $(r = 1, 2, ..., m)$, $(i, j = 1, 2, ..., n)$ are arbitrary real numbers, with $C_{ij}^k = C_{ji}^k$, $\hat{C}_{ij}^k = \hat{C}_{ji}^k$. Also it is supposed that $\tilde{\lambda}_{ij}^k = \tilde{\lambda}_{ji}^k$.

In addition, $\tilde{v}_r$ $(r = 1, 2, ..., m)$ are fuzzy parameters in the right-hand side of the constraints $g_r(x, \tilde{v}_r)$.

Now, it is appropriate to recall that a real fuzzy number $\tilde{P}$ is a continuous fuzzy subset of the real line R^1 whose membership function $\mu_{\tilde{P}}(p)$ is defined by:

(1) A continuous mapping from R^1 to the closed interval [0, 1],

(2) $\mu_{\tilde{P}}(p) = 0$ for all $p \in (-\infty, p_1]$,

(3) strictly increasing on $[p_1, p_2]$,

(4) $\mu_{\tilde{P}}(p) = 1$ for $p \in [p_2, p_3]$,

(5) strictly decreasing on $[p_3, p_4]$,

(6) $\mu_{\tilde{P}}(p) = 0$ for all $p \in [p_4, +\infty)$.

Fig. 1 illustrates a possible shape of the fuzzy number $\tilde{P}$.

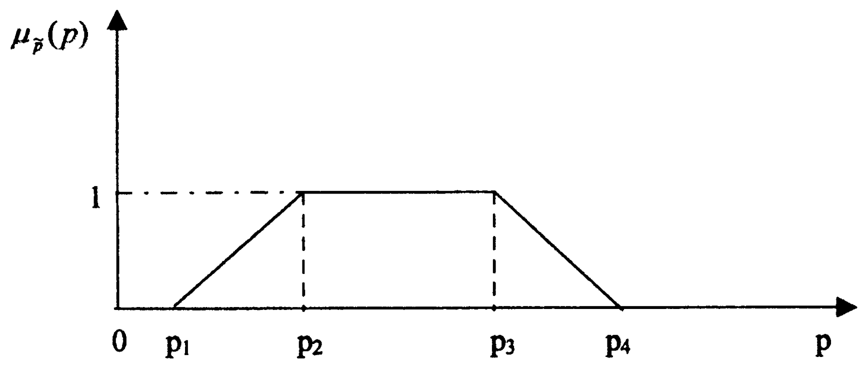

FIG. 1. Membership function of a fuzzy number $\tilde{P}$

It is assumed that $\tilde{\lambda}_{ij}^{k}$, $(i, j = 1, 2, ..., n)$, $\tilde{\eta}_{j}^{k}$, $(j = 1, 2, ..., n)$, $(k = 1, 2, ..., K)$ and $\mu_{\tilde{\upsilon}_r}$ $(r = 1, 2, ..., m)$ in the (FMOQP) are fuzzy numbers whose membership functions are $\mu_{\tilde{\lambda}_{ij}^k}(\lambda_{ij}^{k})$, $\mu_{\tilde{\eta}_j^k}(\eta_{j}^{k})$ and $\mu_{\tilde{\upsilon}_r}(\upsilon_r)$, respectively.

For the simplicity in the notions, let us define :

$$\lambda^k = \begin{bmatrix} \lambda_{11}^k & \lambda_{12}^k \cdots & \lambda_{1n}^k \\ \lambda_{21}^k & \lambda_{22}^k \cdots & \lambda_{2n}^k \\ . & & \\ . & & \\ \lambda_{n1}^k & \lambda_{n2}^k \cdots & \lambda_{nn}^k \end{bmatrix}, \qquad \tilde{\lambda}^k = \begin{bmatrix} \tilde{\lambda}_{11}^k & \tilde{\lambda}_{12}^k \cdots & \tilde{\lambda}_{1n}^k \\ \tilde{\lambda}_{21}^k & \tilde{\lambda}_{22}^k \cdots & \tilde{\lambda}_{2n}^k \\ . & & \\ . & & \\ \tilde{\lambda}_{n1}^k & \tilde{\lambda}_{n2}^k \cdots & \tilde{\lambda}_{nn}^k \end{bmatrix}$$

$$\lambda = (\lambda^1, \lambda^2, ..., \lambda^k) \quad , \quad \tilde{\lambda} = (\tilde{\lambda}^1 \tilde{\lambda}^2, ..., \tilde{\lambda}^K)$$

$$\eta^k = \left(\eta_1^k, \eta_2^k, ..., \eta_n^k\right) \quad , \quad \tilde{\eta}^k = \left(\tilde{\eta}_1^k, \tilde{\eta}_2^k, ..., \tilde{\eta}_n^k\right)$$

$$\eta = (\eta^1, \eta^2, ..., \eta^K) \quad , \quad \tilde{\eta} = (\tilde{\eta}^1, \tilde{\eta}^2, ..., \tilde{\eta}^K)$$

$$\upsilon = (\upsilon_1, \upsilon_2, ..., \upsilon_m) \qquad \tilde{\upsilon} = (\tilde{\upsilon}_1, \tilde{\upsilon}_2, ..., \tilde{\upsilon}_m)$$

In what follows, we give the definition of α-level set or α-cut of the fuzzy numbers $\tilde{\lambda}_{ij}^{k}, \tilde{\eta}_{ij}^{k}$ $(i, j = 1, 2, ..., n)$, $(k = 1, 2, ..., K)$ and $\tilde{\upsilon}_r$ $(r = 1, 2, ..., m)$, (see[9]).

Definition 1 — The α-level set of the fuzzy numbers $\tilde{\lambda}_{ij}^{k}, \tilde{\eta}_{ij}^{k}$ amd $\tilde{\upsilon}_r$ is defined as the ordinary set $L_\alpha(\tilde{\lambda}, \tilde{\eta}, \tilde{\upsilon})$ for which the degree of their membership functions exceed the level $\alpha \in [0, 1]$:

$$L_\alpha(\tilde{\lambda}, \tilde{\eta}, \tilde{\upsilon}) = \Big\{ (\lambda, \eta, \upsilon) / \mu_{\tilde{\lambda}^k_{ij}}(\lambda^k_{ij}) \geq \alpha, \ (i, j = 1, 2, ..., n),$$

$$\mu_{\tilde{\eta}^k_j}(\eta^k_j) \geq \alpha, \ (j = 1, 2, \dots n), \ (k = 1, 2, ..., K),$$

$$\{ \mu_{\tilde{\upsilon}_r}(\upsilon_r) \geq \alpha \quad , (r = 1, 2, ..., m) \}.$$

It is clear that the level set $L_\alpha(\tilde{\lambda}, \tilde{\eta}, \tilde{\upsilon})$ defined above is a convex set[13].

It should be noted that the fuzzy vector quantity $F(x, \tilde{\lambda}, \tilde{\eta})$ is minimized subject to the fuzzy constraint set $M(\tilde{\upsilon})$, where $\tilde{\lambda}$, $\tilde{\eta}$ and $\tilde{\upsilon}$ are fuzzy parameters.

For a certain degree α, the (FMOQP) can be understood as the following nonfuzzy α-multi objective problem:

$$(\alpha\text{-MOP}): \ \min \ F(x, \lambda, \eta) = \Big(f_1(x, \lambda^1, \eta^1), (f_2(x, \lambda^2, \eta^2), ..., (f_K(x, \lambda^K, \eta^K) \Big)$$

subject to

$$M(\upsilon) = \Big\{ x \in R^n / g_r(x, \upsilon) = \sum_{j=1}^{n} a_{rj} x_j \leq b_r + \upsilon_r, \ (r = 1, 2, ..., m),$$

$$x_j \geq 0, \ j = 1, 2, ..., n \ \},$$

$$(\lambda, \eta, \upsilon) \in L_\alpha(\tilde{\lambda}, \tilde{\eta}, \tilde{\upsilon}).$$

Notice that the parameters λ, η, υ are treated as decision variables rather than constants in the α-MOP and this problem is the deterministic version corresponding to problem (FMOQP) together with

α-level set $L_\alpha(\tilde{\lambda}, \tilde{\eta}, \tilde{\upsilon})$.

Based on the definition of the α-level set of the fuzzy numbers, we introduce the concept of α-Pareto optimal solution to the (α-MOP) in the following definition.

Definition 2 — A point $x^* \in M(\upsilon)$ is said to be an α-Pareto optimal to the (α-MOP)[5], if and only if there does not exist another $x \in M(\upsilon)$, $(\lambda, \eta, \upsilon) \in L_\alpha(\tilde{\lambda}, \tilde{\eta}, \tilde{\upsilon})$ such that $f_k(x, \lambda^k, \eta^k) \geq (f_k(x, \lambda^{*k}, \eta^{*k})$, $k = 1, 2, ..., K$, with strict inequality holding for at least one k, where $(\lambda^*, \eta^*, \upsilon^*)$ are called the α-level optimal parameters.

Now, problem (α-MOP) will be treated using the weighting method[5], therefore we have the following single-objective quadratic programming problem:

$$p(\omega): \min \sum_{k=1}^{K} \omega_k \left(\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \left(C_{ij}^{k} + \lambda_{ij}^{k} \hat{C}_{ij}^{k} \right) x_i x_j + \sum_{j=1}^{n} \left(p_j^k + \hat{p}_j^k \eta_j^k \right) x_j \right)$$

subject to

$x \in M(\upsilon)$,

$(\lambda, \eta, \upsilon) \in L_\alpha(\tilde{\lambda}, \tilde{\eta}, \tilde{\upsilon})$

where

$$\omega_k \geq 0, (k = 1, 2, ..., K), \omega = (\omega_1, \omega_2, ..., \omega_K) \neq 0.$$

In what follows, we have four different cases:

Case I : When $\lambda^k = 0$ and $\eta^k = 0$ for all $k = 1, 2, ..., K$.

In this case, problem $p(\omega)$ becomes a parametric quadratic programming problem and it takes the following form:

$$p_{Q_1}(\omega): \min \sum_{k=1}^{K} \omega_k \left(\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} C_{ij}^{k} x_i x_j + \sum_{j=1}^{n} p_j^k x_j \right)$$

subject to

$x \in M(\upsilon)$,

$\upsilon \in L_\alpha(\tilde{\upsilon})$.

where $\omega_k \geq 0, (k = 1, 2, ..., K), \omega = (\omega_1, \omega_2, ..., \omega_K) \neq 0$.

Case II : When $\lambda^k = 0$ and $\eta^k \geq 0$ $(k = 1, 2, ..., K)$, $\eta \neq 0$.

In this case, problem $p(\omega)$ becomes a parametric quadratic programming problem and it takes the following form:

$$p_{Q_{II}}(\omega): \min \sum_{k=1}^{K} \omega_k \left(\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} C_{ij}^{k} x_i x_j + \sum_{j=1}^{n} \left(p_j^k + \hat{p}_j^k \eta_j^k \right) x_j \right)$$

subject to

$x \in M(\upsilon)$,

$(\eta, \upsilon) \in L_\alpha(\tilde{\eta}, \tilde{\upsilon})$.

where

$$\omega_k \geq 0, (k = 1, 2, ..., K), \omega = (\omega_1, \omega_2, ..., \omega_K) \neq 0.$$

Case III : When $\lambda^k \geq 0$ and $\eta^k = 0$ $(k = 1, 2, ..., K)$, $\lambda \neq 0$.

In this case, problem $p(\omega)$ becomes a parametric nonlinear programming problem and it can be written as

$$p_{\mathrm{NL}_1}(\omega): \min \sum_{k=1}^{K} \omega_k \left(\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \left(C_{ij}^{k} + \lambda_{ij}^{k} \hat{C}_{ij}^{k} \right) x_i x_j + \sum_{j=1}^{n} p_j^k x_j \right)$$

subject to

$x \in M(\upsilon)$,

$(\lambda, \upsilon) \in L_\alpha(\tilde{\lambda}, \tilde{\upsilon})$.

where

$$\omega_k \geq 0, (k = 1, 2, \ldots, K), \omega = (\omega_1, \omega_2, \ldots, \omega_K) \neq 0.$$

Case IV : When $\lambda^k \geq 0$ and $\eta^k \geq 0$ $(k = 1, 2, \ldots, K)$, $\lambda \neq 0$.

In this case, problem $p(\omega)$ becomes a parametric nonlinear programming problem and it can be written as

$$p_{\mathrm{NL}_{\mathrm{II}}}(\omega): \min \sum_{k=1}^{K} \omega_k \left(\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \left(C_{ij}^{k} + \lambda_{ij}^{k} \hat{C}_{ij}^{k} \right) x_i x_j + \sum_{j=1}^{n} \left(p_j^k + \hat{p}_j^k \eta_j^k \right) x_j \right)$$

subject to

$x \in M(\upsilon)$,

$(\lambda, \eta, \upsilon) \in L_\alpha(\tilde{\lambda}, \tilde{\eta}, \tilde{\upsilon})$

where

$$\omega_k \geq 0, (k = 1, 2, \ldots, K), \omega = (\omega_1, \omega_2, \ldots, \omega_K) \neq 0.$$

Let

$$E(\omega) = \left\{ x^* \in R^n / \sum_{k=1}^{K} \omega_k f_k(x^*, \lambda^{*k}, \eta^{*k}) = \max_{\substack{x \in M(\upsilon), \\ (\lambda, \eta, \upsilon) \in L_\alpha(\tilde{\lambda}, \tilde{\eta}, \tilde{\upsilon})}} \sum_{k=1}^{K} \omega_k f_k(x, \lambda, \eta) \right\}$$

It should be noted that x^* is an α-Pareto optimal solution for problem (α-MOP) with the corresponding α-level optimal parameters $(\lambda^*, \eta^*, \upsilon^*)$ if there exists $\omega^* \geq 0$ such that $(x^*, \lambda^*, \eta^*, \upsilon^*)$ solves $p(\omega^*)$ and either one of the following two conditions holds:

(*i*) $\omega_k^* > 0$ for all $k = 1, 2, \ldots, K$.

(*ii*) $(x^*, \lambda^*, \eta^*, \upsilon^*)$ is the unique minimizer of $p(\omega^*)$.

3. Basic Stability Notions in (FMOQP)

3.1. *The solvability set*

Definition 3 — The solvability set B of problem (α-MOP) is defined as :

$B = \{\omega \in R^{k}_{+} - \{0\} /$ there exists α-Pareto optimal solution x^* of problem

(α-MOP), $x^* \in E(\omega)\}$.

3.2. *The stability set of the first kind of problem (α-MOP)*

Definition 4 — Suppose that $\omega^* \in B$ with the corresponding α-Pareto optimal solution $E(\omega^*)$, then the stability set of the first kind $S(x^*)$ of problem (α-MOP) corresponding to x^* is defined as:

$$S(x^*) = \{\omega \in B / x^* \in E(\omega) \text{ is an } \alpha\text{-Pareto optimal solution of problem } (\alpha\text{-MOP})\}.$$

4. Determination of the Set $S(x^*)$

In what follows, an algorithm for the determination of the (SSK1) to problem (α-MOP) will be described in a finite number of steps. This algorithm is based mainly upon the concept of the membership function of the fuzzy parameters in the (FMOQP). The more general situation for problem (α-MOP) is Case IV and will be considered only throughout this section.

The (α-MOP) problem can be considered as a parametric nonlinear problem in the form of $p_{\text{NLII}}(\omega)$ which can be solved at certain $\omega = \omega^* \in B$ using any available nonlinear programming package, for example, GINO[2,7] or Super GINO[1].

Problem $p_{\text{NLII}}(\omega)$ can be reformulated as follows:

$$p_{\text{NLII}}(\omega) \ \min \sum_{k=1}^{K} \omega_k f_k(x, \lambda^k, \eta^k)$$

subject to

$$g_r(x, \upsilon_r) = \sum_{j=1}^{n} a_{rj} x_j \le b_r + \upsilon_r, \quad r = 1, 2, \ldots, m,$$

$$\mu_{\tilde{\lambda}^k_{ij}}\left(\lambda^k_{ij}\right) \ge \alpha, \ (i, j = 1, 2, \ldots, n), \ (k = 1, 2, \ldots, K),$$

$$\mu_{\tilde{\eta}^k_j}\left(\eta^k_j\right) \ge \alpha, \ (j = 1, 2, \ldots, n), \ (k = 1, 2, \ldots, K),$$

$$\mu_{\tilde{\upsilon}_r}(\upsilon_r) \ge \alpha, \ (r = 1, 2, \ldots, m),$$

$$x_j \ge 0, \ j = 1, 2, \ldots, n.$$

The Kuhn-Tucker necessary optimality conditions corresponding to the $p_{\mathrm{NLII}}(\omega)$ problem are:

$$\left\{\begin{array}{l}
\sum_{k=1}^{K} \omega_k \dfrac{\partial f_k}{\partial x_\alpha} + \sum_{r=1}^{m} \delta_r \dfrac{\partial g_r}{\partial x_\alpha} = 0, \qquad \alpha = 1, 2, \ldots, n, \\[2ex]
\sum_{k=1}^{K} \omega_k \dfrac{\partial f_k}{\partial \lambda_{st}^{q}} - \sum_{k=1}^{K} \sum_{i=1}^{n} \sum_{j=1}^{n} u_{ij}^{k} \dfrac{\partial \mu_{\tilde{\lambda}_{ij}}^{k} (\lambda_{ij}^{k})}{\partial \lambda_{st}^{q}} = 0, \; q = 1, 2, \ldots, K, \; s, t = 1, 2, \ldots, K \\[2ex]
\sum_{k=1}^{K} \omega_k \dfrac{\partial f_k}{\eta_\beta^{p}} - \sum_{k=1}^{K} \sum_{j=1}^{n} \rho_j^{k} \dfrac{\partial \mu_{\tilde{\eta}_j}{}^{k} (\eta_j^{k})}{\eta_\beta^{p}} = 0, \; p = 1, 2, \ldots, K, \; \beta = 1, 2, \ldots, n \\[2ex]
\sum_{r=1}^{m} \delta_r \dfrac{\partial g_r}{\partial \upsilon_r} - \sum_{r=1}^{m} \upsilon_r \dfrac{\partial \mu_{\tilde{\upsilon}_r} \tilde{\upsilon}_r}{\partial \upsilon_r} = 0, \; \gamma = 1, 2, \ldots, m
\end{array}\right\} \quad \ldots \text{(I)}$$

$$\left\{\begin{array}{ll}
g_r(x, \upsilon_r) \leq 0, & r = 1, 2, \ldots, m, \\
\mu_{\tilde{\lambda}_{ij}^{k}} (\lambda_{ij}^{k}) \geq \alpha, & k = 1, 2, \ldots, K, \; i, j = 1, 2, \ldots, n, \\
\mu_{\tilde{\eta}_j^{k}} (\eta_j^{k}) \geq \alpha, & k = 1, 2, \ldots, K, \; j = 1, 2, \ldots, n, \\
\mu_{\tilde{\upsilon}_r} (\upsilon_r) \geq \alpha, & r = 1, 2, \ldots, m, \\
\delta_r g_r(x, \upsilon_r) = 0, & r = 1, 2, \ldots, m
\end{array}\right\} \quad \ldots \text{(II)}$$

$$\left\{\begin{array}{ll}
u_{ij}^{k}\left[-\mu_{\tilde{\lambda}_{ij}^{k}}\left(\lambda_{ij}^{k} \right) + \alpha \right] = 0, & k = 1, 2, \ldots, K, \; i, j = 1, 2, \ldots, n, \\[2ex]
\rho_j^{k}\left[-\mu_{\tilde{\eta}_j^{k}}\left(\eta_j^{k} \right) + \alpha \right] = 0, & k = 1, 2, \ldots, K, \; i, j = 1, 2, \ldots, n, \\[2ex]
V_r\left[-\mu_{\tilde{\upsilon}_r}(\upsilon_r) + \alpha \right] = 0, & k = 1, 2, \ldots, K,
\end{array}\right\} \quad \ldots \text{(III)}$$

$$\delta_r, \mu_{ij}^{k}, \rho_j^{k}, V_r, x_j \geq 0, \; k = 1, 2, \ldots, K, \; r = 1, 2, \ldots, m, \; j, j = 1, 2, \ldots, n.$$

where δ_r, μ_{ij}^{k}, ρ_j^{k} and V_r are the Lagrange multipliers and all the above expressions of the Kuhn-

Tucker conditions are evaluated at $\left(x_{\alpha}^{*}, \lambda_{ij}^{k*}, \eta_{j}^{k*}, \upsilon_{r}^{*}\right)$.

Notice that the Kuhn-Tucker conditions consist of three sets of relations. The set of constraint I together with the set III represent a polytope T for which its vertices can be determined using any algorithm based on the simplex method, for example, Balinski's algorithm[3]. According to whether any of the Lagrange multipliers are zero or positive, the stability set of the first kind $S_{NL}(x^*)$ of the problem $p_{NLII}(\omega)$ will be determined.

SOLUTION ALGORITHM

The algorithm to determin set $S_{NL}(x^*)$ of problem $p_{NL\,II}(\omega)$ can be described as follows:

Step 0 : Set $\alpha^* = 0$.

Step 1 : Elicit a membership function for the fuzzy parameters in the (FMOQP).

Step 2 : Formulate the parametric nonlinear problem $p_{NL\,II}(\omega)$.

Step 3 : Choose certain $\omega = \omega^* \in B$, then solve $p_{NL\,II}(\omega)$ by using any available nonlinear programming package, for example GINO[2,7] or Super GINO[1], to get the optimal solution x^* of $p_{NL\,II}(\omega)$ with the α-cut optimal parameters $(\lambda^*, \eta^*, \upsilon^*)$.

Step 4 : Substitute with $x^*, \lambda^*, \eta^*, \upsilon^*$ in the Kahn-Tucker necessary Optimality condition and solve the resulting system using Balinski's algorithm[3].

Step 5 : Determine the set $S_{NL}(x^*)$ according to the values of the Lagrange multipliers.

Step 6 : Set $\alpha^* = (\alpha^* + step) \in [0, 1]$ and go to step 1.

Step 7 : Repeat the above procedure until the interval [0, 1] is fully exhausted. Then, stop.

5. A NUMERICAL EXAMPLE

In this section we provide a numerical example to demonstrate the theory developed in the paper. Our example is given in the more general form of the FMOQP, since it contains fuzzy parameters in the objective functions and in the constraints.

$$\text{(FMOQP)} : \min\ (x, \tilde{\lambda}, \tilde{\eta}) = \left(f_1(x, \tilde{\lambda}^1, \tilde{\eta}^1),\ f_2(x, \tilde{\lambda}^2, \tilde{\eta}^2)\right)$$

subject to

$$-x_1 + x_2 \le 1 + \tilde{\upsilon}_1,$$

$$-x_1 + 23x_2 \le 10 - \tilde{\upsilon}_2,$$

$$x_2 \le 4,$$

$$x_1, x_2 \ge 0,$$

where

$$f_1(x, \tilde{\lambda}^1, \tilde{\eta}^1) = (1 + \tilde{\lambda}^1_{11}) x_1^2 - x_1 + (3 + \tilde{\eta}^1_2) x_2,$$

$$f_2(x, \tilde{\lambda}^2, \tilde{\eta}^2) = (3 + \tilde{\lambda}^1_{22}) x_2^2 - (1 + \tilde{\eta}^2_1) x_1 + x_2.$$

Let the fuzzy parameters be characterized by the following fuzzy numbers:

$\tilde{\lambda}^1_{11} = (0, 1, 4, 6)$, $\tilde{\lambda}^2_{22} = (0, 2, 3, 5)$,

$\tilde{\eta}^1_2 = (0, 2, 3, 5)$, $\tilde{\eta}^2_1 = (0, 1, 4, 6)$,

$\tilde{\upsilon}_1 = (0, 1, 4, 6)$ and $\tilde{\upsilon}_2 = (0, 2, 3, 5)$.

Assume that the membership functions corresponding to the fuzzy numbers take the following form:

$$\mu_p(p) = \begin{cases} 0, & p \le p_1, \\ 1 - \left(\dfrac{p - p_2}{p_2 - p_1}\right)^2, & p_1 \le p \le p_2, \\ 1, & p_2 \le p \le p_3, \\ 1 - \left(\dfrac{p - p_3}{p_4 - p_3}\right)^2, & p_3 \le p \le p_4, \\ 0, & p \ge p_4. \end{cases}$$

Let $\alpha = 0.36$ then we get:

$$0.4 \le \lambda^1_{11} \le 5.6,$$

$$0.2 \le \lambda^2_{22} \le 4.6,$$

$$0.2 \le \eta^1_2 \le 4.6,$$

$$0.4 \le \eta^2_1 \le 5.6,$$

$$0.4 \le \upsilon_1 \le 5.6,$$

$$0.2 \le \upsilon_2 \le 4.6.$$

The non fuzzy α-multiobjective problem can be written as follows:

$$(\alpha\text{-MOP}): \quad \min \left\{ f_1\left(x, \lambda^1, \eta^1\right), f_2\left(x, \lambda^2, \eta^2\right) \right\}$$

subject to

$$\left\{\begin{array}{lcl} -x_1 + x_2 & \le & 1 + \upsilon_1, \\ -x_1 + 3x_2 & \le & 10 - \upsilon_2, \\ x_2 & \le & 4, \\ 0.4 \le \lambda_{11}^1 & \le & 5.6, \\ 0.2 \le \lambda_{22}^2 & \le & 4.6, \\ 0.2 \le \eta_2^1 & \le & 4.6, \\ 0.4 \le \eta_1^2 & \le & 5.6, \\ 0.4 \le \upsilon_1 & \le & 5.6, \\ 0.3 \le \upsilon_2 & \le & 4.6, \\ x_1, x_2 \ge 0. & & \end{array}\right\} \quad \ldots (*)$$

where

$$f_1\left(x, \lambda^1, \eta^1\right) = \left(1 + \tilde{\lambda}_{11}^1\right)x_1^2 - x_1 + \left(3 + \tilde{\eta}_2^1\right)x_2,$$

$$f_2\left(x, \lambda^2, \eta^2\right) = \left(3 + \tilde{\lambda}_{22}^2\right)x_2^2 - \left(3 + \tilde{\eta}_1^2\right) + x_2.$$

Using the weighting mehtod, then the (α-MOP) problem becomes:

$$p'_{NL\,II}(w)\text{: min } \left\{ \omega_1 \left\{ \left(1 + \lambda_{11}^1\right)x_1^2 - x_1 + \left(3 + \eta_2^1\right)x_2 \right\}\right.$$

$$\left. + \ \omega_2 \left\{ \left(3 + \lambda_{22}^2\right)x_2^2 - \left(1 + \eta_1^2\right)x_1 + x_2 \right\}\right.$$

subject to the set of constraints (*).

with $\omega_1 \ge 0$, $\omega_2 \ge 0$ and X and $\omega_1 + \omega_2 = 1$.

Now, for $\omega_1 = \omega_1^* = \frac{1}{2}$ and $\omega_2 = \omega_2^* = \frac{1}{2}$ so we have:

$$p'_{NL\,II}(\omega^*) \text{ min: } \left\{ \frac{1}{2}x_1^2 + \frac{3}{2}x_2^2 + \frac{1}{2}\lambda_{11}^1 x_1^2 + \frac{1}{2}\lambda_{22}^2 x_2^2 + \frac{1}{2}\eta_2^1 x_2 - \frac{1}{2}\eta_1^2 x_1 - x_1 + 2x_2 \right\}$$

subject to the set of constraints (*).

The Kuhn-Tucker necessary optimality conditions corresponding to the $p'_{NL\,II}(\omega^*)$ problem are:

$$\omega_1\left[2\left(1+\lambda_{11}^1\right)x_1-1\right]+\omega_2\left[-\left(1+\eta_1^2\right)\right]-\delta_1-\delta_2-\delta_4=0,$$

$$\omega_1\left[\left(3+\eta_2^1\right)\right]+\omega_2\left[2\left(3+\lambda_{22}^2\right)x_2+1\right]+\delta_1+3\delta_2+\delta_3-\delta_5=0,$$

$$\omega_1 x_1 + u_1 - u_2 = 0,$$

$$\omega_2 x_2^2 + u_3 - u_4 = 0,$$

$$\omega_1 x_2^2 + \rho_1 - \rho_2 = 0,$$

$$-\omega_2 x_1 + \rho_3 - \rho_4 = 0,$$

$$-\delta_1 + V_1 - V_2 = 0,$$

$$\delta_2 + V_3 - V_4 = 0,$$

$$-x_1 + x_2 - \upsilon_1 + \xi_1 = 1,$$

$$-x_1 + 3x_2 + \upsilon_2 + \xi_2 = 10,$$

$$x_2 + \xi_3 = 4,$$

$$\lambda_{11}^1 + \xi_4 = 5.6,$$

$$\lambda_{11}^1 - \psi_1 = 0.4,$$

$$\lambda_{22}^2 + \xi_5 = 4.6,$$

$$\lambda_{22}^2 - \psi_2 = 0.2,$$

$$\eta_2^1 + \xi_6 = 4.6,$$

$$\eta_1^2 - \psi_3 = 0.2,$$

$$\eta_1^2 + \xi_7 = 5.6,$$

$$\eta_1^2 - \psi_4 = 0.4,$$

$$\upsilon_1 + \xi_8 = 5.6,$$

$\upsilon_1 - \psi_5 = 0.4,$

$\upsilon_2 + \xi_9 = 4.6,$

$\upsilon_2 - \psi_6 = 0.2,$

$\delta_1 \xi_1 = 0,$

$\delta_2 \xi_2 = 0,$

$\delta_3 \xi_3 = 0,$

$\delta_4 x_1 = 0,$

$\delta_5 x_2 = 0,$

$\upsilon_1 \xi_4 = 0,$

$\upsilon_2 \psi_1 = 0,$

$\upsilon_3 \xi_5 = 0,$

$\upsilon_4 \psi_2 = 0,$

$\rho_1 \xi_6 = 0,$

$\rho_2 \psi_3 = 0,$

$\rho_3 \xi_7 = 0,$

$\rho_4 \psi_4 = 0,$

$V_1 \xi_8 = 0,$

$V_2 \psi_5 = 0,$

$V_3 \xi_9 = 0,$

$V_4 \psi_6 = 0,$

all variables ≥ 0.

The α-Pareto optimal solution : $\left(\overset{*}{x}_1, \overset{*}{x}_2\right)$ = (2.714286,0) together with the α-level optimal parameters $\left(\overset{*1}{\lambda}_{11}, \overset{*2}{\lambda}_{22}, \overset{*1}{\eta_2}, \overset{2*}{\eta_1}, \overset{*}{\upsilon}_1, \overset{*}{\upsilon}_2\right)$ = (0.4, 1.149955, 1.186390, 5.6, 1.308642, 1.320988) and the optimal objective value of problem $p'_{NL_{II}}(\omega^*)$ equals –5.517143.

From the Kuhn-Tucker necessary optimality conditions to problem $p'_{NL_{II}}(\omega)$ we get:

$$6.6000008\ \omega_1 - 6.6\omega_2 = 0,$$

$$4.186390\ \omega_1 + \omega_2 > 0\,,$$

$$\omega_1 + \omega_2 = 1,$$

$$\omega_1 > 0,\ \ \omega_2 > 0.$$

Therefore, the stability set of the first kind of problem (α-MONLP) is given by:

$$S(2.714286,0) = \{\omega \in R^2 / 6.6000008\ \omega_1 - 6.6\omega_2 = 0,$$

$$4.186390\ \omega_1 + \omega_2 > 0\,,$$

$$\omega_1 + \omega_2 = 1,$$

$$\omega_1 > 0,\ \ \omega_2 > 0\}.$$

6. Concluding Remarks

In this paper, we have suggested a procedure for solving fuzzy multiobjective quadratic programming problems (FMOQP). Some basic stability notions have been characterized for FMOQP. An algorithm in finite steps have been proposed to determine the set SSK1) for the FMOQP. Also a numerical example has been given to clarify the developed theory and the proposed algorithm. In our opinion, many aspects and general questions remain to be studied and explored in the area of fuzzy multiobjective quadratic programming problems. There are, however, several particular unsolved problems that should be studied in the future. Some of these problems are:

(*a*) An algorithm is needed for solving large-scale fuzzy multiobjective quadratic programming problems.

(*b*) An algorithm is needed for solving stochastic multiobjective quadratic programming problems.

(*c*) A study should be carried out on the interactive approaches for treating stochastic and fuzzy multiobjective quadratic programming problems.

(*d*) A comparison study between stochastic and fuzzy multiobjective quadratic programming problems should be carried out.

(*e*) Computer codes are needed for solving large-scale stochastic and fuzzy multiobjective quadratic programming problems.

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