Indian J. pure appl. Math., **35**(6) : 723-735, June 2004

THE SIX-VERTEX THEOREM FOR CLOSED PLANAR CURVE WHICH BOUNDS AND IMMERSED SURFACE

NURAL YÜKSEL AND MEHMET ÖZDEMIR

Department of Mathematics, Erciyes University 38039, *Kayseri, Turkey*
e-mail : ozdemirm@erciyes.edu.tr and yukseln@erciyes.edu.tr

(*Received* 25 *July* 2000; *after final revision* 8 *June* 2003; *accepted* 14 *February* 2004)

We prove that the number of vertices is at least 6 for closed planar curve which bounds an immersed surface with non-zero genus.

Key Words: Vertex; Shell; Genus; Curvature

1. INTRODUCTION

A vertex of a smooth planar curve γ of class C^2 is a stationary point of its curvature. The classical four-vertex theorem states that every simple closed curve has at least 4 vertices. More generally, Pinkall showed in[1] that if a closed curve bounds an immersed surface, then it has at least 4 vertices. In[2] it is shown that for each natural number $g \geq 1$ there is a curve that bounds a surface of genus g and has only 6 vertices. The following was made in[2] as a conjecture. The number of vertices of a smooth planar curve is at least 6, if it bounds a surface other than the disc.

In this note, we show their conjecture is true.

Let $\gamma = \gamma(s) : R \rightarrow R^2$ be a l-periodic curve of class C^2 parametrized by arc-length. We take the normal vector field $n = n(s)$ of γ such that the frame $\{\gamma, n\}$ is positively oriented. The curvature function $k = k(s)$ is defined by the Frenet-Serret equation $\ddot{\gamma} = kn$.

The curve γ is said to have a vertex at $s \in R$ if $k'(s) = 0$. A vertex at s is called an honest vertex if k fails to be a strictly monotonic function in every neighbourhood of s.

A geometric meaning of vertex is the following[3]: The curve γ has a maximal (resp. minimal) vertex at $s = s_0$ if and only if there exists a positive number $\varepsilon > 0$ such that the image $\gamma|_{[s_0 - \varepsilon, s_0 + \varepsilon)}$ is contained in the right (resp. left) hand side of the osculating circle at $s = s_0$. Since the number of the maximal and the minimal vertices are the same, the number of vertices are even unless it is infinitive.

An orientation preserving Mobius transformation T is written in the form

$$T(z) = \frac{az + b}{cz + d}$$

where $ad - bc = 1$, a, b, c, $d \in C$. Here we identify the plane $(R^2; x, y)$ with C and set $z = x + iy$. T maps an oriented circle to a circle with the same orientation, the following lemma is obvious.

Lemma 1.1 — Let γ be a planar curve. Then any orientation preserving Mobius transformation maps maximal (resp. minimal) vertices of γ to the same one of $T \circ \gamma$.

2. Shell

Definition 2.1 — The planar curve γ is said to have a shell on the open interval $I = (a, b)$, if $\gamma(a) = \gamma(b)$, $(0 \leq a < b \leq l)$ and γ has no self-intersection on the interval I. If γ has shells on intervals $I_1, I_2, \ldots, I_n$ respectively, then these shells are called independent if

$$\pi(I_i) \cap \pi(I_j) = \phi, \ (1 \leq i < j \leq n)$$

holds, where $\pi : R \to R \mid Z$ is the canonical projection. The following lemma is obtained from the fact: If $\gamma(a) = \gamma(b)$, $(0 \leq a < b \leq l)$, then there exist an open subinterval I of (a, b) such that the restriction $\gamma|_I$ has no self-intersection.

Lemma 2.1 — Let γ be a closed planar curve which has a self-intersection. Then γ has at least two independent shells.

The each shell $\gamma|_I$ of γ bounds a domain D which is called the interior of the shell. The internal angle μ to D at the intersection point is called the internal angle of shell $\gamma|_I$ (Fig. 2.1). We remark that each shell admits two possible orientations: If the bounded domain D lies in left-hand (resp. right-hand) side of γ, the shell is said to have positive (resp. negative) orientation (Fig. 2.2).

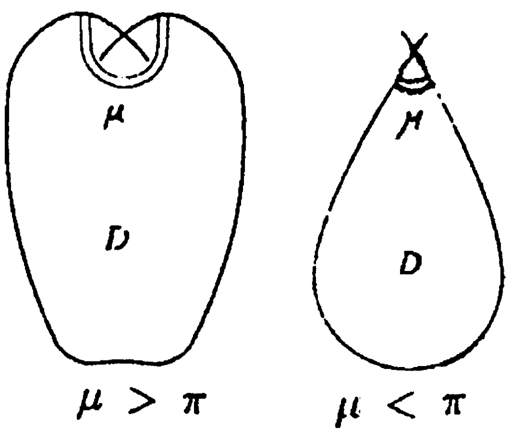

Fig. 2.1

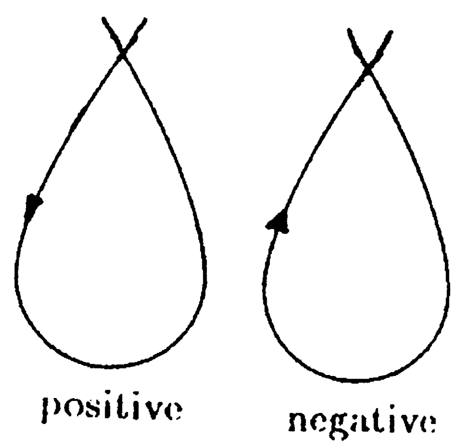

Fig. 2.2

Lemma 2.2 — $\gamma : [a, b] \to R^2$ be a planar curve such that it has a shell with internal angle $\mu \leq \pi$ on the interval (a, b). If the orientation of the shell is positive (resp. negative), there is a maximal (resp. minimal) vertex on (a, b).

The γ has a shell with internal angle $\mu \geq \pi$ on the internal (a, b). If the orientation of the shell is positive (resp. negative), there is a minimal (resp. maximal) vertex on (a, b)[3].

Corollary 2.1 — Let $\gamma: [a, b] \to R^2$ be a planar curve such that it has a shell with internal angle π on the interval (a, b). Then there are at least a pair of maximal and minimal vertices on (a, b).

We identify $R^2 \cup \{\infty\}$ with the unit sphere S^2 by the stereographic projection at the north pole. According to Pinkall[1], we say that a closed planar curve γ bounds an immersed surface of genus g, if there is a compact orientable surface M^2 of genus g with connected boundary $\partial M = S'$ and an immersion $f: M^2 \to R^2 \cup \{\infty\}$ such that $\gamma = f|_{\partial M}$. In this study, we assume that the local image of the surface $f(M^2)$ lies on the left-hand side of the curve γ.

Lemma 2.3 — Let γ be a closed planar curve of class C^2 which bounds an immersed surface. Suppose that γ has a positive shell of internal angle $\mu \le \pi$ on a interval (a, b) $(0 \le a < b \le l)$. Then there exists an interval (c, d) $(0 \le c < d \le l)$, such that the image of $\gamma|_{[c, d]}$ is contained in the interior D of the shell on (a, b).

PROOF : We suppose that the interior D does not contain any point on the curve γ. Let $f: M^2 \to R^2$ be an associated immersion such that $.\gamma = f|_{\partial M}$. By our conventions the local image of the surface $f(M^2)$ lies on the left-hand side on the curve γ and hence the inverse image $f^{-1}(D)$ is not empty.

Let V be a connected component of the inverse image $f^{-1}(D)$ whose closure meets ∂M. Then the assumption yields that $f(V) = D$ and it can be easily checked that the restriction $f|_{\upsilon}: V \to D$ induces a topological finite covering structure. Since D is simply connected, $f|_{\upsilon}$ is a diffeomorphism.

On the other hand, we set

$$\gamma_{\varepsilon}(s) = \gamma(s) + \varepsilon \,.\, n(s)$$

for $s \in R$ where $n(s)$ is the oriented normal vector field on γ. Since the statement is purely topological, by a suitable differentiable small perturbation on the shell, we may assume that the internal angle μ is less than π. Then for a suitable small $\varepsilon > 0$, γ_{ε} has a self-intersection point in D (Fig. 2.3). Since the inverse image of γ_{ε} in V is contained in the boundary of ε-collar of surface M^2, this contradicts the fact that $f|_{\upsilon}$ is a diffeomorphism.

Lemma 2.4 — Let γ be a closed planar curve of class C^2 which bounds an immersed surface. Suppose that γ has a positive shell of internal angle $\mu \le \pi$ on an interval (a, b) $(0 \le a < b \le l)$. Then there exists a shell on some inteval (c, d) $(0 \le c < d \le l)$, such that the image of $\gamma|_{[c, d]}$ is contained in the interior D of the shell on (a, b).

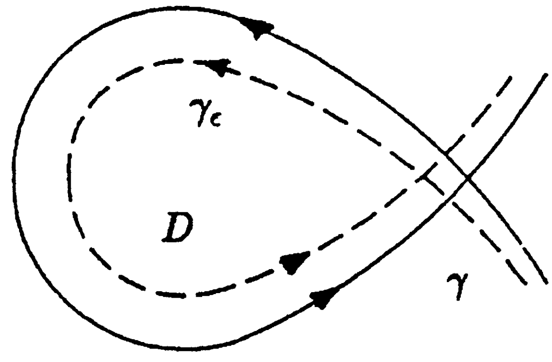

FIG. 2.3

PROOF : We suppose that the interior D does not contain any shell on the curve γ. By Lemma 2.3, we may suppose that there exists an interval I such that the image of $\gamma|_I$ are contained in D. Since the statement is purely topological, by a suitable differentiable small perturbation on the shell, we may assume that the internal angle μ is less than π. Since $\mu < \pi$, for a suitable small $\varepsilon_0 > 0$, γ_ε, $(0 < \varepsilon < \varepsilon_0)$ defined by Fig. 2.1 makes positive shell of internal angle $(< \pi)$ on some subinterval (a', b') of (a, b) such that the image of $\gamma_\varepsilon|_{[a', b']}$ are contained in D (Fig. 2.3). We denote by D_ε the interior of the new shell $\gamma_\varepsilon|_{(a', b')}$.

Let $\{I_\lambda\}_{\lambda \in \Lambda}$ be the splitting of the inverse image $\gamma^{-1}(D)$ as a family of disjoint open subsets of $(0, l)$. Since the length of the closed curve γ is finite, only finite components, say

$$\gamma|_{I_1}, \gamma|_{I_2}, \dots, \gamma|_{I_n}$$

meet the closure of $D_{\varepsilon/2}$ where $\{I_j\}_{j=1,2,\dots,n}$ is a finite subset of $\{I_\lambda\}_{\lambda \in \Lambda}$. Then for each $\gamma|_{I_j}$, one can patch a simply connected domain smoothly, to avoid the domain D (Fig. 2.4). After these n-times surgeries, one can easily see that D_ε does not contain any point on γ_ε. Though the curve γ_ε is in the class C^1, the modified argument in the proof of Lemma 2.3 can be applied as follows: We may suppose $\varepsilon < \varepsilon_0/2$. Let $f: M^2_\varepsilon \to R^2$ be an associated immersion such that $\gamma = f|_{\partial M_\varepsilon}$. Let V_ε be a connected component of the inverse image $f^{-1}(D_\varepsilon)$ whose boundary is contained in ∂M_ε. Then we can conclude that $f(U_\varepsilon) = D_\varepsilon$ and it can be easily checked that the restriction $f|_{V_\varepsilon}: V_\varepsilon \to D_\varepsilon$ induces a topological finite covering structure, where ε is as in the proof of Lemma 2.4. Since D_ε is simply connected, $f|_{V_\varepsilon}$ is a diffeomorphism. Since $\gamma_{2\varepsilon}$ has a self-intersection in D_ε, this makes a contradiction (Fig. 2.4).

Lemma 2.5 — Let γ be a closed planar curve which has a shell of internal angle $\mu > \pi$ on

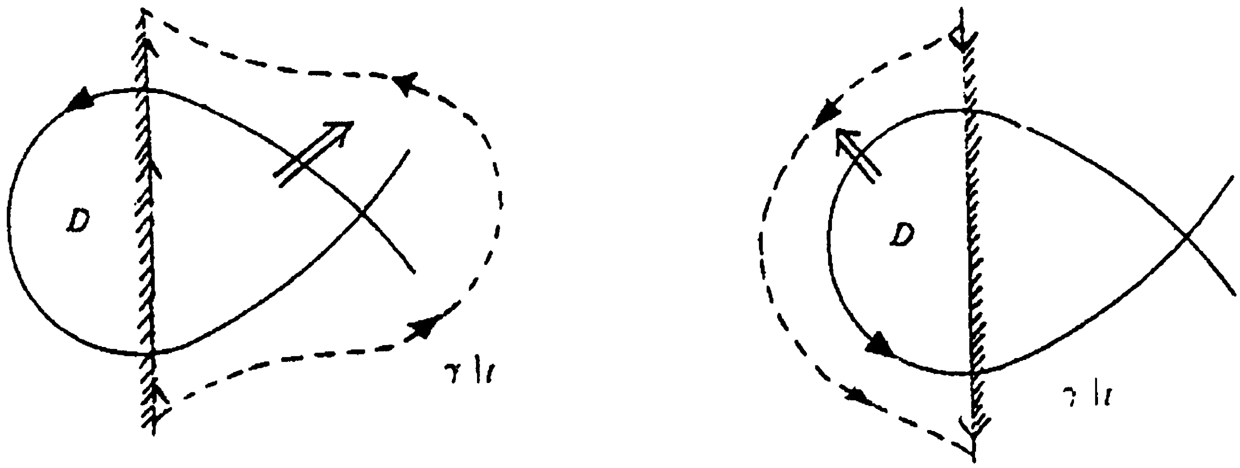

FIG. 2.4

the interval (a, b) $(0 \le a < b \le l)$ and D interior of its shell. Suppose that there exists a number such that $\gamma(s_0) \notin \overline{D}$, where $\overline{D}$ is the closure of D. Then there exists another shell of internal angle $\mu \le \pi$ on some interval (c, d) $(0 \le c < d \le l)$ which is contained in $\overline{D}$.

PROOF : Since γ is periodic, we may assume that $s_0 > b$. Then the only two possibilities occur.

(a) There exists an interval (c, d) $(a < c < b < d < s_0)$ such that $\gamma|_{(c,d)}$ is a shell of internal angle $\mu \le \pi$ in $\overline{D}$ (see Fig. 2.5a).

(*b*) There exists an interval (c, d) $(b < c < d < s_0)$ such that $\gamma|_{(c,d)}$ is a shell of internal angle $\mu \le \pi$ in D (see Fig. 2.5b).

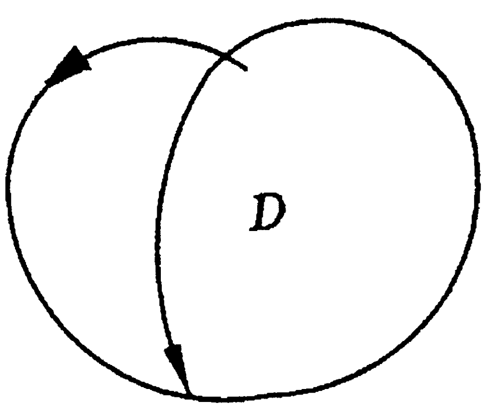

FIG. 2.5a

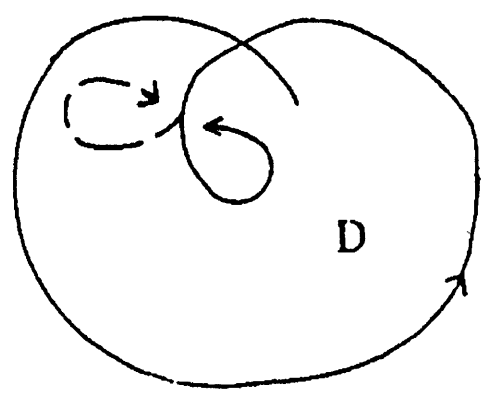

FIG. 2.5b

This proves the lemma.

Theorem 2.1 — *Let γ be a closed planar curve of period l, which bounds an immersed surface. Suppose that γ has a positive shell of internal angle $\mu \leq \pi$ on an interval (a, b) $(0 \leq a < b \leq l)$. Then there exists a negative shell of internal angle $\mu \leq \pi$ on some interval (c, d) $(0 \leq c < d \leq l)$, such that the image of $\gamma|_{(c,d)}$ is contained in the interior D of the shell on (a, b). In particular, two intervals (a, b) and (c, d) are disjoint (Fig. 2.6).*

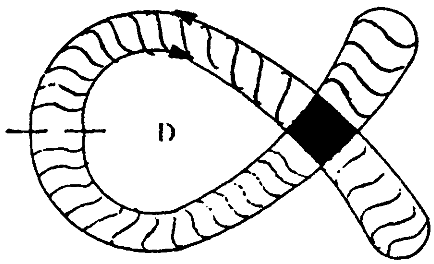

FIG. 2.6

PROOF : By Lemma 2.3 and Lemma 2.5, γ has a shell of internal angle $\mu \leq \pi$ on some interval (c, d) which is contained in the original shell. If the new shell is positive, we can apply this argument again. Since γ is an immersion with compact image, the number of independent shells is finite. So we find a negative shell of internal angle $\mu \leq \pi$ contained in D by repeating the argument.

As an application of Theorem 2.1, we get the followings:

Corollary 2.2 — Let $\gamma : R \to R^2$ be a closed planar curve of period l, which bounds an immersed surface. Suppose that γ has at most 4 vertices. Then for any shell on γ associated with an interval (a, b) $(0 \leq a < b \leq l)$, there exists $c \in (a, b)$ which attains a minimal vertex.

PROOF : When γ is a simple closed curve, the proposition is obvious because it has a unique shell $(0, l)$.

So we may assume that γ has a self-intersection. By a suitable Mobius transformation, we may also assume that the shell $\gamma|_{(a,b)}$ has internal angle $\mu \leq \pi$. If the shell has negative orientation, then it has a minimal vertex by Lemma 2.2. So we assume that the shell $\gamma|_{(a,b)}$ has positive orientation. Moreover, by Lemma 2.2 we may also assume that the internal angle μ is less than π. Then there exists a number s_0 $(a < s_0 < b)$ such that $\gamma(s_0)$ is a maximal vertex. By Theorem 2.1, there is a negative shell $\gamma|_{(c,d)}$ of internal angle $(\leq \pi)$ contained in the interior D of the original shell $\gamma|_{(a,b)}$. By Lemma 2.2, there exists a number s_1 $(c < s_1 < d)$ such that $\gamma(s_1) \in D$ is a minimal vertex. Since γ is periodic, we may assume that $s_1 > b$. Since γ has at most 4 vertices, either $\gamma|_{(s_0, s_1)}$ or $\gamma|_{(s_1, s_{0+l})}$ has no vertices. Without loss of generality, we may assume that $\gamma|_{(s_0, s_1)}$ has no vertices. Since any shell has at least one vertex, $\gamma|_{(s_0, s_1)}$ has no self-intersection. Since $\gamma(s_1) \in D$ and by the assumption $\mu < \pi$, there exist $t_0 \in (a, b)$ and $t_1 \in (s_0, s_1)$ such that $\gamma(t_0) = \gamma(t_1)$ and $\gamma|_{(b, t_1)}$ lies in the compliment of $\overline{D}$. Then γ has a positive shell of internal angle $(\geq \pi)$ on (t_0, t_1). By Lemma 2.2, there is a minimal vertex on (t_0, t_1). Since γ has no vertex on (s_0, s_1), the minimal vertex lies in a subinterval (t_0, s_0) of (a, b) (Fig. 2.7).

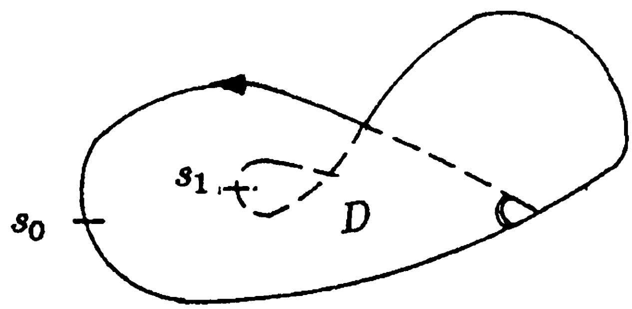

FIG. 2.7

Corollary 2.3 — Let γ be a closed planar curve which bounds an immersed surface. Then the curve γ has at least 4 vertices.

PROOF : When γ is a simple closed curve the corollary reduces to the classical vertex theorem. So we may assume that γ has a self-intersection. By Lemma 1.1, γ has at least two shells. By Corollary 2.2, the number of minimal vertices of γ is greater than or equal to 2. This proves the corollary.

3. SEMI-SIMPLE CLOSED CURVES

Definition 3.1 — A closed planar curve $\gamma : R \to R^2$ of period l, is called semi-simple if there exist numbers s_0 and s_1 $(s_0 < s_1)$ such that the restrictions $\gamma|_{[s_0, s_1]}$ and $\gamma|_{[s_1, s_{0+l}]}$ are embeddings (Fig. 3.1).

Theorem 3.1 — *Let γ be a smooth closed planar curve of period l which bounds an immersed surface. Suppose that γ has exactly 4 vertices. Then it is semi-simple.*

PROOF : Let $\gamma(s_0)$ and $\gamma(s_1)$ be the minimal vertices of γ. Then by Corollary 2.2, $\gamma|_{[s_0, s_1)}$ and $\gamma|_{[s_1, s_{0+l})}$ are embeddings.

Theorem 3.2 — *Let γ be a closed planar curve of period l, which bounds an immersed surface. Suppose that γ is semi-simple. Then it bounds only the disc.*

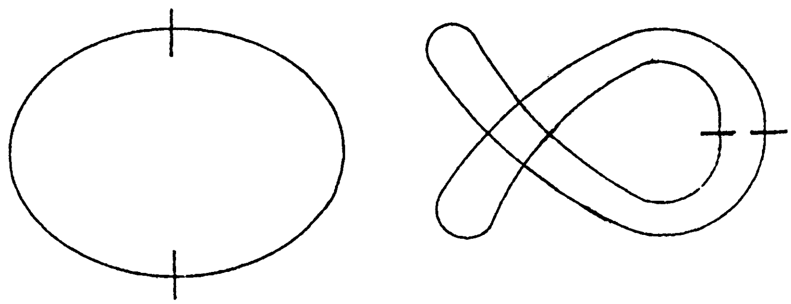

FIG. 3.1

The 6-vertex theorem stated in introduction immediately follows from Theorem 3.1 and Theorem 3.2.

In contrast to the semi-simple case, for each genus g, there is a closed curve which bounds an immersed surface of genus g such that it can be divided by three embedding parts. It is realized by the examples in[2].

When the curve γ is a simple closed curve, the theorem is obvious. So we assume that γ is semi-simple, but has a self-intersection. Let $\gamma : R \to R^2$ be a closed planar curve of period l. We assume that γ is semi-simple. Then by the definition, there exist numbers s_0 and s_1 ($s_0 < s_1$) such that $\gamma|_{[s_0, s_1]}$ and $\gamma|_{[s_1, s_{0+l}]}$ are embeddings. This division of closed curve may not be unique. We fix this division and call $\gamma(s_0)$ and $\gamma(s_1)$ connecting points of semi-simple closed curve γ. We denote $\gamma_1 = \gamma|_{[s_0, s_l]}$ and $\gamma_2 = \gamma|_{[s_1, s_{0+l}]}$.

The following properties are elementary conclusions of definition of semi-simple closed curve.

(1) After performing a small perturbation, we may assume that all self-intersection of γ are transversal[4].

Here after, we assume that any semi-simple closed curve has the property 1.

(2) The number of self-intersections is finite and every self-intersection is a double point.

PROOF : Suppose that there is a triple point. Then the third image of γ does not belong to γ_1 and γ_2, which yields a contradiction.

(3) The number of independent shells is exactly two.

PROOF : Since γ_1 and γ_2, do not have self-intersection. γ has at most two independent shells. Thus Lemma 2.1 yields the conclusion.

(4) If γ bounds an immersed surface. γ has even number of self-intersections. Moreover, it can be assumed that γ has a negative shell of internal angle ($< \pi$).

PROOF : By Lemma 5.3 in[8], γ has even number of self-intersections. By a suitable Mobius transformation, we may assume that γ has a shell of internal angle ($< \pi$). If this shell is positive, we find a negative shell of internal angle ($< \pi$) by the Theorem 2.1 and the Property 1.

Lemma 3.1 — Let γ be a semi-simple closed curve, which bounds an immersed surface. Suppose that there are two disjoint closed intervals $[a, b]$ and $[c, d]$ contained in $[0, l]$ such that

(*a*) $\gamma(a) = \gamma(d) = p$, $\gamma(b) = \gamma(c) = q$.

(*b*) $\gamma|_{(a, b)}$ and $\gamma|_{(c, d)}$ have no intersection on itself and each other.

(*c*) The two internal angles of the domain D bounded by $\gamma|_{[a, b]}$ and $\gamma|_{[c, d]}$ are both less than π (Fig. 3.2).

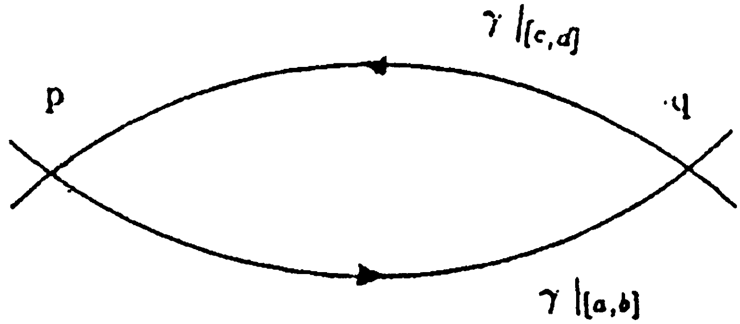

FIG. 3.2

Then by an image homotopic deformation of the immersed surface, certain two intersection points of γ can be cancelled.

PROOF : We call such a domain D a leaf. If the leaf D contains n_0 points on γ, then two intersection $\gamma|_{[a, b]}$ and $\gamma|_{[c, d]}$ can be cancelled, obviously.

Now we assume that D contains points on γ. By the property 2, the inverse image $\gamma^{-1}(\overline{D})$ is expressed by a finite union of disjoint closed subintervals in $[0, l]$;

$$\gamma^{-1}(\overline{D}) = I_1 \cup I_2 \cup \cdots \cup I_n.$$

Step 1 — First we consider the case that D contains points on γ which are not the connecting points. Then each interval

$I_j = [\alpha_j, \beta_j]$ ($j = 1, 2, \ldots, n$) is a subinterval of $[s_0, s_1)$ or $[s_1, s_{0+} l]$. When $I_j \subset [s_0, s_1]$ (resp. $I_j \subset [s_1, s_0 + l]$), the restriction of the curve $\gamma|_{I_j}$ belongs γ_1 (resp. γ_2) and hence $\gamma|_{I_j}$ meets $\gamma|_{[c, d]}$ (resp. $\gamma|_{[a, b]}$) at $s = \alpha_j$ and $s = \beta_j$.

So we find a new leaf D' bounded by the curves $\gamma|_{I_j}$ and the restriction of $\gamma|_{[c, d]}$ (resp. $\gamma|_{[a, b]}$), because internal angles are both less than π (Fig. 3.3a). Replacing D by D', we can apply this process. Since the intersection points are finite, repeating this process, we find two disjoint intervals $[a', b']$ and $[c', d']$ such that

(*a*) $\gamma|_{(a', b')}$ and $\gamma|_{(c', d')}$ have no intersection on itself and each other.

(*b*) The interior D'' bounded by $\gamma|_{[a', b']}$ and $\gamma|_{[c', d']}$ contains no points on γ.

Thus two intersection $\gamma|_{[a', b']}$ and $\gamma|_{[c', d']}$ can be cancelled.

Step 2 — Next we consider the case that D contains connecting points on γ. It suffices to show the existence of other two disjoint leaves. Since there are only two connecting points, at least one leaf does not contain them and the discussion reduces to Step 1. Without loss of generality, we may assume that the connecting point $\gamma(s_0)$ is contained in D. Then there exist numbers α and $\beta\,(s_0 < \alpha < a,\ d. < \beta < s_0 + l)$ such that $\gamma|_{[a, a]}$ (resp. $\gamma|_{[d, \beta]}$) meets $\gamma|_{[c, d]}$ (resp. $\gamma|_{[a, b]}$) only at the points $\gamma(\alpha)$ and $\gamma(a)$ (resp. $\gamma(d)$ and $\gamma(\beta)$). Obviously, the interior D_1 (resp. D_2) bounded by $\gamma|_{[\alpha, a]}$ (resp. $\gamma|_{[d, \beta]}$) and a restriction of $\gamma|_{[c, d]}$ (resp. $\gamma|_{[a, b]}$), has internal angles ($< \pi$) and so it is a leaf. Moreover, D and D_1 (resp. D_2) are disjoint (Fig. 3.3b). If D_1 and D_2 are also disjoint, we find three disjoint leaves D, D_1 and D_2. Next we suppose that $D_1 \cap D_2$ is not empty. Then there exists a number $\xi\,(d < \xi < \beta)$ such that $\gamma|_{[d, \xi]}$ meets $\gamma|_{[\alpha, a]}$ only at the points $\gamma(d)$ and $\gamma(\xi)$. Since the domain D_3 bounded by $\gamma|_{[d, \xi]}$ and a restriction of $\gamma|_{[\alpha, a]}$ has internal angles ($< \pi$), if is also a leaf. Obviously, these three leaves D, D_1 and D_3 are mutually disjoint (Fig. 3.3c). This proves the lemma.

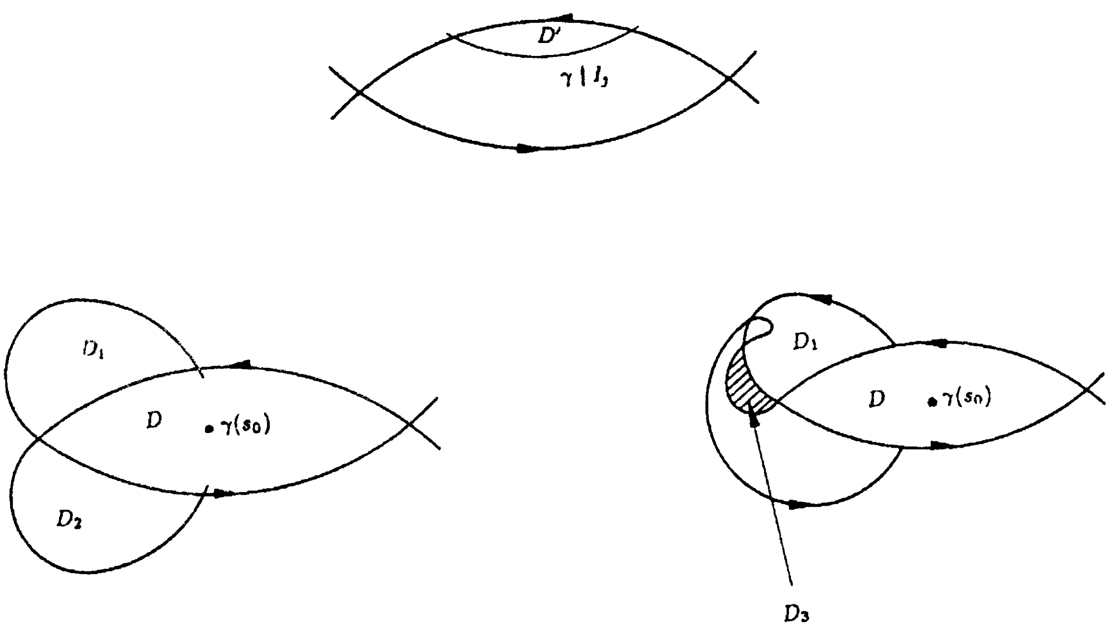

FIG. 3.3a, b and c

Lemma 3.2 — Let γ be a semi-simple closed curve with period l. Suppose that $\gamma(a)=\gamma(b)$ $(a<b<a+l)$ such that $\gamma|_{[a,b]}$ is not a shell. Then there exist two numbers a' and b' $(a<a'<b'<b)$ satisfying the following properties:

(*a*) $\gamma(a')=\gamma(b')$.

(*b*) $\gamma|_{[a,a']}$ and $\gamma|_{[b',b]}$ are embeddings.

(*c*) There is no intersection between $\gamma|_{(a,a')}$ and $\gamma|_{(b',b)}$.

PROOF : Without loss of gnerality, we may assume that the connecting points satisfy the inequality $a < s_0 < b < s_1 < a+l$. Since $\gamma|_{(a,b)}$ is not a shell, there is a subinterval (α,β) of (a, b) such that $\gamma|_{(\alpha,\beta)}$ is a shell. Obviously, $s_0 \in (\alpha,\beta)$. So $\gamma|_{[a,\alpha]}$ (resp. $\gamma|_{[\beta,b]}$ is a restriction of γ_2 (resp. γ_1). Hence $\gamma|_{[a,\alpha]}$ and $\gamma|_{[\beta,b]}$ are embeddings. Now we consider the point $\gamma(a')=\gamma(b')$ $(a<a'\leq\alpha, b<b'\leq\beta)$ at which $\gamma|_{[a,\alpha]}$ meets $\gamma|_{[\beta,b]}$ first time. Then the numbers a' and b' are the desired ones.

Lemma 3.3 — Let γ be a semi-simple closed curve with period l. Suppose that there are two disjoint closed intervals $[a, b]$ and $[c, d]$ $0 \leq a<b<c<d\leq l$ such that

(*a*) $\gamma(a) = \gamma(d)$ $(= p)$, $\gamma(b) = \gamma(c)$ $(= q)$.

(*b*) $\gamma|_{[a,b]}$ and $\gamma|_{[c,d]}$ have no intersection on itself and each other.

(*c*) The internal angles of a domain D bounded by $\gamma|_{[a,b]}$ and $\gamma|_{[c,d]}$ are greater than π at the point p and less than π at the point q.

(*d*) There exist a subinterval (α,β) of (b, c) such that γ has a shell on (α,β) and $\gamma(\alpha)=\gamma(\beta)\notin D$ (Fig. 3.4).

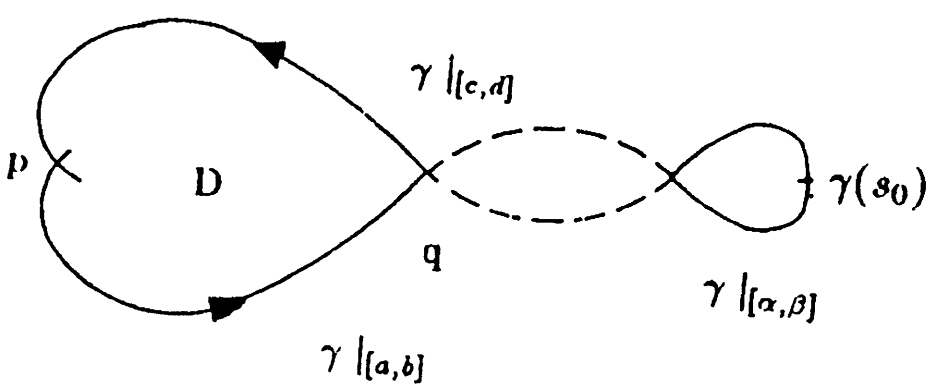

FIG. 3.4

Then there exists a leaf D' as in Fig. 3.2 contained in $\overline{D}$.

PROOF :

Case 1 — First we consider the case $\gamma|_{(d,a+l)}$ is not contained in D. Then there exists a number d' $(d<d'<a+l)$ such that $\gamma(d')\in\partial D$ and $\gamma|_{(d,d')}$ has no self-intersection and contained in D. By the property 2, we have $\gamma(d')\neq p,q$. If $\gamma(d')$ lies on the curve $\gamma|_{(c,d)}$, then we can take $c', c<c'<d$, such that $\gamma(d')=\gamma(c')$ and find a positive shell of internal angle ($<\pi$) on the interval (c',d'). By the Theorem 2.1 and Property 3, there exists a negative shell $\gamma|_I$ on some interval I whose image is contained in the interior of the shell $\gamma|_{(c',d')}$. By the assumption (d) of the lemma, the shells on (α,β), (c',d') and I are independent. But this contradicts the Property 4.

So we may assume that $\gamma(d')$ lies on the curve $\gamma|_{(a,b)}$. Then we can take $a', a<a'<b$, such that $\gamma(a')=\gamma(d')$ and find a leaf as in Fig. 3.2 on the interval (a,a') between (d,d').

Case 2 — Next we consider the case $\gamma|_{(d,a+l)}$ contained in D. If $\gamma|_{(d,a+l)}$ is a shell, by the Property 2, it must be a positive shell of internal angle ($<\pi$). By Theorem 2.1 and Property 2, there exists a negative shell $\gamma|_I$ on some interval I whose image is contained in the interior of the shell $\gamma|_{(d,a+l)}$. By the assumption d of the lemma, the shells on (α,β), $(d,a+l)$ and I are independent. But this contradicts the Property 4. Hence $\gamma|_{(d,a+l)}$ is not a shell. Then by Lemma 3.2, there exist numbers d' and a' $(a'<a<d<d')$ such that

(*a*) $\gamma(a')=\gamma(d')$ $(=p')$, $\gamma(a)=\gamma(d)$ $(=q')$.

(*b*) $\gamma|_{(a',a)}$ and $\gamma|_{(d,d')}$ have no intersection on itself and each other.

(*c*) The internal angle at the point q' of the domain D' bounded by $\gamma|_{[a',a]}$ and $\gamma|_{[d,d']}$ are less than π.

If the internal angle at the point p' is also less than π, then we find a leaf as in Fig. 3.2 on the interval (a',a) between (d,d'). Finally, we consider the case the internal angle at the point p' is greater than π. Then it is easily seen that to closed intervals $[a',a]$ and $[d,d']$ also satisfy the assumptions of Lemma 3.3. Thus we can apply this argument again for the intervals $[a',a]$ and $[d,d']$. Since the number of self-intersections of γ is finite, we find a leaf as in Fig. 3.2 by repeating the argument.

PROOF OF THEOREM 3.2 : Suppose that there exists a semi-simple closed curve which bounds an immersed surface of genus $g>0$. Let m be the minimum number of intersection point of such curves. Since $g\neq 0$, the integer m is positive and we can assume that γ represents the one of absolutely minimum self-intersection m. By Property 3, we can take an interval (α, β) on which γ has a negative shell of internal angle $\mu < \pi$.

If $\gamma|_{[\beta, \alpha+l)}$ is also a shell, γ has only one self-intersection. But this contradicts the fact that any closed curve which bounds immersed surface has even intersections. Hence $\gamma|_{[\beta, \alpha+l)}$ is not a shell and by Lemma 3.2, there exists numbers a and b $(a < \alpha < \beta < b)$ such that

(*a*) $\gamma(a) = \gamma(b) = p$, $\gamma(\alpha) = \gamma(\beta) = q$.

(*b*) $\gamma|_{(a, \alpha)}$ and $\gamma|_{(\beta, b)}$ have no self-intersection on itself and each other.

(*c*) The internal angle at the point q of the domain D bounded by $\gamma|_{(a, \alpha)}$ and $\gamma|_{(\beta, b)}$ are less than π (Fig. 3.5).

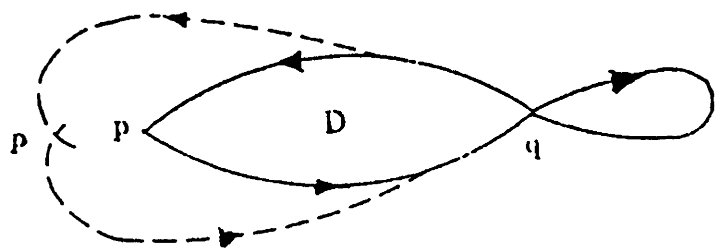

FIG. 3.5

If the internal angle at the point p is also less than π, then certain two intersections can be cancelled by Lemma 3.1. This contradicts the fact that γ has absolutely minimum intersection. Thus the internal angle at the point p is greater than π. Then it is easily seen that two closed intervals $[a, \alpha]$ and $[\beta, b]$ satisfy all the assumptions of Lemma 3.3. Thus we can find a leaf as in (Fig. 3.2), and cancel two intersections. This is also a contradiction. Now we can conclude that there is no semi-simple closed curve which bounds immersed surface of genus $g > 0$.

ACKNOWLEDGEMENT

Authors thank to the referees for all their helpful suggestions.

REFERENCES

1. U. Pinkall, *Aeuqat. math.*, **34** (1987), 221-30.
2. G. Cairns, M. Ozdemir and E. H. Tjaden, *Topology*, **31**(3) (1992), 557-58.
3. S. B. Jackson, *Bull. Amer. Math. Soc.*, **50** (1944), 564-78.
4. H. Whitney, *Compositio Math.*, **4** (1937), 276-86.
5. G. Cairns, M. McIntyre and M. Ozdemir, *Bull. London Math. Soc.*, **25** (1993), 169-76.
6. R. Osserman, *Amer. Math. Monthly*, **92** (1985), 332-37.
7. J. Alexander, *Trans. Amer. Math. Soc.*, **30** (1928), 275-306.
8. L. H. Kauffiman and T. F. Banchoff, *Amer. Math. Monthly*, **84** (1977), 168-85.