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ON GENERALIZED DIFFERENCE PARANORMED STATISTICALLY CONVERGENT SEQUENCES

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Generalized difference paranormed sequence spaces $\bar{c}(\Delta^n, p, q)$, $\bar{c}_0(\Delta^n, p, q)$, $c(\Delta^n, p, q)$, $c_0(\Delta^n, p, q)$, $m(\Delta^n, p, q)$, $m_0(\Delta^n, p, q)$ and $l_\infty(\Delta^n, p, q)$ defined over a seminormed sequence spaces (X, q) are introduced and their different properties studied. Some inclusion results are proved. Some errors appearing in the article of Tripathy and Sen[20] are discussed in detail.

Key Words: Seminorm; Paranorm; Statistical Convergence; Difference Sequence Space

1. INTRODUCTION

Throughout in this paper $w(X), c(X), c_0(X), \bar{c}(X), \bar{c}_0(X), l_\infty(X)$ represent spaces of all, convergent, null, statistically convergent, statistically null and bounded X valued sequence spaces, where (X, q) is a seminormed space, seminormed by q. For $X = C$, the space of complex numbers, these represent the corresponding scalar valued sequence spaces.

The notion of statistical convergence of sequences was introduced by Fast[5] and Schoenberg[15] independently. It is also found in Zygmund[21]. Later on it was studied from sequence space point of view and linked with summability theory by Fridy[6], Connor[1], Šalàt[14], Maddox[10], Rath and Tripathy[13], Tripathy[17,18], Tripathy and Sen[20] and many others.

The notion depends on the density of subsets of the set N of natural numbers. A subset E of N is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ exists,}$$

where χ_E is the characteristic function of E. Clearly all finite subsets of N have density zero and $\delta(E^c) = \delta(N - E) = 1 - \delta(E)$.

A sequence (x_n) is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{k \in N : q(x_k - L) \geq \varepsilon\}) = 0$. We write $x_k \xrightarrow{stat} L$ or stat-lim $x_k = L$.

The notion of paranormed sequence space was introduced by Nakano[11] and Simons[16]. Later on it was further investigated by Maddox[9], Lascarides[8], Rath and Tripathy[12,13], Tripathy and Sen[20] and many others.

The notion of difference sequence spaces was introduced by Kizmaz[7]. It was generalized by Et and Colak[3] as folows:

Let n be a non-negative integer, then

$$X(\Delta^n) = \{(x_k) \in w : (\Delta^n x_k) \in X\},$$

where $X = c, c_0$ and l_∞ and $\Delta^n x_k = \Delta^{n-1} x_k - \Delta^{n-1} x_{k+1}$, $\Delta^0 x_k = x_k$ for all $k \in N$. Further we have

$$\Delta^n x_k = \sum_{v=0}^{n} (-1)^v \binom{m}{v} x_{k+v}. \qquad \dots (1)$$

Later on the generalized difference sequence space was investigated by Et and Nuray[4], Tripathy, Altin and Et[19] and others.

2. Definitions and Background

A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ amd for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in N$.

A sequence space E is said to be symmetric if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$, where $\pi(k)$ is a permutation of N.

A sequence space E is said to be monotone if it contains the canonical pre-images of its step spaces.

We introduce the following definitions in this article. Let $p = (p_k)$ be a sequence of strictly positive real numbers. Then

$$\bar{c}(\Delta^n, p, q) = \left\{(x_k) \in w(X) : \left[q\left(\Delta^n x_k - L\right)\right]^{p_k} \xrightarrow{stat} 0, \text{ for some } L \in X\right\}$$

$$\bar{c}_0(\Delta^n, p, q) = \left\{(x_k) \in w(X) : \left[q\left(\Delta^n x_k\right)\right]^{p_k} \xrightarrow{stat} 0\right\}$$

$$c(\Delta^n, p, q) = \left\{(x_k) \in w(X) : \left[q\left(\Delta^n x_k - L\right)\right]^{p_k} \to 0, \text{ for some } L \in X\right\}$$

$$c_0(\Delta^n, p, q) = \left\{(x_k) \in w(X) : \left[q\left(\Delta^n x_k\right)\right]^{p_k} \to 0\right\}$$

$$l_\infty(\Delta^n, p, q) = \left\{ (x_k) \in w(X) : \sup_k \left[q\left(\Delta^n x_k\right)\right]^{p_k} < \infty \right\}$$

we write

$$m(\Delta^n, p, q) = \bar{c}(\Delta^n, p, q) \cap l_\infty(\Delta^n, p, q)$$

and

$$m_0(\Delta^n, p, q) = \bar{c}_0(\Delta^n, p, q) \cap l_\infty(\Delta^n, p, q).$$

The following inequality will be used throughout this paper. Let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \le sup\, p_k = G, D = Max\,(1, 2^{G-1})$. Then for $a_k, b_k \in C$ for all $k \in N$, we have

$$\left| a_k + b_k \right|^{p_k} \le D\left\{ \left| a_k \right|^{p_k} + \left| b_k \right|^{p_k} \right\}.$$

Two sequences (x_k) and (y_k) are said to be equal for almost all k, written as $x_k = y_k$ for $a.a.\, k$ if $\delta(\{k \in N : x_k \ne y_k\}) = 0$.

For $p_k = 1$ for all $k \in N$ and $q(x) = |x|$, these spaces rduce to the $\bar{c}(\Delta^n)$, $\bar{c}_0(\Delta^n)$ introduced and studied by Et and Naury[4] and the spaces $c(\Delta^n)$, $c_0(\Delta^n)$ and $l_\infty(\Delta^n)$ introduced and studied by Et and Colak[3]. For $n = 0$ and $q(x) = |x|$, these spaces reduced to the spaces $\bar{c}(p), \bar{c}_0(p), m(p)$ and $m_0(p)$ introduced and studied by Tripathy and Sen[20].

MAIN RESULTS

First we establish some results, those will be used in establishing the results of this article.

Lemma 3.1 — (Tripathy and Sen[20], Theorem 3). For two sequences (p_k) and (t_k) we have $m_0(p) \supseteq m_0(t)$ if and only if $\lim\text{-}\inf_{k \in K} \frac{p_k}{t_k} > 0$, where $K \subseteq N$ such that $\delta(K) = 1$.

Lemma 3.2 — (Tripathy and Sen[20], Theorem 4). Let $h = inf\, p_k$, then the following are equivalent :

(*i*) $G < \infty$ and $h > 0$.

(*ii*) $m(p) = m$.

Lemma 3.3 — (Tripathy and Sen[20], Lemma). Let $K = \{n_1, n_2, n_3, \ldots\}$ be an infinite subset of N such that $\delta(K) = 0$. Let

$$T = \{(x_k) : x_k = 0 \text{ or } 1 \text{ for } k = n_i, i \in N \text{ and } x_k = 0, \text{ otherwise}\}.$$

Then T is uncountable.

***Theorem* 3.1** — *$\bar{c}(\Delta^n, p, q), \bar{c}_0(\Delta^n, p, q)\ c(\Delta^n, p, q), c_0(\Delta^n, p, q),\ l_\infty(\Delta^n, p, q),\ m(\Delta^n, p, q),$ $m_0(\Delta^n, p, q)$ are linear spaces.*

PROOF : Let $(x_k), (y_k) \in \bar{c}(\Delta^n, p, q)$ and α, β be scalars. Then

$$\left[q\left(\Delta^n x_k - L\right)\right]^{p_k} \xrightarrow{stat} 0$$

and

$$\left[q\left(\Delta^n y_k - J\right)\right]^{p_k} \xrightarrow{stat} 0.$$

Then we have

$$\left[q\left(\Delta^n(\alpha x_k + \beta y_k) - (\alpha L + \beta J)\right)\right]^{p_k}$$

$$\leq \left[q\left(\alpha(\Delta^n x_k - L) + \beta(\Delta^n y_k - J)\right)\right]^{p_k}$$

$$\leq D \{ \max(1, \sup \alpha^{p_k}) \left[q\left(\Delta^n x_k - L\right)\right]^{p_k}$$

$$+ \max(1, \sup \beta^{p_k} \left[q\left(\Delta^n y_k - J\right)\right]^{p_k} \xrightarrow{stat} 0.$$

Hence $\bar{c}(\Delta^n, p, q)$ is a linear space.

The rest of the cases follow similarly.

The proof of the following result is a routine work.

***Theorem* 3.2** — *The spaces $c(\Delta^n, p, q), c_0(\Delta^n, p, q), l_\infty(\Delta^n, p, q),\ m(\Delta^n, p, q),\ m_0(\Delta^n, p, q)$ are paranormed spaces, paranormed by*

$$g_\Delta(x) = \sum_{i=1}^{n} q(x_i) + \sup_k \left[q\left(\Delta^n x_k\right)\right]^{\frac{p_k}{M}}$$

where $M = (1, \sup p_k)$.

***Theorem* 3.3** — *Let (X, q) be a complete seminormed space, then the spaces $c(\Delta^n, p, q),$ $c_0(\Delta^n, p, q),\ l_\infty(\Delta^n, p, q),\ m(\Delta^n, p, q),\ m_0(\Delta^n, p, q)$ are complete.*

PROOF : We prove for $l_\infty(\Delta^n, p, q)$ and for the other spaces it follows on applying similar arguments. Let (x^s) be a Cauchy sequence in $l_\infty(\Delta^n, p, q)$. Then

$$g_\Delta(x^s - x^t) \to 0 \text{ as } s \text{ and } t \to \infty$$

$$\Rightarrow \text{for } \varepsilon > 0, \text{ there exists } n_0 > 0 \text{ such that } g_\Delta(x^s - x^t) < \varepsilon \text{ for all } s, t > n_0.$$

$$\Rightarrow \sum_{i=1}^{n} q\left(x_i^s - x_i^t\right) + \sup_k \left[q\left(\Delta^n\left(x_k^s - x_k^t\right)\right)\right]^{\frac{p_k}{M}} < \varepsilon \text{ for all } s, t > n_0.$$

$\Rightarrow \left(x_i^s\right)_{s=1}^{\infty}$ (for $i \le n$) and $\left(\Delta^n\left(x_i^s\right)\right)_{s=1}^{\infty}$ for all $k \in N$, are Cauchy sequences in X.

Since X is complete, so these sequences converge in X. Let

$$x_i^s \to x_i, \text{ as } s \to \infty \text{ for all } i \le n. \qquad \text{... (2)}$$

Let

$$\Delta^n\left(x_k^s\right) \to y_k \text{ for all } k \in N \text{ in } X. \qquad \text{... (3)}$$

Now using (1), (2) and (3) and the linearity of the space X we can show that $\left(x_{n+1}^s\right)_{s=1}^{\infty}$ converges in X, say to x_{n+1}. Proceeding in this way inductively one can show that $x_i^s \to x_i$, as $s \to \infty$ for all $k \in N$.

Thus, $\left(\Delta^n x_k^s\right)$ will converge to $\Delta^n x_k$ for all $k \in N$ in X. We have $(x^s - x)$ and $(x^s) \in l_\infty(\Delta^n, p, q)$. Hence by the linearity of $l_\infty(\Delta^n, p, q)$ it follows that $x = x^s - (x^s - x) \in l_\infty(\Delta^n, p, q)$. Hence $l_\infty(\Delta^n, p, q)$ is complete.

Since the inclusion relations $m(\Delta^n, p, q) \subset l_\infty(\Delta^n, p, q)$ and $m_0(\Delta^n, p, q) \subset l_\infty(\Delta^n, p, q)$ are strict, so we have the following result.

Theorem 3.4 — *The spaces $m(\Delta^n, p, q)$ and $m_0(\Delta^n, p, q)$ are nowhere dense subsets of $l_\infty(\Delta^n, p, q)$.*

The proof of the following result is obvious.

Theorem 3.5 — *We have for all* $0 \le i \le n$, $Z(\Delta^i, p, q) \subseteq Z(\Delta^n, p, q)$, *where* $Z = \bar{c}$, c, m, c_0, $\bar{c}_0$, m_0 *and* l_∞.

Theorem 3.6 — *The spaces* $Z(\Delta^n, p, q)$, *are not solid for* $n > 0$, *where* $Z = \bar{c}$, c, m, c_0, $\bar{c}_0$, m_0 *and* l_∞.

PROOF : Let $n = 1$, $p_k = 1$ for all $k \in N$ and $q(x) = |x|$. Then $(x_k) = (k) \in Z(\Delta)$ for $Z = \bar{c}, c, m$ and l_∞. Let $\alpha_k = (-1)^k$, then $(\alpha_k x_k) = \notin Z(\Delta)$. Under the above restrictions, $(x_k) = (1) \in Z(\Delta)$ for $Z = c_0, \bar{c}_0$ and m_0. Let $\alpha_k = (-1)^k$, then $(\alpha_k x_k) \notin Z(\Delta)$.

Remark : For $n = 0$, the spaces $Z(p, q)$ will be solid for $Z = c_0, \bar{c}_0$, m_0 and l_∞ and hence are monotone.

Theorem 3.7 — *The spaces* $Z(\Delta^n, p, q)$ *are not symmetric for* $n > 0$, *where* $Z = \bar{c}, c, m, \bar{c}_0, m_0$ *and* l_∞, *where as the space* $c_0(\Delta^n, p, q)$ *is symmetric for* $n \le 1$ *and not symmetric for* $n \ge 2$.

PROOF : The proof that the space $c_0(\Delta^n, p, q)$ is symmetric for $n = 0$ and 1 is a routine work. Let $n = 1$, $p_k = 1$ for all $k \in N$ and $q(x) = |x|$. Then $(x_k) = (k) \in Z(\Delta)$ for $Z = \bar{c}, c, m$ and l_∞. Let (y_k) be a rearrangement of (x_k), which is defined as follows:

$$(y_k) = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, \ldots\}$$

Then $(y_k) \notin Z(\Delta)$ for $Z = \bar{c}, c, m$ and l_∞.

From the above example, it follows that the space $c_0(\Delta^n, p, q)$ is not symmetric for $n \ge 2$.

Let $n = 1$, $p_k = 1$ for all $k \in N$ and $q(x) = |x|$. For the spaces $\bar{c}_0(\Delta)$ and $m_0(\Delta)$ consider the sequence (x_k) defined by $x_k = 2$, for $(2i-1)^2 \le k < (2i)^2$, $i \in N$ and $x_k = 5$ otherwise. Then $(x_k) \in \bar{c}_0(\Delta)$ and $m_0(\Delta)$. Consider (y_k) the rearrangement of (x_k) defined as follows:

$$(y_k) = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, \ldots\}.$$

Then $(y_k) \notin \bar{c}_0(\Delta)$ and $(y_k) \notin m_0(\Delta)$. Thus the spaces $Z(\Delta^n, p, q)$, where $Z = \bar{c}$, c, m, $\bar{c}_0$, m_0 and l_∞ are not symmetric in general.

Taking $y_k = q(\Delta^n x_k)$ for all $k \in N$, we have the following three results, which follow from Lemma 3.1 and Lemma 3.2.

Proposition 3..8 — For two sequences (p_k) and (t_k) we have $m_0(\Delta^n, p, q) \supseteq m_0(\Delta^n, t, q)$ if and only if $\lim\text{-}\inf_{k \in K} \frac{p_k}{t_k} > 0$, where $K \subseteq N$ such that $\delta(K) = 1$.

The following result is a consequence of the above result.

Corollary 3.9 — For two sequences (p_k) and (t_k) we have $m_0(\Delta^n, p, q) = m_0(\Delta^n, t, q)$ if and only if $\lim\text{-}\inf_{k \in K} \frac{p_k}{t_k} > 0$ and $\lim\text{-}\inf_{k \in K} \frac{t_k}{p_k} > 0$, where $K \subseteq N$ such that $\delta(K) = 1$.

Proposition 3.10 — Let $h = \inf p_k$, then the following are equivalent:

(*i*) $G < \infty$ and $h > 0$.

(*ii*) $m(\Delta^n, p, q) = m(\Delta^n, q)$.

The following result is obvious in view of Lemma 3.3.

Proposition 3.11 — The spaces $m(\Delta^n, p, q)$ and $m_0(\Delta^n, p. q)$ are not separable.

4. Errors in the Article of Tripathy and Sen[20]

For the sake of completeness of discussion, we procure the following results:

Lemma 4.1 (Fridy[6], Theorem 1) — The following statements are equivalent:

(*i*) (x_k) is a statistically convergent sequence

(*ii*) (x_k) is a statistically Cauchy sequence

(*iii*) (x_k) is a sequence for which there is a convergent sequence (y_k) such that $x_k = y_k$ for *a. a. k*.

Lemma 4.2 (Šalát[14], Lemma 1.1) — A sequence (x_k) statistically converges to L if and only if there exists such a set $K = \{k_1 < k_2 < k_3 < \ldots\} \subset N$ that $\delta(K) = 1$ and $\lim_{n \to \infty} x_{k_n} = L$.

Lemma 4.3 (Connor[1], Theorem 2.3) — If x is statistically convergent to L, then there is a convergent sequence y and a statistically null sequence z such that y is convergent to L, $x = y + z$ and $\delta(\{k \in N : z_k \neq 0\}) = 0$. Moreover if x is bounded then z is bounded and $\|z\|_\infty \leq \|x\|_\infty + |L|$.

Lemma 4.4 (Tripathy and Sen[20], Theorem 1) — Let $0 < \inf p_k \leq p_k \leq p_k < \infty$, then the following are equivalent:

(*a*) $(x_k) \in m(p)$

(*b*) $(x_k - L) \in m_0(p)$

(*c*) there exists $(y_k) \in c(p)$ such that $x_k = y_k$ for *a. a. k*.

(*d*) there exists a subset $K = \{k_1, k_2 \ldots\}$ of N such that $\delta(K) = 1$ and $\lim_{n \to \infty} \left| x_{k_n} - L \right|^{p_{k_n}} = 0$.

(*e*) there exists sequences (y_k) and (z_k) such that $x_k = y_k + z_k$ for all $k \in N$ and $(y_k) \in c(p)$, $(z_k) \in m_0(p)$.

The following are the errors:

Error 1 : The statement of the Lemma 4.4 is incorrect. In Lemma 4.1, Lemma 4.2 and Lemma 4.3, the boundedness of (x_k) is not taken, where as it is taken in Lemma 4.4. The implications $(a) \Rightarrow (c)$, $(a) \Rightarrow (d)$ and $(a) \Rightarrow (e)$ are true if $(x_k) \in m(p)$ will be taken in the statement of Lemma 4.4. The reverse implication $(e) \Rightarrow (a)$ will be true too, but the reverse implications $(c) \Rightarrow (a)$ and $(d) \Rightarrow (a)$ will be false if $(x_k) \in m(p)$ will be taken. The reason is that (x_k) may be unbounded. Further the absence of $l_\infty(p)$ will lighten the restrictions on (p_k) i.e. inf $p_k > 0$ is not necessary. Hence the statement of the theorem will be as follows:

Theorem 4.5 — *Let* $\sup p_k < \infty$, *then the following are equivalent:*

(a) $(x_k) \in \bar{c}(p)$.

(b) $(x_k - L) \in \bar{c}_0(p)$.

(c) there exists $(y_k) \in c(p)$ such that $x_k = y_k$ for a.a.k.

(d) there exists a subset $K = \{k_1, k_2 \ldots\}$ of N such that $\delta(K) = 1$ and $\lim_{n \to \infty} \left| x_{k_n} - L \right|^{p_{k_n}} = 0$.

(*e*) there exists sequences (y_k) and (z_k) such that $x_k = y_k + z_k$ for all $k \in N$ and $(y_k) \in c(p)$, $(z_k) \in \bar{c}_0(p)$.

Error 2 : It can be taken as a slip of pen. The middle two lines in the proof of the Theorem 1 of Tripathy and Sen[20] reads like this "Let $(x_k) \in m(p)$. Then there exists L such that $\left| x_k - L \right|^{p_k} \xrightarrow{stat} 0$. Let $z_k = \left| x_k - L \right|^{p_k}$, then $(z_k) \in m_0(p)$". The middle two lines should read like this "Let $(x_k) \in \bar{c}(p)$. Then there exists L such that $\left| x_k - L \right|^{p_k} \xrightarrow{stat} 0$. Let $w_k = \left| x_k - L \right|^{p_k}$, then $(w_k) \in \bar{c}_0$.

Since z_k appears in the statement (*e*) of the theorem, so it should not be repeated. Further the statements of the Theorem is changed, so we have the following error.

Error 3 : In the Remark, the justification for inf $p_k > 0$ should not be given. The reason is that the Theorem is now free from $l_\infty(p)$.

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