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ON NONDIFFERENTIABLE SECOND ORDER SYMMETRIC DUALITY IN MATHEMATICAL PROGRAMMING

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Usual duality results are established for a pair of Wolfe type non-differentiable second order symmetric dual nonlinear programs. Then these are used to investigate symmetric duality for minimax version of nondifferentiable second order symmetric dual models wherein some of the primal and dual variables are constrained to belong to some arbitrary sets, e.g., the sets of integers. Also self duality for this pair has been discussed.

Key Words: Nonlinear Programming; Symmetric Duality; η-Bonvexity; Minimax; Mixed Integer Programming

1. INTRODUCTION

In mathematical programming, a pair of primal and dual problems is called symmetric if the dual of the dual is the primal problem; that is, if the dual problem is expressed in the form of the primal problem, then its dual is the primal problem. However, the majority of dual formulations in nonlinear programming do not possess this property. The first symmetric dual formulation for quadratic programming was proposed by Dorn[6]. Dantzig *et al.*[5] and Mond[13] formulated a pair of symmetric dual programs for a real valued function $K(x, y)$, that is convex in the first variable and concave in the second variable. Later, Chandra and Husain[4] studied a pair of Wolfe type nondifferentiable symmetric dual programs assuming convexity-concavity of the scalar function $K(x, y)$. Subsequently, Chandra *et al.*[3] weakened these assumptions to pseudoconvexity-pseudoconcavity.

The study of second order duality is significant due to the computational advantage over the first order duality as it provides tighter bounds for the value of the objective, function, when approximations are used (see[9,10,12]). Mangasarian[10] considered a nonlinear program and discussed second order duality using certain inequalities. Mond[12] introduced the concept of second order convex function, which was named as bonvex function by Bector and Chandra[2]. Later, Yang[16] discussed second order Mangasarian type duality under generalized representation conditions. Pandey[15] introduced the concept of η-convex functions as a generalization of bonvex functions. Mishra[11] also studied second order symmetric duality under second order F-convexity and second order F-pseudoconvexity for Wolfe and Mond-Weir type models, respectively.

The purpose of this paper is to study nondifferentiable second order symmetric duality under η-bonvexity for Wolfe type model. These duality results are then used to investigate nondifferentiable symmetric duality for minimax mixed integer programs. Self duality has also been discussed.

2. Notations and Prerequisites

Let R^n be the n-dimensional Euclidean space. Let $K(x, y)$ be a real valued thrice continuously differentiable function defined on an open set in $R^n \times R^m$. Let $K_x(\bar{x}, \bar{y})$ denote the gradient vector of K with respect to x at $(\bar{x}, \bar{y})$. Also let $K_{xx}(\bar{x}, \bar{y})$ denote the Hessian matrix with respect to x evaluated at $(\bar{x}, \bar{y})$. $K_y(\bar{x}, \bar{y})$, $K_{yx}(\bar{x}, \bar{y})$ and $K_{yy}(\bar{x}, \bar{y})$ are defined similarly.

Definition 1 — A real twice differentiable function defined on $X \times Y$, where X and Y are open sets in R^n and R^m respectively, is said to be η_1-bonvex in the first variable at $u \in X$, if there exists a function $\eta_1 : X \times X \to R^n$ such that for $v \in Y, r \in R^n, x \in X$,

$$K(x, v) - K(u, v) \geq \eta_1^t(x, u)\,[K_x(u, v) + K_{xx}(u, v)\, r]$$

$$-\frac{1}{2} r^t K_{xx}(u, v)\, r.$$

and $K(x, y)$ is said to be η_2-bonvex in the second variable at $v \in Y$, if there exists a function $\eta_2 : Y \times Y \to R^m$ such that for $u \in X, p \in R^m, y \in Y$,

$$K(u, y) - K(u, v) \geq \eta_2^t(y, v)\,[K_y(u, v) + K_{yy}(u, v)\, p]$$

$$-\frac{1}{2} p^t K_{yy}(u, v)\, p.$$

3. Wolfe Type Second Order Symmetric Duality

We now state the following pair of Wolfe type second order nondifferentiable symmetric programs and establish weak and strong duality theorems.

Primal (WP) :

$$\text{Minimize } F(x, y, p) = K(x, y) + (x^t Bx)^{\frac{1}{2}}$$

$$- y^t K_y(x, y) - y^t K_{yy}(x, y)\, p - \frac{1}{2} p^t K_{yy}(x, y)\, p$$

$$\text{subject to } K_y(x, y) - Cw + K_{yy}(x, y)\, p \leq 0 \quad \text{... (1)}$$

$$w^t Cw \leq 1 \quad \text{... (2)}$$

$$x \geq 0. \quad \text{... (3)}$$

Dual (WD) :

$$\text{Maximize } G(u, v, r) = K(u, v) - (v^t Cv)^{\frac{1}{2}}$$

$$- u^t K_x (u, v) - u^t K_{xx} (u, v)\, r - \frac{1}{2} r^t K_{xx} (u, v)\, r$$

subject to

$$K_x (u, v) + Bz + K_{xx} (u, v)\, r \geq 0 \qquad \text{... (4)}$$

$$z^t Bz \leq 1 \qquad \text{... (5)}$$

$$v \geq 0, \qquad \text{... (6)}$$

where B and C are positive semidefinite matrices of order n and m respectively.

Remark : Let $B = 0$, $C = 0$. Then (WP) and (WD) are reduced to symmetric dual pair of Mond[12]. If in addition, p and r are set to zero then (WP) and (WD) become the first order symmetric dual programs of Dantzing *et al.*[5].

Lemma 1 — (Generalized Schwartz inequality) — Let A be a positive semidefinite symmetric matrix of order n. Then for all $x, z \in R^n$,

$$x^t Az \leq (x^t Ax)^{\frac{1}{2}} (z^t Az)^{\frac{1}{2}}.$$

Theorem 1 *(Weak duality) — Let (x, y, w, p) be feasible for (WP) and (u, v, z, r) be feasible for (WD). Let*

(i) $K(x, y) + x^t Bz$ *be* η_1*-bonvex in the first variable at u,*

(ii) $K(x, y) - y^t Cw$ *be* η_2*-bonvex in the second variable at y,*

(iii) $\eta_1(x, u) + u \geq 0$ and $\eta_2(v, y) + y \geq 0$.

Then

$$\inf \text{ (WP)} \quad \geq \quad \sup \text{ (WD)}.$$

PROOF : By the hypotheses (*i*) and (*ii*),

$$K(x, v) + x^t Bz - K(u, v) - u^t Bz \geq \eta_1^t (x, u)$$

$$[K_x (u, v) + Bz + K_{xx} (u, v)\, r] - \frac{1}{2} r^t K_{xx} (u, v)\, r,$$

and

$$K(x, v) - v^t Cw - K(x, y) + y^t Cw \leq \eta_2^t (v, y)$$

$$[K_y (x, y) - Cw + K_{yy} (x, y)\, p] - \frac{1}{2} p^t K_{yy} (x, y)\, p.$$

Adding these inequalities, we get

$$K(x, y) - K(u, v) - \frac{1}{2} p^t K_{yy}(x, y) p$$

$$+ \frac{1}{2} r^t K_{xx}(u, + v) r + r^t Cw - y^t Cw + x^t Bz - u^t Bz$$

$$\geq \eta_1^t (x, u) [K_x(u, v) + Bz + K_{xx}(u, v) r]$$

$$- \eta_2^t (v, y) [K_y(x, y) - Cw + K_{yy}(x, y) p],$$

or

$$\left[K(x, y) + (x^t Bx)^{\frac{1}{2}} - y^t K_y(x, y) - y^t K_{yy}(x, y) p - \frac{1}{2} p^t K_{yy}(x, y) p \right]$$

$$- \left[K(u, v) - (v^t Cv)^{\frac{1}{2}} - u^t K_x(u, v) - u^t K_{xx}(u, v) r - \frac{1}{2} r^t K_{xx}(u, v) r \right]$$

$$\geq (\eta_1(x, u) + u)^t [K_x(u, v) + Bz + K_{xx}(u, v) r]$$

$$- (\eta_2(v, y) + y)^t [K_y(x, y) - Cw + K_{yy}(x, y) p] \quad \text{(by (2), (5) and Lemma 1)}$$

$$\geq 0 \quad \text{(using (1), (4) and hypotheses } (iii)),$$

and hence

$$\inf \text{(WP)} \geq \sup \text{(WD)}.$$

***Theorem* 2** *(Strong duality) — Let $K : R^n \times R^m \to R$ be thrice differentiable and let $(\bar{x}, \bar{y}, \bar{w}, \bar{p})$ be a local optimal solution for (WP). If*

(i) $K_{yy}(\bar{x}, \bar{y})$ is nonsingular; and

(ii) one of the matrices $\frac{\partial}{\partial y_i}(K_{yy})$, $i = 1, 2, \ldots, m$ is positive or negative definite, then there exists $\bar{z} \in R^n$ such that $\bar{p} = 0$, $(\bar{x}, \bar{y}, \bar{z}, \bar{r} = 0)$ is feasible for (WD) and

$$F(\bar{x}, \bar{y}, \bar{p}) = G(\bar{x}, \bar{y}, \bar{r}).$$

Also, if the hypotheses of Theorem 1 are satisfied for all feasible solutions of (WP) and (WD), then $(\bar{x}, \bar{y}, \bar{w}, \bar{p})$ and $(\bar{x}, \bar{y}, \bar{z}, \bar{r})$ are global optimal solutions for (WP) and (WD) respectively.

PROOF : Since $(\bar{x}, \bar{y}, \bar{w}, \bar{p})$ is a local optimal solution for (WP), by Fritz-John[14] optimality condiions, there exists $\alpha \in R, \beta \in R^m$, $\gamma \in R$ and $\delta \in R^n$ such that (for simplicity, we write K_x, K_{yx} instead of $K_x(\bar{x}, \bar{y})$, $K_{yx}(\bar{x}, \bar{y})$ etc.).

$$\alpha(K_x + B\bar{z}) + K_{yx}(\beta - \alpha\bar{y}) + (K_{yy}\bar{p})_x\left(\beta - \alpha\bar{y} - \frac{1}{2}\alpha\bar{p}\right) - \delta = 0 \quad \text{... (7)}$$

$$K_{yy}(\beta - \alpha\bar{y} - \alpha\bar{p}) + (K_{yy}\bar{p})_y\left(\beta - \alpha\bar{y} - \frac{1}{2}\alpha\bar{p}\right) = 0 \quad \text{... (8)}$$

$$K_{yy}(\beta - \alpha\bar{y} - \alpha\bar{p}) = 0 \quad \text{... (9)}$$

$$2\bar{w}\,\gamma C = \beta C \quad \text{... (10)}$$

$$\beta^t(K_y - C\bar{w} + K_{yy}\bar{p}) = 0 \quad \text{... (11)}$$

$$\gamma^t(\bar{w}^t C\bar{w} - 1) = 0 \quad \text{... (12)}$$

$$\bar{x}^t\delta = 0 \quad \text{... (13)}$$

$$\bar{z}^t B\bar{z} \le 1 \quad \text{... (14)}$$

$$(\bar{x}^t B\bar{x})^{\frac{1}{2}} = \bar{x}^t B\bar{z} \quad \text{... (15)}$$

$$(\alpha, \beta, \gamma, \delta) \geq 0, \quad \text{... (16)}$$

$$(\alpha, \beta, \gamma, \delta) \neq 0. \quad \text{... (17)}$$

Since K_{yy} is nonsingular, (9) yields

$$\beta = \alpha\bar{y} + \alpha\bar{p}. \quad \text{... (18)}$$

Suppose $\alpha = 0$. Then (18) implies $\beta = 0$. Therefore, eqs. (7) and (10) imply $\delta = 0$ and $\gamma = 0$ respectively. Thus $(\alpha, \beta, \gamma, \delta) = 0$, a contradiction to (17). Hence

$$\alpha > 0. \quad \text{... (19)}$$

Using (18) in (8), we get

$$\frac{1}{2}\alpha\bar{p}^t(K_{yy}\bar{p})_y = 0,$$

which by the hypothesis (*ii*) yields

$$\bar{p} = 0. \quad \text{... (20)}$$

Relations (18), (19) and (20) imply

$$\bar{y} = \frac{\beta}{\alpha} \geq 0. \qquad \text{... (21)}$$

From (7), (18), (19) and (20), we obtain

$$K_x + B\bar{z} = \frac{\delta}{\alpha} \geq 0. \qquad \text{... (22)}$$

Therefore, $(\bar{x}, \bar{y}, \bar{z}, \bar{r} = 0)$ is feasible for (WD).

Now let $\frac{2\gamma}{\alpha} = a$. Then $a \geq 0$ and from (10) and (21),

$$C\bar{y} = aC\bar{w},$$

which is the condition for equality in Lemma 1. Therefore,

$$\bar{y}^t C\bar{w} = (\bar{y}^t C\bar{y})^{\frac{1}{2}} (\bar{w}^t C\bar{w})^{\frac{1}{2}}. \qquad \text{... (23)}$$

Now from (12) either $\bar{w}^t C\bar{w} = 1$ or $\gamma = 0$ and hence $C\bar{y} = 0$. Therefore, in either case

$$\bar{y}^t C\bar{w} = (\bar{y}^t C\bar{y})^{\frac{1}{2}}. \qquad \text{... (24)}$$

To show the equality of the two objective functions, multiplying (7) by $\bar{x}$ and using (13), (18) and (20), we get

$$\bar{x}^t K_x + \bar{x}^t B\bar{z} = 0. \qquad \text{... (25)}$$

Hence

$$\begin{aligned} F(\bar{x}, \bar{y}, \bar{w}, \bar{p} = 0) &= K + (\bar{x}^t B\bar{x})^{\frac{1}{2}} - \bar{y}^t K_y \\ &= K + \bar{x}^t B\bar{z} - \bar{y}^t C\bar{w} \text{ (using (11), (15), (18) and (19))} \\ &= K - \bar{x}^t K_x - (\bar{y}^t C\bar{y})^{\frac{1}{2}} \text{ (using (24 and (25))} \\ &= G(\bar{x}, \bar{y}, \bar{z}, \bar{r} = 0). \end{aligned}$$

Also, by Theorem 1, $(\bar{x}, \bar{y}, \bar{w}, \bar{p})$ and $(\bar{x}, \bar{y}, \bar{z}, \bar{r})$ are global optimal solutions for (WP) and (WD) respectively.

4. MINIMAX MIXED INTEGER PROGRAMMING

Let U and V be two arbitrary sets of integers in R^n and R^m respectively. In the present section, we constrain some of the components of x and y to belong to arbitrary sets of integers as in Balas[1] and Gulati and Ahmad[7]. Suppose the first n_1 $(0 \leq n_1 \leq n)$ components of x belong to U and the first m_1 $(0 \leq m_1 \leq m)$ components of y belong to V. Then we write $[x, y] = \left[x^1, x^2, y^1, y^2\right]$, where

$x^1 = \left[x_1, x_2, ..., x_{n_1}\right]$ and $y^1 = \left[y_1, y_2, ..., y_{m_1}\right]$, x^2 and y^2 being the vector of the remaining components of x and y respectively.

For twice differentiable function K in the components of x^2 and y^2, let $K_{x^2}(\bar{x}, \bar{y})$ and $K_{y^2}(\bar{x}, \bar{y})$ be the partial derivatives of K in the components of x^2 and y^2 respectively, evaluated at $(\bar{x}, \bar{y})$. Also let $K_{x^2 x^2}(\bar{x}, \bar{y})$ denote the Hessian matrix with respect to x^2 evaluated $(\bar{x}, \bar{y})$. $K_{y^2}(\bar{x}, \bar{y})$, $K_{y^2 y^2}(\bar{x}, \bar{y})$ and $K_{x^2 y^2}(\bar{x}, \bar{y})$ are defined similarly.

In the sequel, we require the following notion of separability of a vector function (Balas[1] and Gulati and Ahmad[7]).

Definition 2 — Let $s^1, s^2, ..., s^r$ be elements of an arbitrary vector space. A vector function $H(s^1, s^2, ..., s^r)$ will be called additively separable with respect to s^1 if there exist vector functions $M(s^1)$ (independent of $s^2, s^3, ..., s^r$) and $N(s^2, s^3, ..., s^r)$ (independent of s^1) such that

$$H(s^1, s^2, ..., s^r) = M(s^1) + N(s^2, s^3, ..., s^r).$$

We consider the following pair of Wolfe type nondifferentiable minimax mixed integer symmetric programs.

Primal (SP):

$$\text{Max}_{x^1} \ \text{Min}_{x^2, y} \ K(x, y) + ((x^2) Bx^2)^{\frac{1}{2}} - (y^2)^t K_{y^2}(x, y) - (y^2)^t K_{y^2 y^2}(x, y)\, p - \frac{1}{2} p^t K_{y^2 y^2}(x, y)\, p$$

subject to

$$K_{y^2}(x, y) - Cw + K_{y^2 y^2}(\dot{x}, y)\, p \leq 0 \qquad ... (26)$$

$$w^t Cw \leq 1 \qquad ... (27)$$

$$x^2 \geq 0 \qquad ... (28)$$

$$x^1 \in U, y^1 \in V.$$

Dual (SD) :

$$\text{Min}_{v^1} \ \text{Max}_{u, v^2} \ K(u, v) - ((v^2)^t Cv^2)^{\frac{1}{2}} - (u^2)^t K_{x^2}(u, v) - (u^2)^t K_{x^2 x^2}(u, v)\, r - \frac{1}{2} r^t K_{x^2 x^2}(u, v)\, r$$

subject to

$$K_{x^2}(u, \upsilon) + Bz + K_{x^2 x^2}(u, \upsilon)\, r \geq 0 \qquad \dots (29)$$

$$z^t Bz \leq 1 \qquad \dots (30)$$

$$\upsilon^2 \geq 0 \qquad \dots (31)$$

$$u^1 \in U, \upsilon^1 \in V,$$

where p, w and r, z are $m - m_1$ and $n - n_1$ dimensional vector variables.

Remark : If in the problems (SP) and (SD), the second order terms are omitted, i.e., p and r are set to zero vectors, then they become the first order non-differentiable symmetric dual mixed integer programs of Gulati *et al.*[8]. If, in addition $B = 0$ and $C = 0$, then we get the symmetric dual mixed integer problems of Balas[1].

Theorem* 3 *(Symmetric duality) — Suppose that $(\bar{x}, \bar{y}, \bar{w}, \bar{p})$ is an optimal solution for (SP). Also let

(i) $K(x, y)$ be additively separable with respect to x^1 or y^1;

(ii) $K(x, y)$ be thrice differentiable in x^2 and y^2;

(iii) $K(x, y) + (x^2)^t B\bar{z}$ be η_1-bonvex in x^2 for each (x^1, y) and $K(x, y) - (y^2)^t Cw$ be η_2-boncave in y^2 for each (x, y^1);

(iv) $K_{y^2 y^2}(\bar{x}, \bar{y})$ is nonsingular;

(v) one of the matrices $\frac{\partial}{\partial y_i^2} K_{y y^2}(\bar{x}, \bar{y}), i = m_1 + 1, m_2 + 2, \dots, m$ be positive or negative definite;

(vi) $\eta_1(x^2, u^2) + u^2 \geq 0$, and

(vii) $\eta_2(\upsilon^2, y^2) + y^2 \geq 0$, for all $(x, y, w, p, u, \upsilon, z, r)$ feasible for (SP) and (SD).

Then there exists $\bar{z}$ such that $(\bar{x}, \bar{y}, \bar{z}, \bar{r} = 0)$ is optimal for (SD) and the two objectives are equal.

PROOF : Let

$$g = \mathrm{Max}_{x^1}\ \mathrm{Min}_{x^2, y}$$

$$\left[K(x, y) + ((x^2)^t Bx^2)^{\frac{1}{2}} - (y^2)^t K_{y^2}(x, y) - (y^2)^t K_{y^2 y^2}(x, y)\, p - \frac{1}{2} p^t K_{y^2 y^2}(x, y)\, p : (x, y, w, p) \in S \right]$$

and

$$h = \text{Min}_{v^1}\ \text{Max}_{u,v^2}$$

$$\Big[K(u, v) - ((v^2)^t C v^2)^{\frac{1}{2}} - (u^2)^t K_{x^2}(u, v) - (u^2)^t K_{x^2 x^2}(u, v)\, r$$

$$- \frac{1}{2} r^t K_{x^2 x^2}(u, v)\, r : (u, v, z, r) \in T \Big]$$

where S and T are feasible regions of (SP) and (SD) respectively.

As $K(x, y)$ is taken to be additively separable with respect to x^1 or y^1 (say with respect to x^1), it follows that

$$K(x, y) = K^1(x^1) + K^2(x^2, y).$$

Therefore $K_{y^2}(x, y) = \overset{2}{K}_{y^2}(x^2, y)$ and g can be written as

$$g = \text{Max}_{x^1}\ \text{Min}_{x^2, y} \Big[K^1(x^1) + K^2(x^2, y) + ((x^2)^t B x^2)^{\frac{1}{2}} - (y^2)^t \overset{2}{K}_{y^2}(x^2, y)$$

$$- (y^2)^t \overset{2}{K}_{y^2 y^2}(x^2, y) p - \frac{1}{2} p^t \overset{2}{K}_{y^2 y^2}(x^2, y)\, p :$$

$$\overset{2}{K}_{y^2}(x^2, y) - Cw + \overset{2}{K}_{y^2 y^2}(x^2, y)\, p \leq 0,$$

$$w^t C w \leq 1, x^2 \geq 0, x^1 \in U, y^1 \in V \Big]$$

$$= \text{Max}_{x^1}\ \text{Min}_{y^1}\ \text{Min}_{x^2, y^2} \Big[K^1(x^1) + K^2(x^2, y) + ((x^2)^t B x^2)^{\frac{1}{2}} - (y^2)^t \overset{2}{K}_{y^2}(x^2, y)$$

$$- (y^2)^t \overset{2}{K}_{y^2 y^2}(x^2, y) p - \frac{1}{2} p^t \overset{2}{K}_{y^2 y^2}(x^2, y)\, p :$$

$$\overset{2}{K}_{y^2}(x^2, y) - Cw + \overset{2}{K}_{y^2 y^2}(x^2, y)\, p \leq 0,$$

$$w^t C w \leq 1, x^2 \geq 0, x^1 \in U, y^1 \in V \Big].$$

Or

$$g = \text{Max}_{x^1}\ \text{Min}_{y^1} \Big[K^1(x^1) + \phi(y^1) : x^1 \in U, y^1 \in V \Big],$$

where

$$\phi(y^1) = \text{Min}_{x^2, y^2} \Big[K^2(x^2, y) + ((x^2)^t B x^2)^{\frac{1}{2}} - (y^2)^t \overset{2}{K}_{y^2}(x^2, y)$$

$$- (y^2)^t \overset{2}{K}_{y^2 y^2}(x^2, y) p - \frac{1}{2} p^t \overset{2}{K}_{y^2 y^2}(x^2, y)\, p \cdot$$

$$K^2_{y^2}(x^2, y) - Cw + K^2_{y^2 y^2}(x^2, y)\, p \leq 0,$$

$$\left. w^t Cw \leq 1, x^2 \geq 0, x^1 \in U, y^1 \in V \right]. \quad \text{... (32)}$$

Similarly h can be written as

$$h = \text{Min}_{v^1}\ \text{Max}_{u^1} \left[K^1(u^1) + \psi(v^1) : u^1 \in U, v^1 \in V \right],$$

where

$$\psi(v^1) = \text{Max}_{u^2, v^2} \Big[K^2(u^2, v) - ((v^2)^t Cv^2)^{\frac{1}{2}} - (u^2)^t K^2_{x^2}(u^2, v)$$

$$- (u^2)^t K^2_{x^2 x^2}(u^2, v)\, r - \frac{1}{2} r^t K^2_{x^2 x^2}(u^2, v)\, r :$$

$$K^2_{x^2}(u^2, v) + Bz + K^2_{x^2 x^2}(u^2, v)\, r \geq 0,$$

$$\left. z^t Bz \leq 1, v^2 \geq 0, u^1 \in U, v^1 \in V \right]. \quad \text{... (33)}$$

For any given y^1, (32) and (33) are a pair of Wolfe type nondifferentiable second order symmetric dual programs given in previous section and hence, in view of hypotheses assumed here, Theorem 1 and 2 of Section 3 become applicable. Therefore, for $y^1 = \bar{y}^1$, $\phi(\bar{y}^1) = \psi(\bar{y}^1)$. The proof of the remaining part of the theorem is same as that of Gulati and Ahmad[7].

5. SELF DUALITY

A mathematical programming problem is said to be self dual, if it is formally identical with its dual, that is, the dual can be recast in the form of the primal. If we assume

(*i*) K to be skew symmetric, that is, $K(x, y) = -K(y, x)$, and

(*ii*) $B = C$,

then we shall show that the programs (WP) and (WD) are self dual. By recasting the dual problem (WD) as maxmin problem, we have

$$\text{Max}_{v^1}\ \text{Min}_{u\ v^2}$$

$$-K(u, v) + ((v^2)\ C\, v^2)^{\frac{1}{2}} + (u^2)^t K_{x^2}(u, v) + (u^2)^t K_{x^2 x^2}(u, v)\, r$$

$$+ \frac{1}{2} r^t K_{x^2 x^2}(u, v)\, r$$

subject to

$$-K_{x^2}(u, v) - Bz - K_{x^2 x^2}(u, v)\, r \leq 0$$

$z^t Bz \leq 1$

$v^2 \geq 0$

$u^1 \in U, v^1 \in V.$

Since K is skew symmetric,

$$K_{x^2}(u,v) = -K_{y^2}(v, u), \quad K_{x^2 x^2}(u, v) = -K_{y^2 y^2}(v, u).$$

Also, since $B = C$, the above program becomes:

$\text{Max}_{v^1} \ \text{Min}_{u,\, v^2}$

$$K(u, v) + ((v^2)^t C v^2)^{\frac{1}{2}} + (u^2)^t K_{x^2}(u, v) + (u^2)^t K_{x^2 x^2}(u, v)\, r$$

$$+ \frac{1}{2} r^t K_{x^2 x^2}(u, v)\, r$$

subject to

$K_{y^2}(v, u) - Cz + K_{y^2 y^2}(v, u)\, r \leq 0$

$z^t Cz \leq 1$

$v^2 \geq 0$

$u^1 \in U, v^1 \in V,$

which is the primal problem (WP). Thus (WP) is a self dual.

We now state the following self duality theorem. Its proof follows on the lines of Theorem 3 in[7] and Theorem 3 in [8].

Theorem 4 *(Self duality) — Let* $K : R^n \times R^n \to R$ *be skew symmetric and* $B = C$. Then (WP) is a self dual. Furthermore, if (WP) and (WD) are dual programs and $(\bar{x}, \bar{y}, \bar{w}, \bar{p})$ is an optimal solution for (WP), then $(\bar{x}, \bar{y}, \bar{w}, \bar{p} = 0)$ and $(\bar{y}, \bar{x}, \bar{z}, \bar{r} = 0)$ are optimal solutions for (WP) and (WD) respectively, and each objective function value is equal to zero.

REFERENCES

1. E. Balas, *In: Integer and nonlinear programming*, (Editor, J Abadie, North-Holland, Amsterdam, 1970).
2. C. R. Bector and S. Chandra, *Opsearch*, **24** (1987), 143-54.
3. S. Chandra, B. D. Craven and B. Mond, *Optimization* **16** (1985), 653-62.
4. S. Chandra and I. Husain, *Bull. Aust. Math. Soc.*, **24** (1981), 295-307.
5. G. B. Dantzig, E. Eisenberg and R. W. Cottle, *Pacific J. Math.*, **15** (1965), 809-12.
6. W. S. Dorn, *J. Oper. Res. Soc. Japan*, **2** (1960), 93-97.
7. T. R. Gulati and I. Ahmad, *European J. Oper. Res.*, **101** (1997), 122-29.
8. T. R. Gulati, I. Husain and I. Ahmad, *Optimization*, **39** (1997), 69-84.
9. M. A. Hanson, *Opsearch*, **30** (1993), 313-20.
10. O. L. Mangasarian, *J. Math. Anal. Appl.*, **51** (1975), 607-20.

11. S. K. Mishra, *European J. Oper. Res.*, **127** (2000), 507-18.
12. B. Mond, *Opsearch*, **11** (1974), 92-99.
13. B. Mond, *Quart. J. of Appl. Math.*, **23** (1965), 265-69.
14. B. Mond, *J. Math. Anal. Appl.*, **46** (1974), 169-74.
15. S. Pandey, *Opsearch*, **28** (1991), 36-43.
16. X. Q. Yang, *Math. Prog.*, **81** (1998), 327-47.