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SYSTEM OF GENERALIZED EQUILIBRIUM PROBLEMS AND APPLICATIONS IN GENERALIZED CONVEX SPACES*

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By applying a collectively fixed point theorem for a family of set-valued mappings defined on the product space of generalized convex spaces due to authors, some new equilibrium existence theorems for a system of generalized equilibrium problems are proved in the product space of generalized convex spaces. Some applications to losided saddle point problem and Nash equilibrium problem are also obtained. These theorems improve and generalize a number of known results in recent literature.

Key Words: Collectively Fixed Point Theorem; System of Generalized Equilibrium Problems; Generalized Convex Space

1. INTRODUCTION

Pang[1] have shown that a variety of equilibrium models, for example, the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem and the general equilibrium programming problem can be uniformly modelled as a variational inequality defined on the product sets. He decomposed the original variational inequality into a system of variational inequalities which are easy to solve. The method of decomposition was also used by Zhu and Marcotte[2] to solve a variational inequality problem defined on a set of inequality constraints. By the motivated and inspired factors, Cohen and Chaplais[3], and Ansari and Yao[4,5] studied the system of variational inequalities defined on the product set and proved some existence theorems of solutions under various different assumptions.

In this paper, we shall introduce and study a system of generalized equilibrium problems in a product space of generalized convex spaces which includes the system of generalized variational inequalities, generalized equilibrium problems and generalized variational inequality problems as many special cases.

Let I be a finite or infinite index set, $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ be two family of topological spaces. Let $X=\prod_{i\in I} X_i$ and $Y=\prod_{i\in I} Y_i$. For each $i\in I$, let 2^{Y_i} be the family of all subsets of Y_i, $T_i : X\to 2^{Y_i}$ be a set-valued mapping, and $\varphi_i : Y_i\times X_i\times X_i\to \mathbf{R}$ be a real valued function.

A system of generalized equilibrium problems (in short, SGEP) is to find $\hat{x}=(\hat{x}_i)_{i\in I}\in X$ such that for each $i\in I$,

$$\forall z_i\in X_i,\quad \exists\ \hat{y}_i\in T_i(\hat{x})\ \text{satisfying}\ \varphi_i(\hat{y}_i, \hat{x}_i, z_i)\le 0. \qquad \dots (1)$$

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The SGEP (1) includes the abstract generalized variational inequality, the system of generalized Variational-like inequalities, the system of generalized variational inequalities, and the system of variational inequalities as special cases, for example, see[4-10].

The numerical methods for finding approximate solutions of the system of variational inequalities were investigated by Pang[1], and Cohn and Chaplais[3]. While the existence of solutions for the system of variational inequalities were established by Ansari and Yao[4,5].

In this paper, by applying a new collectively fixed point theorem for a family of set-valued mappings defined on the product space of generalized convex spaces due to author, some new equilibrium existence theorems for the SGEP (1) are proved in the product space of generalized convex spaces. As applications, some existence results for losided saddle point problems and Nash equilibrium problems are obtained in generalized convex spaces. These results unify and generalize many key known results in recent literature.

2. PRELIMINARIES

Let X and Y be two nonempty sets. We denote by 2^Y and $\mathcal{F}(X)$ the family of all subsets of Y, the family of all nonempty finite subsets of X respectively. For a subset A of X, we denote by A^c the complement of A in X.

The notion of a generalized convex (G-convex) space was introduced by Park and Kim[11] under an extra isotonic condition. Recently, Part[12], by removed the extra condition, gave the following definition of a G-convex space which coincide with the L-convex space introduced by Ben-El-Mechaiekh *et al.*[13]

A G-convex space (X, Γ) consists of a topological space X and a set-valued mapping $\Gamma : \mathcal{F}(X) \to 2^X \setminus \{\phi\}$ such that for each $A \in \mathcal{F}(X)$ with $|A| = n+1$, there exists a continuous mapping $\varphi_A : \Delta_n \to \Gamma(A)$ such that $B \in \mathcal{F}(A)$ with $|B| = |J| + 1$, implies $\varphi_A(\Delta_J) \subset \Gamma(B)$, where Δ_J denotes the face of Δ_n corresponding $B \in \mathcal{F}(A)$. A set $D \subset X$ is said to be G-convex if for each $A \in \mathcal{F}(D), \Gamma(A) \subset D$.

Let (X, Γ) be a G-convex space and $f : X \to \mathbf{R}$ be a real valued function. f is said to be G-quasiconcave (resp. G-quasiconvex) if for any $\lambda \in \mathbf{R}$, the set $\{x \in X : f(x) > \lambda\}$ (resp., $\{x \in X : f(x) < \lambda\}$) is G-convex.

It is clear that the notion of G-convex space includes the H-spaces, G-H-spaces and many topological spaces with various convexity structure as special cases and the converse are not true in general, for example, see[11-13].

In order to prove our main results we need the following results:

Lemma 2.1[14] — Let X and Y be two topological spaces, $W : X \times Y \to \mathbf{R}$ be function and $G : Y \to 2^X$ be a set-valued mapping such that

(i) W is upper semicontinuous (respectively, lower semicontinuous) on $X \times Y$,

(ii) G is upper semicontinuous at $y_0 \in Y$ and $G(y_0)$ is compact. Then the marginal function

$$V(y) = \sup_{x \in G(y)} W(x, y) \text{ (respectively, } \alpha(y) = \inf_{x \in G(y)} W(x, y))$$

is upper semicontinuous (respectively, lower semicontinuous) at y_0.

The following collectively fixed point theorem is a special case of Theorem 2.2 of Ding and Park[16].

Lemma 2.2 — Let $(X_i, \Gamma_i)_{i \in I}$ be a family of G-convex spaces where I is an (finite or infinite) index set. Let $X = \prod_{i \in I} X_i$ and for each $i \in I$, let $G_i : X \to 2^{X_i}$ be a set-valued mappings such that for each $i \in I$, the following conditions hold:

(*i*) for each $x \in X$, $G_i(x)$ is G-convex,

(*ii*) for each non-empty compact subset K of X, $K = \bigcup_{y_i \in X_i} (\text{int}\, G_i^{-1}(y_i) \cap K)$,

(*iii*) there exists a non-empty subset X_i^0 of X_i such that for each $N_i \in \mathcal{F}(X_i)$, there is a compact G-convex subset L_{N_i} containing $(X_i^0 \cup N_i)$ and the set $D_i = \bigcap_{y_i \in X_i^0} (\text{int}\, G_i^{-1}(y_i))^c$ is empty or compact in X where $(\text{cint}\, G_i^{-1}(y_i))^c$ denotes the complement of $(\text{int}\, G_i^{-1}(y_i))$ in X.

Then there exists a point $\hat{x} = (\hat{x}_i)_{i \in I}$ such that $\hat{x}_i \in G_i(\hat{x})$ for $i \in I$.

Remark 2.1 : If the (X_i, Γ_i) is a compact G-convex space, by letting $X_{i.0} = L_{N_i} = X_i$ for all $N_i \in \mathcal{F}(X_i)$, then the condition (*iii*) is satisfied trivially. Hence we have the following results.

Lemma 2.3 — Let $(X_i, \Gamma_i)_{i \in I}$ be a family of compact G-convex spaces where I is an (finite or infinite) index set. Let $X = \prod_{i \in I} X_i$ and for each $i \in I$, let $G_i : X \to 2^{X_i}$ be a set-valued mappings such that for each $i \in I$, the following conditions hold:

(*i*) for each $x \in X$, $G_i(x)$ is G-convex,

(*ii*) $X = \bigcup_{y_i \in X_i} \text{int}\, G_i^{-1}(y_i)$,

Then there exists a point $\hat{x} = (\hat{x}_i)_{i \in I}$ such that $\hat{x}_i \in G_i(\hat{x})$ for each $i \in I$.

Remark 2.2 : Lemmas 2.2 and 2.3 unify and generalize Theorem 3.1 of Tarafdar[17,18], Theorems 2.4 and 2.5 of Chang, Lee, Wu, Cho and Lee[19], Theorem 3 and Corollary 4 of Lin and Park[20], Theorem 3 of Park[12], Theorems 2.2 and 2.4 of Lan and Webb[21], and Theorem 1 and Corollary 1 of Ansari and Yao[4] to G-convex spaces.

3. Existence of Solutions for *SGEP* (1)

We shall first prove the following equilibrium existence theorem for the *SGEP* (1).

***Theorem* 3.1** — *Let $(X_i, \Gamma_i)_{i\in I}$ be a family of compact G-convex spaces, $\{Y_i\}_{i\in I}$ be a family of topological spaces. Let $X=\prod_{i\in I} X_i$ and $Y=\prod_{i\in I} Y_i$. For each $i\in I$, let $T_i : X \to 2^{Y_i}$ be an upper semicontinuous set-valued mapping with nonempty compact values, and $\varphi_i : Y_i \times X_i \times X_i \to \mathbf{R}$ be a real valued function such that for each*

(*i*) for $(y_i, x_i) \in Y_i \times X_i$, $z_i \mapsto \varphi_i (y_i, x_i, z_i)$ is G-quasiconcave,

(*ii*) for each $z_i \in X_i$, $(y_i, x_i) \mapsto \varphi_i (y_i, x_i, z_i)$ is lower semicontinuous on $Y_i \times X_i$,

(*iii*) φ_i is upper semicontinuous on $Y_i \times X_i \times X_i$,

(*iv*) for each $x \in X$ and $y_i \in T_i(x)$, $\varphi_i (y_i, x_i, x_i) \le 0$,

Then there exists $\hat{x} \in X$ such that for each $i \in I$,

$$\forall z_i \in X_i,\ \exists\ \hat{y}_i \in T_i(\hat{x}) \text{ satisfying } \varphi_i(\hat{y}_i, \hat{x}_i, z_i) \le 0.$$

PROOF : For each $i \in I$ and $n = 1, 2, \ldots$, define a set-valued mapping $G_{i,n} : X \to 2^{X_i}$ by

$$G_{i,n}(x) = \left\{ z_i \in X_i : \exists\, y_i \in T_i(x) \text{ such that } \varphi_i(y_i, x_i, z_i) > \max_{u_i \in X_i} \varphi_i(y_i, x_i, u_i) - \frac{1}{n} \right\}.$$

Then the condition (*i*) implies that for each $x \in X$, $G_{i,n}(x)$ is G-convex. Since φ_i is upper semicontinuous on $Y_i \times X_i \times X_i$ and X_i is compact, we have $G_{i,n}(x) \ne \phi$ for each $x \in X$ and by the condition (*ii*) and Lemma 2.1 with the function $(y_i, x_i) \mapsto \max_{u_i \in X_i} \varphi_i (y_i, x_i, u_i)$ is continuous on $Y_i \times X_i$. By the condition (*ii*), we have that for each fixed $z_i \in X_i$, the function

$$(y_i, \pi_i(x)) \mapsto \varphi_i (y_i, \pi_i(x), z_i) - \max_{u_i \in X_i} \varphi_i (y_i, \pi_i(x), u_i)$$

is lower semicontinuous on $Y_i \times X_i$ where π_i is the projection X onto X_i. Since T_i has compact values by Lemma 2.1 the function

$$x \mapsto \inf_{y_i \in T_i(x)} \left[\varphi_i (y_i, \pi_i(x), z_i) - \max_{u_i \in X_i} \varphi_i (y_i, \pi_i(x), u_i) \right]$$

is lower semicontinuous on X. It follows that for each $i \in I$ and $z_i \in X$, the set

$$G_{i,n}^{-1}(z_i) = \left\{ x \in X : \exists\, y_i \in T_i(x) \text{ such that } \varphi_i(y_i, x_i, z_i) > \max_{u_i \in X_i} \varphi_i(y_i, x_i, u_i) - \frac{1}{n} \right\}$$

$$= \left\{ x \in X : \inf_{y_i \in T_i(x)} [\, \varphi_i (y_i, \pi_i(x), z_i) - \max_{u_i \in X_i} \varphi_i(y_i, x_i, u_i) > -\frac{1}{n} \right\}$$

is open in X. Since for each $i \in I$ and for all $x \in X$, $G_{i,n}(x) \ne \phi$ we have

$$X = \bigcup_{z_i \in X_i} \overset{-1}{G_{i,n}}(z_i) = \bigcup_{z_i \in X_i} \operatorname{int} \overset{-1}{G_{i,n}}(z_i).$$

By Lemma 2.3, there exists a point $\hat{x}_n = (\hat{x}_{i,n})_{i \in I} \in X$ such that for each $i \in I$,

$$\hat{x}_{i,n} \in G_{i,n}(\hat{x}_n), \quad \forall n = 1, 2, \ldots .$$

that is, there exists $\hat{y}_{i,n} \in T_i(\hat{x}_n)$ such that

$$\varphi_i(\hat{y}_{i,n}, \hat{x}_{i,n}, \hat{x}_{i,n}) > \max_{u_i \in X_i} \varphi_i(\hat{y}_{i,n}, \hat{x}_{i,n}, u_i) - \frac{1}{n}, \quad \forall n = 1, 2, \ldots .$$

Since each X_i is compact and hence X is compact, we can assume that $\hat{x}_n \to \hat{x}$ as $n \to \infty$, that is $\hat{x}_{i,n} \to \hat{x}_i$ for each $i \in I$. Since each T_i is upper semicontinuous with compact values, for each $i \in I$, $T_i(X)$ is compact in X_i and so we may assume that $\hat{y}_{i,n} \to \hat{y}_i$ as $n \to \infty$ for each $i \in I$. From the upper semicontinuity of T_i it follows that $\hat{y}_i \in T_i(\hat{x})$ for each $i \in I$. By the conditions (*ii*) and (*iii*), we have

$$\varphi_i(\hat{y}_i, \hat{x}_i, \hat{x}_i) \geq \overline{\lim}_{n \to \infty} \varphi_i(\hat{y}_{i,n}, \hat{x}_{i,n}, \hat{x}_{i,n})$$

$$\geq \overline{\lim}_{n \to \infty} \left[\max_{u_i \in X_i} \varphi_i(\hat{y}_{i,n}, \hat{x}_{i,n}, u_i) - \frac{1}{n} \right]$$

and hence, by the condition (*iv*),

$$0 \geq \varphi_i(\hat{y}_i, \hat{x}_i, \hat{x}_i) = \max_{u_i \in X_i} \varphi_i(\hat{y}_{i,n}, \hat{x}_{i,n}, u_i).$$

This shows that there exists $\hat{x} \in X$ such that for each $i \in I$,

$$\forall z_i \in X_i, \; \exists \; \hat{y}_i \in T_i(x) \text{ satisfying } \varphi_i(\hat{y}_i, \hat{x}_i, \hat{z}_i) \leq 0.$$

Remark 3.1 : Theorem 3.1 generalizes Theorem 2.1 and Corollary 2.1 of Ansari and Yao[5] from topological vector spaces to G-convex spaces without linear structure and our proof is different from that in[5].

If I is a singleton in Theorem 3.1, then we obtain the following result.

Corollary 3.1 — Let (X, Γ) be a compact G-convex space, Y be a topological space. Let $T : X \to 2^Y$ be a upper semicontinuous set-valued mapping with nonempty compact values and $\varphi_i : Y \times X \times X \to \mathbf{R}$ be a real valued function such that

(*i*) for $(y, x) \in Y \times X$, $z \mapsto \varphi(y, x, z)$ is G-quasiconcave,

(*ii*) for each $z \in X$, $(y, x) \mapsto \varphi_i(y, x, z)$ is lower semicontinuous on $Y \times X$,

(*iii*) φ is upper semicontinuous on $Y \times X \times X$,

(*iv*) for each $x \in X$ and $y \in T(x)$, $\varphi(y, x, x) \leq 0$, Then there exists $\hat{x} \in X$ such that

$$\forall z \in X, \ \exists \ \hat{y} \in T(\hat{x}) \text{ satisfying } \varphi(\hat{y}, \hat{x}, z) \leq 0.$$

Remark 3.2 : For the closely related results with Corollary 3.1, we may consult Corollary 4.1 of Ding[9], Theorem 2.2 of Wu and Xu[8], Theorem 1 of Wu[6] and Theorem 1 of Wu[7].

For each $i \in I$, when X_i is not necessarily compact, we have the following results.

Theorem 3.2 — *Let $(X_i, \Gamma_i)_{i \in I}$ be a family of G-convex spaces, $\{Y_i\}_{i \in I}$ be a family of topological spaces. Let $X = \prod_{i \in I} X_i$ and $Y = \prod_{i \in I} Y_i$. For each $i \in I$, let $T_i : X \to 2^{Y_i}$ be an upper semicontinuous set-valued mapping with nonempty compact values, and $\varphi_i : Y_i \times X_i \times X_i \to \mathbf{R}$ be a real valued function such that for each $i \in I$,*

(i) for $(y_i, x_i) \in Y_i \times X_i$, $z_i \mapsto \varphi_i(y_i, x_i, z_i)$ is G-quasiconcave,

(ii) for each $z_i \in X_i$, $(y_i, x_i) \mapsto \varphi_i(y_i, x_i, z_i)$ is lower semicontinuous on $Y_i \times X_i$,

(iii) φ_i is upper semicontinuous on $Y_i \times X_i \times X_i$,

(iv) for each $x \in X$ and $y_i \in T_i(x)$, $\varphi_i(y_i, x_i, x_i) \leq 0$,

(v) there exists a nonempty compact subset K_i of X_i and for each $N_i \in \mathcal{F}(X_i)$, there exists a compact G-convex subset L_{N_i} of X_i containing N_i such that for each $x \in L_N \setminus K$, there is a $z_i \in L_{N_i}$ satisfying

$$\varphi_i(y_i, x_i, z_i) > 0, \quad \forall y_i \in T_i(x),$$

where
$$L_N = \prod_{i \in I} L_{N_i} \text{ and } K = \prod_{i \in I} K_i.$$

Then there exists $\hat{x} \in K$ such that for each $i \in I$,

$$\forall z_i \in X_i, \ \exists \ \hat{y}_i \in T_i(\hat{x}) \text{ satisfying } \varphi_i(\hat{y}_i, \hat{x}_i, z_i) \leq 0.$$

PROOF : For each $i \in I$, define a set-valued mapping $F_i : X_i \to 2^X$ by

$$F_i(z_i) = \left\{ x \in K : \exists \ y_i \in T_i(x) \text{ such that } \varphi_i(y_i, x_i, x_i) \leq 0 \right\}.$$

Since $T_i(x)$ is compact, by (*ii*) and Lemma 2.1, for each fixed $z_i \in X_i$, the function $x \mapsto \inf_{y_i \in T_i(x)} \varphi_i(y_i, \pi_i(x), z_i)$ is lower semicontinuous on X and hence $F_i(z_i)$ is closed for each $z_i \in X_i$.

For each $i \in I$ and for any $N_i \in \mathcal{F}(X_i)$, let K_i and L_{N_i} be subsets of X_i in the condition (*v*).

Then for each $i \in I, L_{N_i}$ is compact G-convex space containing N_i. By Theorem 3.1, there exists $\hat{x} \in L_N$ such that for each $i \in I$,

$$\forall z_i \in L_{N_i}, \exists \hat{y}_i \in T_i(\hat{x}) \text{ satisfying } \varphi_i(\hat{y}_i, \hat{x}_i, z_i) \leq 0.$$

The condition (*v*) implies that $\hat{x} \in K$. Note that for each $i \in I, N_i \subset L_{N_i}$, we obtain

$$\forall z_i \in N_i, \exists \hat{y}_i \in T_i(\hat{x}) \text{ satisfying } \varphi_i(\hat{y}_i, \hat{x}_i, z_i) \leq 0.$$

Hence for each $i \in I$, the family $\{F_i(z_i) : z_i \in X_i\}$ has the finite intersection property. Since K is compact, we must have that for each $i \in I$, $\bigcap_{z_i \in X_i} F_i(z_i) \neq \phi$. Thus there exists $\hat{x} \in K$ such that for each $i \in I$,

$$\forall z_i \in X_i, \exists \hat{y}_i \in T_i(\hat{x}) \text{ satisfying } \varphi_i(\hat{y}_i, \hat{x}_i, z_i) \leq 0.$$

This completes the proof.

Remark 3.3 : Theorem 3.2 generalizes Theorem 2.2 of Ansari and Yao[5] from topological vector spaces to G-convex spaces without linear sturcture under much weaker assumptions. It is easy to see that Corollaries 2.2-2.4 of Ansari and Yao[5] are very special cases of Theorem 3.2.

Theorem 3.3 — *Let* $(X_i, \Gamma_i)_{i \in I}$ *be a family of G-convex spaces,* $\{Y_i\}_{i \in I}$ *be a family of topological spaces. Let* $X = \prod_{i \in I} X_i$ *and* $Y = \prod_{i \in I} Y_i$. *For each* $i \in I$, *let* $T_i : X \to 2^{Y_i}$ *has a continuous selection* $g_i : X \to Y_i$ *and* $\varphi_i : Y_i \times X_i \times X_i \to \mathbf{R}$ *be a real valued function such that for each* $i \in I$,

(i) for $(y_i, x_i) \in Y_i \times X_i, z_i \mapsto \varphi_i(y_i, x_i, z_i)$ *is G-quasiconcave,*

(ii) for each $z_i \in X_i, (y_i, x_i) \mapsto \varphi_i(y_i, x_i, z_i)$ *is lower semicontinuous on* $Y_i \times X_i$,

(iii) φ_i *is upper semicontinuous on* $Y_i \times X_i \times X_i$,

(iv) for each $x \in X$ *and* $\varphi_i(g_i(x), x_i, x_i) \leq 0$,

(v) there exists a nonempty compact subset K_i *of* X_i *and for each* $N_i \in \mathcal{F}(X_i)$, *there exists a compact G-convex subset* L_{N_i} *of* X_i *containing* N_i *such that for each* $x \in L_N \setminus K$, *there is a* $z_i \in L_{N_i}$ *satisfying*

$$\varphi_i(g_i(x), x_i, z_i) > 0,$$

where $L_N = \prod_{i \in I} L_{N_i}$ *and* $K = \prod_{i \in I} K_i$.

Then there exists $\hat{x} \in K$ *and* $\hat{y} \in Y$ *such that for each* $i \in I$,

$$\hat{y}_i \in T_i(\hat{x}) \text{ and } \varphi_i(\hat{y}_i, \hat{x}_i, z_i) \leq 0 \quad \forall z_i \in X$$

PROOF : By Theorem 3.2 with $T_i(x) = \{g_i(x)\}$ for each $i \in I$ and $x \in X$, there exists $\hat{x} \in K$ such that for each $i \in I$,

$$\varphi_i(g_i(\hat{x}), \hat{x}_i, z_i) \leq 0 \quad \forall z_i \in X_i.$$

For each $i \in I$, let $\hat{y}_i = g_i(\hat{x})$. Note that g_i is a continuous selection of T_i, we obtain that $\hat{y} \in Y$ such that for each $i \in I$,

$$\hat{y}_i \in T_i(\hat{x}) \text{ and } \varphi_i(\hat{y}_i, \hat{x}_i, z_i) \leq 0, \quad \forall z_i \in X_i.$$

If I is a singleton in Theorem 3.3, then we obtain the following result.

Corollary 3.2 — Let (X, Γ) be a G-convex space, Y be a topological space. Let $T: X \to 2^Y$ has a continuous selection $g: X \to Y$ and $\varphi: Y \times X \times X \to \mathbf{R}$ be a real valued function such that

(*i*) for $(y, x) \in Y \times X, z \mapsto \varphi(y, x, z)$ is G-quasiconcave,

(*ii*) for each $z \in X, (y, x) \mapsto \varphi(y, x, z)$ is lower semicontinuous on $Y \times X$,

(*iii*) φ is upper semicontinuous on $Y \times X \times X$,

(*iv*) for each $x \in X$ and $\varphi(g(x), x, x) \leq 0$,

(*v*) there exists a nonempty compact subset K of X and for each $N \in \mathcal{F}(X)$, there exists a compact G-convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, there is a $z \in L_N$ satisfying

$$\varphi(g(x), x, z) > 0.$$

Then there exists $\hat{x} \in K$ and $\hat{y} \in T(\hat{x})$ such that

$$\varphi(\hat{y}, \hat{x}, z) \leq 0, \quad \forall z \in X.$$

4. APPLICATIONS

We first give an application to losided saddle-point problems.

Theorem 4.1 — *Let* $(X_i, \Gamma_i)_{i \in I}$ *be a family of G-convex spaces,* $\{Y_i\}_{i \in I}$ *be a family of topological spaces. Let* $X = \prod_{i \in I} X_i$ *and* $Y = \prod_{i \in I} Y_i$. *For each* $i \in I$, *let* $T_i: X \to 2^{Y_i}$ *has a continuous selection* $g_i: X \to Y_i$ *and* $f_i: X_i \times y_i \to R$ *be a real valued continuous such that for each* $i \in I$, *(i) for* $y_i \in Y_i, x_i \mapsto f_i(x_i, y_i)$ *is G-quasiconvex,*

(ii) there exists a nonempty compact subset K_i *of* X_i *and for each* $N_i \in \mathcal{F}(X_i)$, *there exists a compact G-convex subset* L_{N_i} *of* X_i *containing* N_i *such that for each* $x \in L_N \setminus K$, *there is a* $z_i \in L_{N_i}$ *satisfying*

$$f_i(x_i, g_i(x)) > f_i(z_i, g_i(x)),$$

where $L_N = \prod_{i \in I} L_{N_i}$ *and* $K = \prod_{i \in I} K_i$.

Then there exists $\hat{x} \in K$ *and* $\hat{y} = Y$ *such that for each* $i \in I$,

$$\hat{y}_i \in T_i(\hat{x}) \text{ and } f_i(\hat{x}_i, \hat{y}_i) \leq f_i(z_i, \hat{y}_i) \quad \forall z_i \in X_i.$$

PROOF : For each $i \in I$, define a function $\varphi_i : Y_i \times X_i \times X_i$ by

$$\varphi_i(y_i, x_i, z_i) = f_i(x_i, y_i) - f_i(z_i, y_i), \qquad \forall (y_i, x_i, z_i) \in Y_i \times X_i \times X_i.$$

Then it is easy to see that all conditions of Theorem 3.3 are satisfied. By Theorem 3.3, there exists $\hat{x} \in X$ and $\hat{y} \in Y$ such that for each $i \in I$,

$$\hat{y}_i \in T_i(\hat{x}) \text{ and } \varphi_i(\hat{y}_i, \hat{x}_i, z_i) \leq 0, \quad \forall z_i \in X_i.$$

Hence we have that for each $i \in I$,

$$\hat{y}_i \in T_i(\hat{x}) \text{ and } f_i(\hat{x}_i, \hat{y}_i) \leq f_i(z_i, \hat{y}_i) \quad \forall z_i \in X_i.$$

If I is a singleton in Theorem 4.1, then we obtain the following result.

Corollary 4.1 — Let (X, Γ) be a G-convex spaces and Y be a topological space. Let $T : X \to 2^Y$ has a continuous selection $g : X \to Y$ and $f : X \times Y \to \mathbf{R}$ be a continuous function such that

(*i*) for $y \in Y, x \mapsto f(x, y)$ is G-quasiconvex,

(*ii*) there exists a nonempty compact subset K of X and for each $N \in \mathcal{F}(X)$, there exists a compact G-convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, there is a $z \in L_N$ satisfying

$$f(x, g(x)) > f(z, g(x)).$$

Then there exists $\hat{x} \in K$ and $\hat{y} \in T(\hat{x})$ such that for each $i \in I$,

$$f(\hat{x}, \hat{y}) \leq f(z, \hat{y}), \quad \forall z \in X.$$

Remark 4.1 : Corollary 4.1 is a improving variant form of Theorem 4.1 of Ding[9] which, in turn, improves Theorem 2.1 of Park[22], Theorem 4 of Yuan[23], Theorem 2 of Ha[24] and Theorem 1 of Ky Fan[25].

Now we state an application to the Nash equilibrium problems.

Theorem 4.2 — *Let* $(X_i, \Gamma_i)_{i \in I}$ *be a family of G-convex spaces,* $X = \prod_{i \in I} X_i$, $X^i = \prod_{j \in I, j \neq i} X_j$ *and* $f_i : X = X_i \times X^i \to \mathbf{R}$ *be a continuous real valued function such that for each* $i \in I$,

(i) for $x^i \in X^i$, $z_i \mapsto f_i(z_i, x^i)$ *is G-quasiconvex,*

(ii) there exists a nonempty compact subset K_i *of* X_i *and for each* $N_i \in \mathcal{F}(X_i)$, *there exists a compact G-convex subset* L_{N_i} *of* X_i *containing* N_i *such that for each* $x \in L_N \setminus K$, *there is a* $z_i \in L_{N_i}$ *satisfying*

$$f_i(x_i, y^i) > f_i(z_i, y^i) \quad \forall y^i \in X^i,$$

where

$$L_N = \prod_{i \in I} L_{N_i} \text{ and } K = \prod_{i \in I} K_i.$$

Then there exists $\hat{x} \in K$ *such that for each* $i \in I$,

$$f_i(\hat{x}) = f_i(\hat{x}_i, \hat{x}^i) \leq f_i(z_i, \hat{x}^i) \quad \forall z_i \in X_i.$$

i.e., $\hat{x}$ *is a Nash equilibrium point of the conventional game* $((X_i, \Gamma_i), f_i)_{i \in I}$.

PROOF : For each $i \in I$, let $Y_i = X^i$ and $T_i(x) = \pi^i(x) = x^i$ where π^i is the projection from X onto X^i. Then for each $i \in I$, T_i is a continuous single-valued mapping. For each $i \in I$, define a function $\varphi_i : X^i \times X_i \times X_i \to \mathbf{R}$ by

$$\varphi_i(y^i, x_i, z_i) = f_i(x_i, y^i) - f_i(z_i, y^i).$$

It is easy to check that all conditions of Theorem 3.3 are satisfied. By Theorem 3.3, there exists $\hat{x} \in K$ such that

$$\varphi_i(\hat{x}^i, \hat{x}_i, z_i) \leq 0, \quad \forall z_i \in X_i.$$

Hence we have

$$f_i(\hat{x}) = f_i(\hat{x}_i, \hat{x}^i) \leq f_i(z_i, \hat{x}^i), \quad \forall z_i \in X_i,$$

and so $\hat{x}$ is a Nash equilibrium point.

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