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DISTANCE GENERATED BY A FUZZY COMPATIBILITY

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Our three-dimensional Euclidean space is an example of a common existence of crisp and fuzzy structures. Although we have a crisp distance available, the notion of nearness also makes sense. More precisely, it is possible to distinguish at least two levels of fuzzy structures: nearness of points (that is connected to our perception) and our expressions about the points (that is connected to the language we use). This is a motivation for our work that studies a kind of fuzzy metric based on a given crisp one. The topology naturally induced by this metric is also investigated.

Key Words: Metric; Fuzzy Compatibility; Fuzzy Topology

1. BASIC NOTIONS

If a metric space (X, ρ) is given, there can be several attitudes how to identify pairs of points at some level. The theory of fuzzy sets is a tool that enables working with vague terms of a natural language, hence also an expression like "the points A and B are quite close to each other" may be a subject of a rigorous research. Of course, it is not a statement of a classical logic, hence it is not possible to assign the truth value 0 or 1 to such an expression. The idea of the fuzzy set theory is to assign "truth values" not only from the set $\{0, 1\}$, but from the unit interval [0; 1]. In other words, a fuzzy subset of a set X is a mapping $f: X \to [0; 1]$. This can be considered as a generalization of the characteristic functions for usual sets. (The term crisp sets is often used to distinguish between fuzzy sets and usual ones).

A fuzzy set A can be identified with the expression "x has the property A". So far example

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the fuzzy set $A :]0, \infty[\to [0;1]$ such that $A(x) = \max\left\{0; 1 - \frac{1}{x}\right\}$ can be (in some context) understood as the fuzzy set of all large numbers. At the same time we can identify its values with degrees of being true of the statement "x is a large positive number". Hence the fuzzy set theory is closely connected to the fuzzy logic, for example the inequality $A(x) \leq B(x)$ for all x can be interpreted as the implication "if an element is A, then it is B".

In fuzzy logic the conjunction of two vague statements (and the inter-section of two fuzzy sets in fuzzy sets theory) is not unique, but depends on the triangular norm (t-norm) used in a particular situation. A triangular norm is a commutative, associative, non-decreasing mapping $T: [0; 1]^2 \to [0; 1]$, such that $T(x, 1) = x$ for each $x \in [0; 1]$. The intersection is then defined by the formula $A \cap_T B(x) = T(A(x), B(x))$. The triangular norms and their properties are in details studied in[9].

The best known fuzzy relation to model approximate equality is probably the relation E of fuzzy T-equality or fuzzy T-equivalence (where T is a given triangular norm), studied e.g. by De Beats and Mesiar in[2]. By fuzzy T-equivalence for a given t-norm T on the set X we mean a mapping $E: X^2 \to [0; 1]$, such that for each $x, y, z \in X$ there is $E(x, x) = 1$, $E(x, y) = E(y, x)$ and $T(E(x, y), E(y, z)) \leq E(x, z)$. However, the relation of fuzzy equivalence used in fuzzy logic is not always suitable when considering distances because of its last property, corresponding to transitivity of a classical equivalence relation. Surely, if a point x is near to y and y is near to z,then x need not to be near to z.

An approach similar to ours was used also in the work by De Beats, Mareš and Mesiar in[1], where fuzzy equivalence is generated by a metric.

A more suitable relation for dealing with distances could be that of resemblance introduced and studied in papers by Kerre and de Cock in[4&5] for general metric spaces and also (called fuzzy nearness) by Kalina, Dobrakovová and Janiš in[6,7&8] for real numbers. Here the transitivity is replaced by a weaker condition.

We try to deal with the problem in a more general frame and assume that fuzzy compatibility is given on X, i.e., a function $C: X^2 \to [0; 1]$ fulfilling the properties $C(x, x) = 1$ (reflexivity) and $C(x, y) = C(y, x)$ (symmetry) for each $x, y \in X$. All the above mentioned relations (T-equivalence, resemblance and fuzzy nearness) are examples of fuzzy compatibility.

Also we suppose that a triangular norm T is given, which will be a formal model for the conjunction, as we have described above.

2. Fuzzy Quantities Generated by a Metric

Let us suppose that a metric ρ, a compatibility C and a left-continuous triangular norm T are given on a space X.

For $a, b \in X$ and $\alpha \in [0; 1]$ we define the st $(\rho^* (a, b))_\alpha$ in the following way: it consists of all those non-negative real numbers x for which there are $a_1, b_1, a_2, b_2 \in X$ such that

$$\rho (a_1, b_1) \geq x, T(C(a, a_1), C(b, b_1)) \geq \alpha$$

and

$$\rho (a_2, b_2) \geq x, T(C(a, a_2), C(b, b_2)) \geq \alpha.$$

Lemma 1 — For any $a, b, c \in X$,

$$\rho (a, c) \in (\rho^* (a, b))_{C(b, c)}.$$

PROOF : It is sufficient to put $x = \rho(a, c), a_1 = a_2 = a, b_1 = b_2 = c$ and $\alpha = C(b, c)$ in the previous definition. The result then follows from the property $T(1, \alpha) = \alpha$ which holds for any triangular norm. □

We can see that the sets $(\rho^* (a, b))_\alpha$ are positive real intervals. Moreover, if $\alpha \geq \beta$, then $(\rho^* (a, b))_\alpha \subseteq (\rho^* (a, b))_\beta$. This condition is, however, not sufficient for these sets to be cuts of a fuzzy quantity, since they are not closed under intersections in general. To avoid this we will define a fuzzy set $\hat{\rho}(a, b)$ putting

$$\hat{\rho}(a, b)(x) = \sup \left\{ \alpha \in [0; 1]\, x \in (\rho^* (a, b))_\alpha \right\},$$

with the convention $\sup \phi = 0$.

Thus we obtain for each $a, b \in X$ a mapping $\hat{\rho}(a, b)$ from the set $[0; \infty]$ to the unit interval. The set of all such mappings will be denoted by F. Note that for each $a, b \in X$ there is $\hat{\rho}(a, b)(\rho(a, b)) = 1$. The elements of F are non-negative LR-fuzzy numbers.

Using Lemma 1, we can observe the following:

Lemma 2 — For any $a, b, c \in X$,

$$\hat{\rho}(a, b)(\rho(a, c)) \geq C(b, c).$$ □

The relationship between $\hat{\rho}$ and ρ^* is described in the following lemma. Here $(\hat{\rho}(a, b))_\alpha$ denotes the α-level (α-cut) of the fuzzy number $\hat{\rho}(a, b)$.

Lemma 3 — If $a, b \in X, \alpha \in]0; 1]$, then

$$(\hat{\rho}(a, b))_\alpha = \bigcap_{\beta < \alpha} (\rho^* (a, b))_\beta.$$

PROOF : If $x \in (\hat{\rho}(a, b))_\alpha$, then $\hat{\rho}(a, b) \geq \alpha$ which is the same as

$$\sup \{\beta; x \in (\rho^* (a, b))_\beta\} \geq \alpha$$

what is a contradiction.

Conversely, let $x \in (\rho^*(a,b))_\beta$ for all $\beta < \alpha$. Then

$$\sup \{\gamma, x \in (\rho^*(a,b))_\gamma\} \geq \alpha,$$

what means that $\rho^*(a,b)(x) \geq \alpha$ or $x \in (\hat{\rho}(a,b))_\alpha$. □

In the sequel it is shown that the mapping $\hat{\rho}: X^2 \to F$ has properties similar to those of a metric. Namely:

Proposition 1 — For each $a \in X$ the fuzzy set $\hat{\rho}(a,a)$ is a fuzzy zero, in a sense that $\hat{\rho}(a,a)(0) = 1$. □

This property follows directly from the definition of $\rho^*(a,a)$ and the definition of $\hat{\rho}$.

Moreover, it is quite easy to see that the 1-cut of $\hat{\rho}(a,a)$ is an interval with the left endpoint at zero and the right endpoint at the diameter of the set $C_a = \{x \in X, C(a,x) = 1\}$.

Proposition 2 — For any $a, b \in X$,

$$\hat{\rho}(a,b) = \hat{\rho}(b,a).$$

This is a consequence of symmetry in C, T and ρ. □

In order to show a property similar to triangular inequality we have to note that we use the Zadeh's extension principle to add two elements of F with the convention $\sup \phi = 0$. We briefly recall this principle in its simple form (for more details see[10]).

If A, B are fuzzy subsets of a real line and $*$ is a binary operation defined on the set of all real numbers and T is a triangular norm, then

$$(A_{*T} B)(x) = \sup \{T(A(y), B(z)); x = y * z\}$$

for all real x.

We will use it only for the case of the addition.

The ordering in F will be undetstood in the following way: if A, $B \in F$, then $A \leq B$ if and only if the right end-points of the cuts $(A)_\alpha$ do not exceed the right end-points of the corresponding cuts $(B)_\alpha$. Considering this ordering we will prove a statement similar to the triangle inequality in a metric space.

Proposition 3 — For any $a, b, c \in X$,

$$\hat{\rho}(a,b) \leq \hat{\rho}(a,c) +_T \hat{\rho}(c,b).$$

PROOF : First we will show that the right end-points of the cuts $\rho^*(a,b)$ do not exceed the right end-points of the corresponding cuts of $\hat{\rho}(a,c) +_T \hat{\rho}(c,b)$.

Let $\alpha \in [0;1]$, let $x \in (\rho^*(a,b))_\alpha$. Then there are $a_1, a_2, b_1, b_2 \in X$ such that

$$\rho(a_1, b_1) \le x \le \rho(a_2, b_2)$$

and

$$T(C(a, a_1), C(b, b_1)) \ge \alpha, T(C(a, a_2), C(b, b_2)) \ge \alpha.$$

We will show that for any such x we can find a number $y \in R, y \ge x$ such that y will belong to the set $((\hat{\rho}(a, c) +_T \hat{\rho}(c, b))_\alpha$.

Let us take $y = \rho(a_2, c) + \rho(c, b_2)$. From the triangle inequality for ρ we get $y \ge \rho(a_2, b_2) \ge x$.

Next, we have

$$\begin{aligned}&(\hat{\rho}(a, c) +_T \hat{\rho}(c, b))(y)\\ &\quad= \sup\left\{T(\hat{\rho}(a, c)(u), \hat{\rho}(c, b)(\upsilon)), y = u + \upsilon\right\}\\ &\quad\ge T(\hat{\rho}(a, c)(\rho(a_2, c)), \hat{\rho}(c, b)(\rho(c, b_2)))\\ &\quad\ge T(C(a, a_2), C(b, b_2)) \ge \alpha.\end{aligned}$$

The first inequality follows from the property of the supermum, the next one from Lemma 2.

Thus the right end-pooints of the cuts of $\rho^*(a, b)$ do not exceed the right end-points of the corresponding cuts of $\hat{\rho}(a, c) +_T \hat{\rho}(c, b)$.

Now let us assume that there is an α such that the right end-point of $(\hat{\rho}(a, b))_\alpha$ exceeds the right end-point of $(\hat{\rho}(a, c) +_T \hat{\rho}(c, b))_\alpha$. Let us denote the former one by x and the later on by y. Take an arbitrary point z between x and y. Then

$$(\hat{\rho}(a, c) +_T \hat{\rho}(c, b))(z) = \gamma < \alpha.$$

Take an arbitrary $\beta \in (\gamma, \alpha)$. Then due to Lemma 3, z belongs to $(\rho^*(a, b))_\beta$ but exceeds the right end-point of $(\hat{\rho}(a, c) +_T \hat{\rho}(c, b))_\beta$ what contradicts previously proved statement. □

The propositions proved above thus justify to call the mapping $\hat{\rho}$ by the term of fuzzymetric generated by ρ and C, considering a triangular norm T. Shortly we will speak of a fuzzy metric.

3. Topological Properties

A concept of fuzzy topological space was defined in the early paper of Chang[3]. This concept has been investigated and developed in a series of papers published up to now. Fuzzy topology T on a set S is defined as a family of fuzzy sets on S satisfying analogous properties as a crisp topology. The ordered pair (S, T) was called fuzzy topological space.

We recall the definition of fuzzy topology given by Chang in [3].

Let S be a non-empty set. A fuzzy topology on S is a family T of fuzzy sets on S, satisfying the following conditions:

(a) $\overline{\phi}, S \in T$;

(b) If $\overline{A}, \overline{B} \in T$, then $\overline{A} \cap \overline{B} \in T$;

(c) If $\overline{A}_i \in T$, for $i \in I$, then $\bigcap_{i \in I} \overline{A}_i \in T$.

In the following the define a fuzzy toplogy, corresponding to the fuzzy metric $\hat{\rho}$ in a natural way.

For every $a \in X$ asnd every x a non-negative real number we define a fuzzy set A_a^x on X as follows:

$$A_a^x(b) = \sup \{\alpha \in [0;\ 1];\ \text{there exists } a_1, b_1; \rho(a_1, b_1) < x;$$
$$C(a, a_1) = \beta, C(b, b_1) = \gamma, T(\beta, \gamma) \geq \alpha\}$$

By the commutativity we obtain $A_a^x(b) = A_b^x(a)$.

Moreover, from $x \leq y$ it follows that $A_a^x \leq A_a^y$.

Obviously, if $\rho(a, b) < x$, then $A_a^x(b) = 1$. Hence, the following statement is evident.

Proposition 4 — The family of fuzzy sets

$$S = \left\{A_a^x(b) : a \in X, x \in R \text{ and } x > 0\right\}$$

is a sub-basis of a fuzzy topology.

Now we take all the finite intersections of elements from S. The obtained family is a basis of fuzzy topology. By the construction, fuzzy topology τ is uniquely determined by the family S.

Taking into consideration the following proposition, we call the fuzzy topology τ topology induced by the fuzzy metric $\hat{\rho}$.

We show that the above topology τ isin close connection with the fuzzy metric $\hat{\rho}$:

Proposition 5 —

$$A_a^x(b) = \sup_{z < x} \hat{\rho}(a, b)(z).$$

PROOF : Let $a, b \in X$ and ˇ a non-negative real number. We consider two subsets of [0, 1]:

$$F_1 = \{\alpha \in [0;\ 1];\ \text{there exists } a_1, b_1; \rho(a_1, b_1) < x;$$
$$C(a, a_1) = \beta, C(b, b_1) = \gamma; T(\beta, \gamma) \geq \alpha\}$$

and

$$F_2 = \{\alpha \in [0;\ 1];\ z \in (\rho^* (a, b))_\alpha \text{ for } z < x\}$$

$$= \alpha \in [0;\ 1];\ \mid \text{there are } a_1, b_1, a_2, b_2 \in X \text{ such that } \rho (a_1, b_1) \geq z,$$

$$T (C (a, a_1), C (b, b_1)) \geq \alpha \text{ and } rhgo\ (a_2, b_2) \leq z, T (C (a, a_2), C (b, b_2)) \geq \alpha\ \}.$$

Suppose that $\alpha \in F_1$. Then, there are $a_1, b_1 \in X$ such that

$$\rho (a_1, b_1) < x;\ C (a, a_1) = \beta, C (b, b_1) = \gamma;\ T (\beta, \gamma) \geq \alpha.$$

We will prove that $\alpha \in F_2$. Indeed, we denote $\rho (a_{1,} b_1)$ by z, and chose a_1, b_1, a_2 and b_2 to be a_1, b_1, a_1 and b_1. In this way, the conditions

$$\rho (a_1, b_1) \geq z, T (C (a, a_1), C (b, b_1)) \geq \alpha$$

and

$$\rho (a_2, b_2) \geq z, T (C (a, a_2), C (b, b_2)) \geq \alpha$$

are satisfied, and $\alpha \in F_2$.

Therefore, $F_1 \subseteq F_2$ and the supremum of F_1 is less than the supremum of F_2. Thus, we proved

$$A^x_a (b) \leq \sup_{z < x} \hat{\rho} (a, b) (z).$$

To prove the converse equality, suppose that $\alpha \in F_2$. There are $z < x$ and a_1, b_1, a_2 and b_2 such that

$$\rho (a_1, b_1) \geq z, T (C (a, a_1), C (b, b_1)) \geq \alpha$$

and

$$\rho (a_2, b_2) \leq z, T (C (a, a_2), C (b, b_2)) \geq \alpha.$$

Since $z < x$ and since there are elements a_2, b_2 such that

$$\rho (a_2, b_2) \leq z < x, T (C (a, a_2), C (b, b_2)) \geq \alpha$$

we have $\alpha \in F_1$. Hence, $F_2 \leq F_1$ and we prove the opposite equality. □

For every fixed $\alpha \neq 0$ there is a level topology defined in a usual way, taking all α-level sets of fuzzy sets from τ. We denote an α level topology bt τ_α.

Now, under assumption that the fuzzy compatibility C satisfies $C (x, y) = 1$ if and only if $x = y$, we state the following.

Proposition 6 — The 1-level topology of the fuzzy topology τ is Housdorff.

PROOF : Let $a, b \in X$, and $a \neq b$. Let $\rho(a, b) = x$. Let $A_{a,1}^{\frac{x}{2}}$ be the 1 level of the fuzzy set $A_a^{\frac{x}{2}}$, and $A_{b,1}^{\frac{x}{2}}$ be the 1 level of the fuzzy set $A_b^{\frac{x}{2}}$.

Suppose that $c \in A_{a,1}^{\frac{x}{2}} \cap A_{b,1}^{\frac{x}{2}}$. Thus, $A_a^{\frac{x}{2}}(c) = 1$ and $A_b^{\frac{x}{2}}(c) = 1$. By the conditions on T and C, $\rho(a, c) < \frac{x}{2}$ and $\rho(b, c) < \frac{x}{2}$, hence $\rho(a, c) + \rho(b, c) < x$, which contradicts the assumption $\rho(a, b) = x$. □

In the sequel, we show that the induced fuzzy topology is in a sense a T_1 fuzzy topology in case when X is finite.

For $a, b \in X$, we say that a is not equal to b on a level α (and denote this by $a \neq_\alpha b$), if

$$\alpha = \{\inf \delta : \text{for all } \beta, \gamma \in [0, 1] \text{ such that } T(\beta, \gamma) \geq \delta,$$

$$\{d \in X ; C(a, d) = \beta\} \cap \{d \in X ; C(b, d) = \gamma\} = \phi\}.$$

Proposition 7 — For every $\alpha \in (0, 1]$, if $a \neq_\alpha b$, then there are open sets O_1 and O_2 in τ_α, such that $a \in O_1, b \in O_2, a \notin O_2, b \notin O_1$.

PROOF : Let $a \neq_\alpha b$. Let $r = \min \rho(a_1, b_1)$, where $a_1 \in \{d \mid C(a, d) = \beta\}$ and $b_1 \in \{d \mid C(b, d) = \gamma\}$ for some $\beta, \gamma \in [0, 1]$ such that $T(\beta, \gamma) \geq \alpha$. Under the assumption that X is finite, such a non-zero minimum exists. By the assumption,

$$\{d \in X ; C(a, d) = \beta\} \cap \{d \in X ; C(b, d) = \gamma\} = \phi.$$

Now, we consider fuzzy sets $A_a^{\frac{r}{2}}$ and $A_b^{\frac{r}{2}}$ and their α levels (which are elements of topology T_α).

We prove that $b \notin A_{a,\alpha}^{\frac{r}{2}}$.

If $b \notin A_{a,\alpha}^{\frac{r}{2}}$, then there exist a_1, b_1 for which

$$\rho(a_1, b_1) < \frac{r}{2} ; C(a, a_1) = \beta, C(b, b_1) = \gamma, T(\beta, \gamma) \geq \alpha,$$

which contradicts the assumption that $a \neq_\alpha b$.

Similarly, we prove that $a \notin A_{b,\alpha}^{\frac{r}{2}}$. □

4. CONCLUSION

In this paper we have introduced a new type of fuzzy metric, generated by a crisp metric and a fuzzy compatibility, modelling fuzzy fonjunctionby a left-continuous t-norm. A fuzzy topology naturally induced by the new metric is also defined and investigated. It turned out that the 1-level topologyof this fuzzy topology is Hausdorff. In finite case, this topology is a T_1 fuzzy topology.

Besides of a theoretical significance, the introduced notions can be applied in various fields in which different types of measures of dissimilarity and distance are applicable, e.g., in biology (taxonomy), sociology and other sciences which use imprecise statements about precisely defined objects.

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