Indian J. pure appl. Math., **35**(5) : 699-708, May 2004
© Printed in India.

SECOND-ORDER DUALITY FOR NONLINEAR PROGRAMMING*

X. M. YANG**,

Department of Mathematics, Chongqing Normal University, Chongqing 400047, *People's Republic of China*

X. Q. YANG, K. L. TEO AND S. H. HOU

Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong

(*Received* 16 *December* 2002; *accepted* 7 *January* 2004)

Four second-order dual models for nonlinear programming are introduced and their duality under generalized second-order F-convexity assumptions are discussed. These results generalize and improve corresponding first-order works of Chandra and Abha[4].

Key Words: Second-Order Dual Model; Duality Theorem; Nonlinear Programming; Generalized Second-Order *F*-Convexity

1. INTRODUCTION

So far a number of duals have been suggested to the nonlinear programming problems among which the Wolfe dual[11] is well known. While studying duality under generalized convexity, Mond and Weir[13] proposed a number of different duals for nonlinear programming problems with nonnegative variables and proved various duality theorems under appropriate pseudo-convexity/quasi-convexity assumptions.

Taking motivation from Bazaraa and Goode[1] and Hanson and Mond[6], Nanda and Das[14] attempted to extend the results of Mond and Weir[13] to cone domains with appropriate pseudo-invexity and quasi-invexity assumptions on objective and constraint functions. However, certain shortcomings were pointed out in the work of Nanda and Das[14] and appropriate modifications were suggested for studying duality under pseudo-invexity assumptions in Chandra and Abha[4].

The study of second order duality is significant due to the computational advantage over first order duality as it provides tighter bounds for the value of the objective function when approximations are used[8,9,12,17]. Mangasarian[9] considered a nonlinear programming and discussed second order duality under inclusion condition. Mond[12] was the first to present second order symmetric dual models and proved second order symmetric duality theorems under second order convexity. Later, Jeyakumar[8] and Yang[17] discussed also second order Mangasarian type dual formulation under ρ-convexity and generalized representation conditions respectively. Hanson had also indicated, by example, that one advantage of second order duality when applicable is that if a feasible point in the primal is given and first order duality conditions do not apply, then we can use second order duality to provide a lower bound of the value of the primal programming problem in[5].

*This research was partially supported by the National Natural Science Foundation of China. The project sponsored by SRF for ROCS, SEM and Natural Science Foundation of Chongqing.

**Current address: Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong.

Therefore, many research works on second-order duality have appeared in references [2-5,7-9,11-12,15-20]. The purpose of this paper is to introduce four second-order dual models for nonlinear programming and to discuss their duality under generalized second-order F-convexity assumptions. Such results generalize and improve corresponding first-order works of Chandra and Abha[4].

2. PRELIMINARIES

Throughout this paper, let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Let C_1, C_2 be closed convex cones with nonempty interior in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $S \subseteq \mathbb{R}^n$ be open and $C_1 \subseteq S$. Let $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}^m$ be twice differentiable functions. Further let $\nabla f(\bar{x})$ and $\nabla^2 f(\bar{x})$, respectively denote the gradient and the Hessian matrix of f evaluated at $\bar{x}$. The symbols $\nabla f(\bar{x})$ and $\nabla^2 f(\bar{x})$ $(i = 1, 2, ..., m)$ are defined similarly. $C_i^*, i = 1, 2$, is the polar cone to $C_i, i = 1, 2$ and is defined by

$$C_i^* = \{z : x^T z \le 0, \forall x \in C_i\}, \ i = 1, 2.$$

Definition 1 — A functional $F : X \times X \times \mathbb{R}^n \to \mathbb{R}$ (where $X \subseteq \mathbb{R}^n$) is sublinear if for all $(x, u) \in X \times X$,

(*i*) $F(x, u, a_1 + a_2) \le F(x, u, a_1) + F(x, u, a_2)$ for all $a_1, a_2 \in \mathbb{R}^n$

(*ii*) $F(x, u, \alpha a) = \alpha F(x, u, a)$ for all $\alpha \in \mathbb{R}_+$ and for all $a \in \mathbb{R}^n$.

For notational convenience, we write $F_{x,u}(a) = F(x, u, a)$.

Now we consider a sublinear functional $F : X \times X \times \mathbb{R}^n \to \mathbb{R}$ and a twice differentiable function $f : X \to \mathbb{R}$.

Definition 2 — f is said to be second order (strictly) F-pseudoconvex at $u \in X$ if

$$(x, p) \in X \times \mathbb{R}^n, \quad F_{x,u}\left[\nabla_u f(u) + \nabla_{uu} f(u)p\right] \geq 0$$

$$\Rightarrow f(x) \ge (>) f(u) - \frac{1}{2} p^T \nabla_{uu} f(u)p \ (\text{for } x \ne u).$$

Definition 3 — f is said to be second order F-quasiconvex at $u \in X$ if

$$(x, p) \in X \times \mathbb{R}^n, \quad f(x) \le f(u) - \frac{1}{2} p^T \nabla_{uu} f(u)\, p.$$

$$\Rightarrow F_{x,u}\left[\nabla_u f(u) + \nabla_{uu} f(u)p\right] \le 0.$$

We consider the following problem:

Problem P

$$\min \quad f(x)$$

$$\text{s. t.} \quad g(x) \in C_2^*, \qquad \text{... (1)}$$

$$x \in C_1. \qquad \text{... (2)}$$

We now state our second order dual models as following:

Problem $(\text{ND})_1$

$$\text{Max } f(u) + y^T g(u) - \frac{1}{2} p \nabla^2 (f(u) + y^T g(u))\, p$$

$$- u^T \left[\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u))\, p \right]$$

$$\text{s. t. } -\left[\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u))\, p \right] \in C_1^*, \qquad \text{... (3)}$$

$$y \in C_2. \qquad \text{... (4)}$$

Problem $(\text{ND})_2$

$$\text{Max} \quad f(u) - \frac{1}{2} p \nabla^2 f(u)\, p$$

$$\text{s. t. } -\left[\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u))\, p \right] \in C_1^*, \qquad \text{... (5)}$$

$$y^T g(u) - \frac{1}{2} p \nabla^2 y^T g(u)\, p - u^T$$

$$\left[\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u))\, p \right] \geq 0 \qquad \text{... (6)}$$

$$y \in C_2. \qquad \text{... (7)}$$

Problem $(\text{ND})_3$

$$\text{Max} \quad f(u) - \frac{1}{2} p \nabla^2 f(u)\, p - u^T$$

$$\left[\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u))\, p \right]$$

$$\text{s. t. } -\left[\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u))\, p \right] \in C_1^*, \qquad \text{... (8)}$$

$$y^T g(u) - \frac{1}{2} p \nabla^2 y^T g(u)\, p \geq 0 \qquad \text{... (9)}$$

$$y \in C_2. \qquad \text{... (10)}$$

Problem $(\mathrm{ND})_4$

$$\text{Max} \quad f(u) + y^T g(u) - \frac{1}{2} p \nabla^2 f(u) + y^T g(u)) p$$

$$\text{s. t.} \ -\left[\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u)) p \right] \in C_1^*, \qquad \text{... (11)}$$

$$u^T \left[\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u)) p \right] \leq 0 \qquad \text{... (12)}$$

$$y \in C_2. \qquad \text{... (13)}$$

3. RESULTS

In this section, we will present various duality results.

Theorem 1 *(Weak duality) — Let x be feasible for (P) and (u, y, p) be feasible for* $(ND)_1$*. Suppose there exist sublinear functionals* $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *such that,* $f(\cdot) + y^T g(\cdot) - (\cdot)^T \left[\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u)) p \right]$ *is second order F-pseudoconvex at u. Then*

$$\inf (P) \geq \sup ND)_1.$$

PROOF : Suppose that

$$f(x) < f(u) + y^T g(u) - \frac{1}{2} p \nabla^2 (f(u) + y^T g(u)) p$$

$$- u^T \left[\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u)) p \right]. \qquad \text{... (14)}$$

From (3) and (2), (1) and (4), we have

$$- x^T \left[\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u)) p \right] \leq 0, \qquad \text{... (15)}$$

$$y^T g(x) \leq 0. \qquad \text{... (16)}$$

Thus, (14), (15) and (16) imply

$$f(x) + y^T g(x) - x^T \left[\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u)) p \right]$$

$$< f(u) + y^T g(u) - \frac{1}{2} p \nabla^2 (f(u) + y^T g(u)) p - u^T$$

$$\left[\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u)) p \right]. \qquad \text{... (17)}$$

By the hypothesis of the second-order F-pseudoconvexity of $f(\cdot)+y^T g(\cdot) \; -(\cdot)^T \left[\nabla(f(u)+y^T g(u))+\nabla^2(f(u)+y^T g(u))p\right]$ at u, we obtain

$$F_{x,u}(0) < 0,$$

which contradicts $F_{x,u}(0) = 0$. Hence,

$$f(x) \geq f(u)+y^T g(u)-\frac{1}{2}p\,\nabla^2(f(u)+y^T g(u))\,p-u^T \left[\nabla(f(u)+y^T g(u))+\nabla^2(f(u)+y^T g(u))\,p\right].$$

That is,

$$\inf (P) \geq \sup (ND)_1. \qquad \square$$

***Theorem* 2 (*Weak duality*) — *Let x be feasible for (P) and (u, y, p) be feasible for* $(ND)_2$. *Suppose there exist sublinear functionals* $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *such that, f is second-order F-pseudoconvex at u and* $y^T g(\cdot) + (\cdot)^T \upsilon$ *is second order F-pseudoconvex at u for* $\upsilon=-\left[\nabla(f(u)+y^T g(u))+(\nabla^2 f(u)+\nabla^2 y^T g(u))p\right]$. *Then*

$$\inf (P) \geq \sup ND)_2.$$

PROOF : Suppose that

$$f(x) < f(u) - \frac{1}{2}p\,\nabla^2 f(u)\,p.$$

This, in view of the second order F-pseudoconvexity of f, yields

$$F_{x,u}(\nabla f(u)+\nabla^2 f(u)\,p) < 0. \qquad \ldots (18)$$

From the dual constraint (5),

$$-\left[\nabla(f(u)+y^T g(u))+\nabla^2(f(u)+y^T g(u))\,p\right] \in C_1^*,$$

and therefore there exists $\upsilon \in C_1^*$ such that

$$\upsilon=-\left[\nabla(f(u)+y^T g(u))+(\nabla^2 f(u)+\nabla^2 y^T g(u))p\right].$$

That is,

$$\upsilon+\nabla y^T g(u)+\nabla^2 y^T g(u)\,p+[\nabla f(u)+\nabla^2 f(u)\,p\,]=0.$$

By sublinearity of F and $F_{x,u}(0)=0$, we have

$$F_{x,u}(\upsilon+\nabla y^T g\,(u)+\nabla^2 y^T g(u)\,p) > 0,$$

which, because of the second-order F-quasiconvexity of $y^T g + \upsilon^T(\cdot)$ for

$$\upsilon = -\left[\nabla (f(u) + y^T g(u)) + (\nabla^2 f(u) + \nabla^2 y^T g(u))p \right], \text{ yields}$$

$$y^T g(x) - x^T\left[\nabla (f(u) + y^T g(u)) + (\nabla^2 f(u) + \nabla^2 y^T g(u))p \right]$$

$$> y^T g(u) - \frac{1}{2} p \nabla^2 y^T g(u)\, p - u^T$$

$$\left[\nabla f(u) + \nabla y^T g(u) + (\nabla^2 f(u) + \nabla^2 y^T g(u))p \right]. \qquad \text{... (19)}$$

But from (1) and (7), (2) and (5), we have that $y^T g(x) \leq 0$ and

$x^T\left[\nabla f(u) + \nabla y^T g(u) + (\nabla^2 f(u) + \nabla^2 y^T g(u))p \right] \leq 0$. Thus, (19) gives

$$y^T g(u) - \frac{1}{2} p \nabla^2 y^T g(u)\, p - u^T$$

$$\left[\nabla f(u) + \nabla y^T g(u) + (\nabla^2 f(u) + \nabla^2 y^T g(u))p \right] < 0,$$

which contradicts (6). □

Theorem 3 *(Weak duality)* — *Let x be feasible for (P) and (u, y, p) be feasible for* $(ND)_3$. *Suppose there exist sublinear functionals* $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *such that, for all* $\upsilon = -\left[\nabla (f(u) + y^T g(u)) + (\nabla^2 f(u) + \nabla^2 y^T g(u))p \right]$, $f + \upsilon^T(\cdot)$ *is second-order F-pseudoconvex at u and* $y^T g$ *is second order F-pseudoconvex at u for . Then*

$$\inf\ (P) \geq \sup\ (ND)_3.$$

PROOF : Suppose that

$$f(x) < f(u) - \frac{1}{2} p \nabla^2 f(u)\, p - u^T$$

$$\left[\nabla (f(u) + y^T g(u) + \nabla^2 (f(u) + y^T g(u))p \right]. \qquad \text{... (20)}$$

From (2) and (8), we have

$$-x^T\left[\nabla (f(u) + y^T g(u) + \nabla^2 (f(u) + y^T g(u))p \right] \leq 0. \qquad \text{... (21)}$$

It follows from (20) and (21) that

$$f(x) - x^T\left[\nabla (f(u) + y^T g(u) + \nabla^2 (f(u) + y^T g(u))p \right]$$

$$< f(u) - \frac{1}{2} p \nabla^2 f(u)\, p - u^T$$

$$\left[\nabla (f(u) + y^T g(u) + \nabla^2 (f(u) + y^T g(u))p \right].$$

By the second order F-pseudoconvexity of $f + v^T(\cdot)$ for

$$v = -\left[\nabla f(u) + \nabla y^T g(u) + (\nabla^2 f(u) + \nabla^2 y^T g(u))p \right] \text{ at } u,$$

we have

$$F_{x,u}(\nabla f(u) + \nabla^2 f(u)\, p + v) < 0$$

From sublinearity of F, $F_{x,u}(0) = 0$ and

$$v + \left[\nabla f(u) + \nabla y^T g(u) + (\nabla^2 f(u) + \nabla^2 y^T g(u))p \right] = 0,$$

we get

$$F_{x,u}(\nabla y^T g(u) + \nabla^2 y^T g(u)\, p) > 0,$$

which by the second order F-pseudoconvexity of $y^T g$ implies that

$$y^T g(x) > y^T g(u) - \frac{1}{2} p\, \nabla^2 y^T g(u)\, p. \qquad \text{... (22)}$$

We note that $y^T g(x) \le 0$ from (1) and (10). Thus, (22) implies that

$$y^T g(u) - \frac{1}{2} p\, \nabla^2 y^T g(u)\, p < 0,$$

which contradicts (9). □

***Theorem* 4** *(Weak duality) — Let x be feasible for (P) and (u, y, p) be feasible for $(ND)_4$. Suppose there exist sublinear functionals $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying*

$$F_{x,u}(a) + a^T u \le 0 \text{ for all } a \in C_1^*, \qquad \text{... (A)}$$

and $f + y^T g$ is second-order F-pseudoconvex at u. Then

$$\inf (P) \ge \sup (ND)_4.$$

PROOF : Suppose that

$$f(x) < f(u) + y^T g(u) - \frac{1}{2} p\, (\nabla^2 f(u) + \nabla^2 y^T g(u))\, p. \qquad \text{... (23)}$$

From (1) and (13), we have

$$y^T g(x) \le 0. \qquad \text{... (24)}$$

It follows from (23) and (24) that

$$f(x) + y^T g(x) < f(u) + y^T g(u) - \frac{1}{2} p \nabla^2 (f(u) + y^T g(u)) p,$$

which by the second-order F-pseudoconvexity of $f + y^T g$ implies that

$$F_{x,u} (\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u)) p) < 0. \quad \dots (25)$$

On the other hand, from (A) and (11), we have

$$F_{x,u}\left(-\left[\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u)) p \right] \right)$$

$$- u^T \left[\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u)) p \right] \leq 0.$$

That is,

$$F_{x,u}\left(-\left[\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u)) p \right] \right)$$

$$\leq u^T \left[\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u)) p \right].$$

From (12), we get

$$F_{x,u}\left(-\left[\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u)) p \right] \right) \leq 0. \quad \dots (26)$$

From sublinearity of F and $F_{x,u}(0) = 0$, (26) implies that

$$F_{x,u}\left(\nabla (f(u) + y^T g(u)) + \nabla^2 (f(u) + y^T g(u)) p \right) \geq 0,$$

which contradicts (25). □

Theorem* 5 *(Strong duality) — Let $\bar{x}$ be a local or global optimal for (P) at which a suitable constraint qualification*[10] *be satisfied. Then there exists $\bar{y} \in \mathbb{R}^m$ such that $(\bar{x}, \bar{y}, \bar{p} = 0)$ is feasible to $(ND)_1$, and the corresponding values of (P) and $(ND)_1$ are equal. Further if the hypothesis of Theorem 1 as stated above is also satisfied, then $\bar{x}$ and $(\bar{x}, \bar{y}, \bar{p} = 0)$ are global optimal solutions for (P) and $(ND)_1$, respectively.

PROOF : The proof of this theorem is the same as given by Chandra and Anha[4] in light of above Theorem 1. □

In fact, we also state and prove strong duality theorems, similar to Theorem 5, for other duals $(ND)_2$-$(ND)_4$ as well.

We now give Mangasarian type[10] strict converse duality theorems as follows:

Theorem 6 *(Strict converse duality) — Let $\bar{x}$ be an optimal solution for (P) and $(\bar{u}, \bar{y}, \bar{p})$ be an optimal solution for $(ND)_1$. If there exists sublinear functionals $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that for all $v \in C_1^*$, $f + y^T g + v^T(\cdot)$ is second-order strictly* F-pseudoconvex at u, then $\bar{x} = \bar{u}$, i.e., $\bar{u}$ is an optimal solution for (P).

PROOF : We assume that $\bar{x} \neq \bar{u}$ and exhibit a contradiction. Since $\bar{x}$ is an optimal solution for (P), it follows by strong duality (Theorem 5) that there exist $y^* \in C_2, p^* = 0$ such that $(\bar{x}, y^*, p^* = 0)$ is an optimal solution for $(ND)_1$. Since $(\bar{u}, \bar{y}, \bar{p})$ is also an optimal solution for $(ND)_1$, it follows that

$$f(\bar{x}) = f(\bar{u}) + \bar{y}^T g(\bar{u}) - \frac{1}{2}\bar{p}\,\nabla^2 (f(\bar{u}) + \bar{y}^T g(\bar{u}))\,\bar{p} - \bar{u}^T$$

$$\left[\nabla (f(\bar{u}) + \bar{y}^T g(\bar{u})) + \nabla^2 (f(\bar{u}) + \bar{y}^T g(\bar{u}))\,\bar{p} \right].$$

From (1) and (4), (2) and (3), we know that

$$\bar{y}^T g(\bar{x}) - \bar{x}^T \left[\nabla (f(\bar{u}) + \bar{y}^T g(\bar{u})) + \nabla^2 (f(\bar{u}) + \bar{y}^T g(\bar{u}))\,\bar{p} \right] \leq 0.$$

Thus, we obtain

$$f(\bar{x}) + \bar{y}^T g(\bar{x}) - \bar{x}^T \left[\nabla (f(\bar{u}) + \bar{y}^T g(\bar{u})) + \nabla^2 (f(\bar{u}) + \bar{y}^T g(\bar{u}))\,\bar{p} \right]$$

$$\leq f(\bar{u}) + \bar{y}^T g(\bar{u}) - \frac{1}{2}\bar{p}\,\nabla^2 (f(\bar{u}) + \bar{y}^T g(\bar{u}))\,\bar{p} - \bar{u}^T$$

$$\left[\nabla (f(\bar{u}) + \bar{y}^T g(\bar{u})) + \nabla^2 (f(\bar{u}) + \bar{y}^T g(\bar{u}))\,\bar{p} \right].$$

This, in view of strict pseudoconvexity of $f + \bar{y}^T g + v^T(\cdot)$ for all $v \in C_1^*$, gives

$$F_{x,u}(0) < 0,$$

which contradicts $F_{x,u}(0) = 0$. Hence, the result. □

We can also state and prove strict converse duality theorems, similar to Theorem 6, for other duals $(ND)_2$-$(ND)_4$ as well.

REFERENCES

1. M. S. Bazaraa and J. J. Goode, *Oper. Research*, **21** (1973), 1-9.
2. C. R. Bector and S. Chandra, *Opsearch*, **23** (1986), 89-95.
3. C. R. Bector, S. Chandra and I. Husain, *Opsearch*, **28** (1991), 249-63.
4. S. Chandra and Abha, *Europ J. Oper. Res.*, **122** (2000), 161-65.
5. M. A. Hanson, *Opsearch*, **30** (1993), 311-20.
6. M. A. Hanson and B. Mond, *J. Inf. Optimizn. Sci.*, **3** (1982), 25-32
7. S. H. Hou and X. M. Yang, *J. Math. Anal. Appl.*, **255** (2001), 491-98.
8. V. Jeyakumar, *Utalitas Mathematics*, **34** (1985), 1-15.

9. O. L. Mangasarian, *J. Math. Anal. Appl.*, **51** (1975), 607-20.
10. O. L. Mangasarian, *Nonlinear Programming*, McGraw-Hill, New York, 1969.
11. S. K. Mishra, *Europ. J. Oper. Res.*, **127** (200), 507-18.
12. B. Mond, *Opsearch*, **11** (1974), 90-99.
13. B. Mond and T. Weir, Generalized concavity and duality, *In*: S. Schaible, W. T. Ziemba (Eds.), *Generalized Concavity in Optimization and Economics*, Academic Press, New York, 1981, pp. 263-79.
14. S. Nanda and L. N. Das, *Europ J. Oper Res.*, **88** (1996), 572-77.
15. D. S. Kim, Y. B. Yun and H. Kuk, *Appl. Math. Lett.*, **10** (1997), 17-22.
16. S. K. Suneja, C. S. Lalitha and S. Khurana, *Europ. J. Oper. Res.*, **144** (2002), 492-500.
17. X. Q. Yang, *Math. Prog.*, **81** (1998), 327-47.
18. X. M. Yang, *Opsearch*, **32** (1995), 205-09.
19. X. M. Yang and S. H. Hou, *Appl. Math. Lett.*, **14** (2001), 587-92.
20. X. M. Yang, X. Q. Yang and K. L. Teo, *Europ. J. Oper. Res.*, **144** (2002), 554-59.