*Indian J. pure appl. Math.*, **35**(6) : 747-757, June 2004


# MINIMAL REDUCING SUBSPACES OF THE UNILATERAL SHIFT OPERATOR ON AN OPERATOR WEIGHTED SEQUENCE SPACE

MUNMUN HAZARIKA

*Department of Mathematical Sciences, Tezpur University, Napaam* 784 028
*e-mail: munmum@tezu.ernet.in*

AND

S. C. ARORA

*Department of Mathematics, University of Delhi, Delhi* 110 007

(*Received* 8 *May* 2003; *accepted* 19 *February* 2004)

A complete description of the minimal reducing subspaces of the unilateral shift operator on a weighted sequence space is obtained.

**Key Words: Operator Weighted Shift; Operator Weighted Sequence Space; Reducing Subspace; Minimal Reducing Subspace**

## 1. INTRODUCTION

The purpose of this paper is to give a complete description of the (minimal) reducing subspaces of the unilateral shift operator on a weighted sequence space. Shift operators have been studied extensively and the related bibliography is quite elaborate. Here we mention two excellent survey[1,2] and which contain many results and references related to the topic of this paper. Recently Stessin and Zhu[3] have classified the reducing subspaces of weighted unilateral shift operators of finite multiplicity. Their results can be easily verified to be a particular finite dimensional case of the results that we establish in this paper.

We begin with a brief description of the operator weighted sequence space to be considered. Let $\left\{B_n\right\}_{n=0}^{\infty}$ be a uniformly bounded sequence of positive invertible self adjoint operators on a separable complex Hilbert space $K$ such that $\left\{B_n^{-1}\right\}_{n=0}^{\infty}$ is also uniformly bounded. We consider the space of vectors $H^2(B) = \{f = (f_0, f_1, \ldots,) : f_i \in K \text{ and } \sum_{i=0}^{\infty} || B_i f_i ||^2 < \infty\}$ with inner product $< f, g>_B = \sum_{n=0}^{\infty} < B_n f_n, B_n g_n >$ and norm $\|f\|_B = \left(\sum_{n=0}^{\infty} || B_n f_n ||^2\right)^{\frac{1}{2}}$. Then $H^2(B)$ is an operator valued weighted sequence space. In fact, $H^2(B)$ is a Hilbert space with respect to the given inner product.

Let $\left\{e_i\right\}_{i=0}^{\infty}$ be an orthonormal basis for $K$. Also let $S$ denote the unilateral shift operator on $H^2(B)$. We denote by $x^i y^j$ the sequence in $K$ that has $e_i$ as the $(j+1)^{th}$ entry and zero as all other entries. Then $\left\{x^i y^j\right\}_{i,j=0}^{\infty}$ is an orthonormal basis for $H^2(B)$, and $S(x^i y^j) = x^i y^{j+1}$ for all $i$ and $j$.

In this paper, we shall make our study for the case where each $B_n$ is a positive diagonal operator on $K$ with the diagonal elements as $\left\{\beta_i^{(n)}\right\}_{i=0}^{\infty}$ respectively. Obviously, $\beta_i^{(n)} > 0$ for all $i$ and $n$.

The paper is primarily concerned with specifying the minimal reducing sub-spaces of the unilateral shift operator $S$ on $H^2(B)$. A closed subspace $X$ of $H^2(B)$ is said to be an invariant subspace of $S$, if $S$ maps $X$ into itself. $X$ is said to be a reducing subspace of $S$ if $X$ is invariant under both $S$ and its adjoint $S^*$. A reducing subspace $X$ of $S$ in $H^2(B)$ will be called minimal if the only reducing subspaces of $S$ contained in $X$ are $\{0\}$ and $X$.

A weight sequenct $\{B_n\}$ is of type I if for each pair of distinct non negative integers $m$ and $n$ $(m \neq n)$ there exists some positive integer $k$ such that $\frac{\beta_m^{(k)}}{\beta_m^{(0)}} \neq \frac{\beta_n^{(k)}}{\beta_n^{(0)}}$.

A weight sequence $\{B_n\}$ is of type II if it is not of type I. Thus $\{B_n\}$ is of type II if there exists distinct non-negative integers $m$ and $n$ $(m \neq n)$ such that $\frac{\beta_m^{(k)}}{\beta_m^{(0)}} \neq \frac{\beta_n^{(k)}}{\beta_n^{(0)}}$ for every positive integer $k$.

We can define a relation ~ on the set $\{0, 1, 2, ...\}$ as follows:

Two non-negative integers $m$ and $n$ are said to be $B$-related (denoted by $m \sim n$) if for every positive integer $k$, we have $\frac{\beta_m^{(k)}}{\beta_m^{(0)}} \neq \frac{\beta_n^{(k)}}{\beta_n^{(0)}}$.

Then ~ is an equivalence relation on the set $\{0, 1, 2, ...\}$.

A weight sequence $\{B_n\}$ of type II is said to be of type III if ~ partitions $\{0, 1, 2, ...\}$ into a finite number of equivalence classes.

We have $x^n = (e_n, 0, 0, ...)$ is in $H^2(B)$ for each $i$. Let $X_n = Span\ \{x^n y^k : k = 0, 1, 2, ...\}$ where we use Span to denote the closed linear span of a set in a Hilbert space. From Lemma 2.2 in the next section, it is obvious that each $X_n$ is a reducing subspace of $S$ in $H^2(B)$.

We can now state our main results.

***Theorem*** A — *If* $\{B_n\}$ *is of type I, then* $X_n$*'s are the only reducing subspaces of* $S$ *in* $H^2(B)$.

***Theorem*** B — *If* $\{B_n\}$ *is of type II, then* $S$ has minimal reducing subspaces other than the $X_n$'s. In fact, S has infinitely many reducing subspaces each generated by some $F = F_0(x)$ which is transparent.

***Theorem*** C — *If* $\{B_n\}$ *is of type III, then every reducing subspace of* $S$ *in* $H^2(B)$ *must contain a minimal reducing subspace.*

## 2. Transparent Sequences

We begin the section with a few definitions.

*Definition* 2.1 — Let $p$ be a polynomial of degree $n, p(x) = \sum_{k=0}^{n} \alpha_k x^k$. $p$ is said to be transparent if for any two non-zero coefficients $\alpha_i$ and $\alpha_j$ of $p$ we have $i \sim j$.

*Definition* 2.2 — $f(x) = \sum_{k=0}^{\infty} \alpha_k x^k$ is said to be transparent if $p_n(x) = \sum_{k=0}^{\infty} \alpha_k x^k$ is transparent for every $n$.

*Definition* 2.3 — If $F = (f, 0, 0, \ldots)$ is in $H^2(B)$ where $f = \sum_i \alpha_i e_i$, then we denote $\sum_i \alpha_i e^i$ as $f(x)$. Using this notation we can write $F = f(x)$. Thus $F = f(x)$ in $H^2(B)$ is transparent if $f(x)$ is transparent as in Definition 2.2.

*Definition* 2.4 — For a non-zero polynomial $p(x) = \sum_{k=0}^{n} \alpha_k x^k$ of degree $n$, order of zero of $p$ at the origin is said to be $m$ if $p^{(m)}(0) = m!\,\alpha_m \neq 0$ but $p^{(k)}(0) = 0\ \forall 0 \leq k < m$.

*Definition* 2.5 — For $f(x) = \sum_{k=0}^{\infty} \alpha_k x^k$, order of zero of $f$ at the origin is said to be $m$ if $f^{(m)}(0) \neq 0$ but $f^{(k)}(0) = 0\ \forall 0 \leq k < m$.

*Definition* 2.6 — If $X = \{F \in H^2(B) : F = f(x)\}$, then $m$ is said to be the order of zero of $X$ at the origin if there exists some $F = f(x) \in X$ such that $f^{(m)}(0) \neq 0$ but $g^{(k)}(0) = 0$ for all

$G = g(x) \in X$ and $0 \le k < m$.

*Definition* 2.7 — If $X$ is a reducing subspace of $S$ in $H^2(B)$, then $x^i \in X$ if and only if $x^i y^j \in X$ $\forall j$. *The non-negative integer* $m$ is said to be the order of zero of $X$ at the origin if there exists $F = (f_0, f_1, \ldots)$ in $X$ with $f_0^{(m)}(0) \neq 0$ and for all $G = (g_0, g_1, \ldots) \in X$, $g_0^{(k)}(0) = 0$ $\forall 0 \le k < m$.

Next we prove a few lemmas in this section.

*Lemma* 2.1 — If ~ partitions the set {0, 1, 2, ...} into equivalence classes $\Omega_1, \Omega_2, \ldots$, then for each $k$, $p_k(z) = \sum_{i \in \Omega_k} \alpha_i z^i$ is transparent.

The proof being obvious is omitted.

*Lemma* 2.2 — For non negative integers $i$ and $k$,

$$S^*(x^i y^k) = \begin{cases} 0 & \text{if } k = 0 \\ \left(\dfrac{\beta_i^{(k)}}{\beta_i^{(k-1)}}\right)^2 x^i y^{k-1} & \text{if } k > 0. \end{cases}$$

PROOF : Let $F = (f_0, f_1, \ldots) \in H^2(B)$ where $f_k = \sum_{i=0}^{\infty} \alpha_i^{(k)} e_i$ for $k = 0, 1, 2, \ldots$ .

Since $\langle SF, x^i \rangle = 0$ $\forall i$, so $S^* x^i = 0$ $\forall i$.

Again $\langle SF, x^i y \rangle = \langle B_1 f_0, B_1 e_i \rangle = \left(\beta_i^{(1)}\right)^2 \alpha_i^{(0)}$

and

$$\langle F, \left(\frac{\beta_i^{(1)}}{\beta_i^{(0)}}\right)^2 x^i \rangle = \langle B_0 f_0, \left(\frac{\beta_i^{(1)}}{\beta_i^{(0)}}\right)^2 B_0 e_i \rangle = \left(\beta_i^{(1)}\right)^2 \alpha_i^{(0)}$$

So $\langle SF, x^i y \rangle = \langle F, \left(\dfrac{\beta_i^{(1)}}{\beta_i^{(0)}}\right)^2 x^i \rangle$ which gives $S^*(x^i y) = \left(\dfrac{\beta_i^{(1)}}{\beta_i^{(0)}}\right)^2 x^i$

Similarly it can be proved that $S^*(x^i y^k) = \left(\dfrac{\beta_i^{(k)}}{\beta_i^{(k-1)}}\right)^2 x^i y^{k-1}$ $\forall$ k > 0. □

Similarly it can be proved that $S^*(x^i y^k) = \left(\dfrac{\beta_i^{(k)}}{\beta_i^{(k-1)}}\right)^2 x^i y^{k-1}\ \forall\ k > 0.$ □

*Lemma* 2.3 For any non-negative integer $k$, $(S^k)^* S^k (x^i y^j) = \left(\dfrac{\beta_i^{(j+k)}}{\beta_i^{(j)}}\right)^2 x^i y^j.$

PROOF : $(S^k)^* S^k (x^i y^j) = (S^k)^* (x^i y^{j+k}) = \left(\dfrac{\beta_i^{(j+k)}}{\beta_i^{(j)}}\right)^2 x^i y^j$, which we get by applying Lemma 2.2, $k$ number of times. □

*Lemma* 2.4 — If $F = f(x)$ in $H^2(B)$ is transparent and if order of zero of $F$ at the origin is $m$, then $(S^k)^* S^k (F) = \left(\dfrac{\beta_m^{(k)}}{\beta_m^{(0)}}\right)^2$ $i$

PROOF : Let $F = f(x) = \sum_{i=m}^{\infty} \alpha_i x^i$. Since $F$ is transparent, so for all $i$ with $\alpha_i \neq 0$, $i \sim m$.

Therefore, $\dfrac{\beta_i^{(k)}}{\beta_i^{(0)}} = \dfrac{\beta_m^{(k)}}{\beta_m^{(0)}}$ for all positive integers $k$. So for all $i$ with $\alpha_i \neq 0$ we have $(S^k)^* S^k (x^i)$

$= \left(\dfrac{\beta_i^{(k)}}{\beta_i^{(0)}}\right)^2 x^i = \left(\dfrac{\beta_m^{(k)}}{\beta_m^{(0)}}\right)^2 x^i$. Therefore, $(S^k)^* S^k (F) = \sum_{i=m}^{\infty} \alpha_i\ (S^k)^*\ S^k (x^i) = \left(\dfrac{\beta_m^{(k)}}{\beta_m^{(0)}}\right)^2$

$\sum_{i=m}^{\infty} \alpha_i x^i \left(\dfrac{\beta_m^{(k)}}{\beta_m^{(0)}}\right)^2 F.$ □

*Definition* 2.8 — Let $\mathcal{S}$ be a vector space consisting of all finite linear combinations of finite products of the operators $S$ and $S^*$. For any non-zero function $F \in H^2(B)$, $\mathcal{S}F = \{TF : T \in \mathcal{S}\}$. Then the closure of $\mathcal{S}F$ in $H^2(B)$ is a reducing subspace of $S$ and is denoted by $X_F$. $X_F$ is called the subsapce generated by $F$. Clearly, $X_F$ is the smallest reducing subspace of $H^2(B)$ containing $F$.

*Lemma* 2.5 — If $F = f(x)$ in $H^2(B)$ is transparent, then $X_F = Span\ \{Fy^k : k = 0, 1, 2, ...\}$.

PROOF : Let $X = Span\ \{Fy^k : k = 0, 1, 2, ...\}$. Then $Fy^k = S^k F \in X_F\ \forall k$.

So, $F \in X \subseteq X_F$. We claim that $X$ is reducing for $S$.

For any $G \in H^2(B), SG = Gy$ and $X \subseteq H^2(B)$. So $X$ is invariant under $S$. Also $S^*(x^i) = 0$ $\forall\ i$ and $F = f(x)$. So $S^*(F) = 0$. For any positive integer $k, S^*(Fy^k) = S^*S(Fy^c)$ where $c = k-1 \geq 0$. If order of zero of $F$ at the origin is $m$, then since $F$ is transparent, so by Lemma 2.4, we have $S^*(Fy^k) = \left(\dfrac{\beta_m^{(k)}}{\beta_m^{(c)}}\right)^2 Fy^c \in X$. Thus for any $G \in X, S^*G \in X$. Therefore, $X$ is reducing under $S$. Since $X_F$ is the smallest reducing subspace of $S$ containing $F$, so we must have $X = X_F$.

□

*Definition* 2.9 — Let $F = f(x) \in H^2(B)$ and $f(x) = \sum_{i=0}^{\infty} \alpha_i x^i$. Let $\Omega_1, \Omega_2, \ldots$ be the disjoint equivalence classes of $\{0, 1, 2, ...\}$. For each $k$, we define $q_k(x) = \sum_{i \in \Omega_k} \alpha_i x^i$. Dropping those $q_k(x)$ which are zero, we list the remaining ones as $\{f_1(x), f_2(x)$ in such a way that for $i < j$, ...$\}$ order of zero of $f_i(x)$ at the origin is less than the order of zero of $f_j(x)$ at the origin. The resulting decomposition $F = f_1(x) + f_2(x) + \ldots$ is called the canonical decomposition of $F$.

If there exists a finite positive integer $n$ such that $F = f_1(x) + f_2(x) + \ldots + f_n(x)$, then $F$ in the above case is said to have a finite canonical decomposition.

*Lemma* 2.6 — Let $X$ be a reducing subspace of $S$ in $H^2(B)$ and $F = f(x)$ be in $X$. If $F$ has a finite canonical decomposition $F = f_1(x) + f_2(x) + \ldots + f_n(x)$ then each $f_i(x)$ is also in $X$.

PROOF : Let $m_i$ be the order of zero of $f_i(x)$ at the origin. Then $f_i(x) = \sum_{j=m_i}^{\infty} \alpha_i^{(i)} x^j$, $\alpha_{m_i}^{(i)} \neq 0$. Also $m_1 < m_2 < \ldots < m_n$ and no two of them are $B$-related. Since $m_1$ and $m_n$ are not $B$-related, so we can choose a positive integer $k$ such that $\frac{\beta_m^{(k)}}{\beta_{m_1}^{(0)}} \neq \frac{\beta_{m_n}^{(k)}}{\beta_{m_n}^{(0)}}$. Now $(S^k)^* S^k f_i(x) = \left(\frac{\beta_{m_i}^{(k)}}{\beta_{m_i}^{(0)}}\right)^2 f_i(x)$ $\forall\, i$ and so $(S^k)^* S^k F = \sum_{i=1}^{n} \left(\frac{\beta_{m_i}^{(k)}}{\beta_{m_i}^{(0)}}\right)^2 f_i(x)$. Also, $\left(\frac{\beta_{m_n}^{(k)}}{\beta_{m_n}^{(0)}}\right)^2 F = \sum_{i=1}^{n} \left(\frac{\beta_{m_n}^{(k)}}{\beta_{m_n}^{(0)}}\right)^2 f_i(x)$. Therefore $\left(\frac{\beta_{m_n}^{(k)}}{\beta_{m_n}^{(0)}}\right)^2 F - (S^k)^* S^k F = \sum_{i=1}^{n-1} \left[\left(\frac{\beta_{m_n}^{(k)}}{\beta_{m_n}^{(0)}}\right)^2 - \left(\frac{\beta_{m_i}^{(k)}}{\beta_{m_i}^{(0)}}\right)^2\right] f_i(x)$.

Let $q(x) = \left(\frac{\beta_{m_n}^{(k)}}{\beta_{m_n}^{(0)}}\right)^2 F - (S^k)^* S^k F$. Then $q(x) \in X$. As $m_1$ and $m_{n-1}$ are not $B$-related so we can choose positiver integer $c$ such that $\frac{\beta_{m_1}^{(c)}}{\beta_{m_1}^{(0)}} \neq \frac{\beta_{m_{n-1}}^{(c)}}{\beta_{m_{n-1}}^{(0)}}$. Also, $h(x) = \left(\frac{\beta_{m_{n-1}}^{(c)}}{\beta_{m_{n-1}}^{(0)}}\right)^2 q(x) - (S^c)$

$$S^c q(x) = \sum_{i=1}^{n-2} \left[\left(\frac{\beta_{m_n}^{(k)}}{\beta_{m_n}^{(0)}}\right)^2 - \left(\frac{\beta_{m_i}^{(k)}}{\beta_{m_i}^{(0)}}\right)^2\right] \left[\left(\frac{\beta_{m_{n-1}}^{(c)}}{\beta_{m_{n-1}}^{(0)}}\right)^2 - \left(\frac{\beta_{m_i}^{(c)}}{\beta_{m_i}^{(0)}}\right)^2\right] f_i(x).$$ Then $h(x) \in X$.

Repeating this process, after $n$–1 steps we have a non-zero constant multiple of $f_1(x)$ which belongs to $X$. So $f_1(x) \in X$. Similarly it can be shown that each $f_i(x)$ is in $X$. □

## 3. AN EXTREMAL PROBLEM

***Theorem*** 3.1 — *Let $X$ be a non-zero reducing subspace of $S$ in $H^2(B)$ and let $m$ be the order of zero of $X$ at the origin. Then the extremal problem sup* $\left\{ \operatorname{Re} f_0^{(m)}(0) : F = (f_0, f_1, \ldots) \in X, \|F\| \leq 1 \right\}$

*has a unique solution* $G = G_0(x)$ *with* $\| G \| = 1$ *and* $G_0^{(m)}(0) > 0$.

PROOF : We define $\varphi : H^2(B) \to C$ as follows:

for $F = (f_0, f_1, ...) \in H^2(B)$, $\varphi(f) = f_0^{(m)}(0)$. Then $\varphi$ is a bounded linear functional on $H^2(B)$. So from elementary functional analysis, it follows that the extremal problem has a unique solution $G = (G_0, G_1, ...)$ in $X$ such that $\| G \| = 1$ and $G_0^{(m)}(0) > 0$. We claim that $G = G_0(x)$.

Let $H = \dfrac{G + SF}{\| G + SF \|}$ for any $F \in X$. Then $H \in X$. Also for $F = (f_0, f_1, ...)$, $G + SF$ = $(G_0, G_1 + f_0, G_2 + f_1, ...)$. Therefore, $H = (H_0, H_1, ...)$ where $H_0 = \dfrac{G_0}{\| G + SF \|}$ and $H_i = \dfrac{G_i + f_{i-1}}{\| G + SF \|}$ $\forall\, i > 0$. Now $\| H \| = 1$ and $H_0^{(m)}(0) = \dfrac{G_0^{(m)}(0)}{\| G + SF \|} > 0$. So by the extremality of $G$, we must have $Re\, H_0^{(m)}(0) \leq Re\, G_0^{(m)}(0)$, which implies that $\| G + SF \| \geq 1$ $\forall\, F \in X$. Thus $G \perp SX$. In particular, since $S^* G \in X$ so $\langle G, SS^* G \rangle = 0 \Rightarrow S^* G = 0$. Since $Ker\, S^* = K \oplus 0 \oplus \ldots$, so we must have $G_i = 0$ $\forall\, i > 0$. That is, $G = G_0(x)$. □

Note : The function $G$ in Theorem 3.1 will be called the extremal function of $X$.

***Theorem*** 3.1 — *If the extremal function of any reducing subspace of $S$ in $H^2(B)$ has a finite canonical decomposition then it must be transparent.*

PROOF : Let $X$ be a reducing subspace of $S$ in $H^2(B)$. Let $G = G_0(x)$ be the extremal function of $X$ with a finite canonical decomposition $G = g_1(x) + g_2(x) + \ldots + g_n(x)$. Each $g_i(x)$ is transparent and also by Lemma 2.6, each of them is in $X$. Let $m$ be the order of zero of $G_0(x)$ at the origin. Then $G_0(x) = \sum_{i=m}^{\infty} \alpha_i x^i$, $\alpha_m \neq 0$. Also $G_0^{(m)}(0) = m!\, \alpha_m$. Clearly $g_1(x)$ contains the term $\alpha_m x^m = \dfrac{G_0^{(m)}(0)}{m!} x^m$. Since $g_i(x)$'s are mutually orthogonal so we have $\| g_1(x) \| \leq \| G_0(x) \| = \| G \| = 1$. Also $g_1^{(m)}(0) = G_0^{(m)}(0)$. By extremality of $G$, we must have $G = G_0(x) = g_1(x)$. Thus $G$ is transparent. □

## 4. MINIMAL REDUCING SUBSPACES

In this section, we study minimal reducing subspaces of the operator $S$ in $H^2(B)$. Note that in general operators may have reducing subspaces that do not contain minimal reducing subspaces. Actually, there are operators which possess lots of reducing subspaces but have no minimal reducing subspaces at all. For example, the operator of multiplication by $z$ on the Lebesgue Space $L^2(\mathbb{D}, dA)$, where $dA$ is area measure.

***Theorem*** 4.1 — *Suppose $X$ is a minimal reducing subspace of $S$ in $H^2(B)$. If $F = F_0(x)$ is in $X$, then $F$ is transparent.*

PROOF : We take $m$ to be the order of zero of $F_0(x)$ at the origin and $F = F_0(x) = \sum_{i=m}^{\infty} \alpha_i x^i$. Let, if possible, $F$ be not transparent. Then there exists a positive integer $k > m$ such that $\alpha_k \neq 0$ and $k$ is not $B$-related to $m$. This implies that there exists a poisitve integer $l$ such that $\frac{\beta_k^{(l)}}{\beta_k^{(0)}} \neq \frac{\beta_m^{(l)}}{\beta_m^{(0)}}$.

Let $G = (S^l)^* S^l F - \left( \frac{\beta_m^{(l)}}{\beta_m^{(0)}} \right)^2 F$. Then $G$ is in $X$. In fact, since $F = F_0(x) = \sum_{i=m}^{\infty} \alpha_i x^i$, so by Lemma 2.4, we have, $G = \sum_{i=m+1}^{\infty} \gamma_i x^i$ where $\gamma_i = \alpha_i \left[ \left( \frac{\beta_i^{(l)}}{\beta_i^{(0)}} \right)^2 - \left( \frac{\beta_m^{(l)}}{\beta_m^{(0)}} \right)^2 \right]$. Since $\gamma_k \neq 0$ so $G \neq 0$. Thus, $0 \neq G \in X$ and so the reducing subspace $X_G$ is contained in $X$. Also $F \in X$ but $F \notin X_G$ since every function in $X_G$ is missing the term $x^m$. So we have a non-zero reducing subspace $X_G$ which is properly contained in $X$. This contradicts the minimality of $X$. Hence $F$ must be transparent. □

As an immediate corollary of the above result we have the following:

*Corollary* 4.1 — The extremal function of a minimal reducing subspace of $S$ in $H^2(B)$ is always transparent.

***Theorem*** 4.2 — *Let $X$ be a reducing subspace of $S$ in $H^2(B)$. Then $X$ is minimal if and only if there exists $F = F_0(x)$ such that $F$ is transparent and $X = X_F$ = Span $\{Fy^n : n = 0, 1, 2, ...\}$.*

PROOF : Let $X$ be minimal and $G = G_0(x)$ be the associated extremal function. Then by Corollary 4.1, $G$ is transparent. Since $X_G \subseteq X$ is also reducing, so the minimality of $X$ together with Lemma 2.5 gives $X = X_G$ = *Span* $\{Gy^n : n = 0, 1, 2, ...\}$.

Conversely, if $F=F_0(x)$ is transparent, then by Lemma 2.5, $X_F = Span\ \{Fy^n : n = 0, 1, 2, ...\}$. We claim that $X_F$ is minimal.

Let $Y$ be a non-zero reducing subspace of $S$ contained in $X_F$. Let $H=H_0(x)$ be the extremal function of $Y$. Then $H \in X_F$ and so $H = Ff(y)$ for some analytic function $f$. Hence $H=H_0(x) = Ff(y)$ which means that $f$ must be constant. Thus, $H$ is a constant multiple of $F$. In particular, $F \in$ Thus, $Y = X_F$. Hence $X_F$ must be minimal. □

*Corollary* 4.2 — Every reducing subspace of $S$ in $H^2(B)$, whose extremal function has a finite canonical decomposition, contains a minimal reducing subspace.

PROOF : Let $X$ be a reducing subspace in $H^2(B)$ whose associated extremal function $G$ has a finite canonical decomposition. By Theorem 3.2, $G$ is transparent and so $X_G$ is a minimal reducing subspace of $S$ which is contained in $X$. □

***Theorem*** 4.3 — *If the weight sequence $\{B_n\}$ is of type I, then $X_n$'s ($n$ = 0, 1, 2, ...) are the only minimal reducing subspaces of $S$ in $H^2(B)$ and the extremal function of every non-zero reducing subspace of $S$ in $H^2(B)$ is a monomial.*

PROOF : Since the weight sequence $\{B_n\}$ is of type I, so any two distinct non-negative integers $m$ and $n$ are not $B$-related. So $G=G_0(x)$ is transparent if and only if $G_0(x)=x^i$ for some non-negative integer $i$. Thus the only minimal reducing subspaces are $X_n = Span\ \{x^n y^k : k = 0, 1, 2, ...\}$ for $n = 0, 1, 2, ...$ .

***Theorem*** 4.4 — *If $\{B_n\}$ is of type II, then $S$ has minimal reducing subspaces other than the $X_n$'s. In fact, $S$ has infinitely many reducing subspaces each generated by some $F=F_0(x)$ which is transparent.*

PROOF : Since the weight sequence $\{B_n\}$ is of type II, so there exist distinct non-negative integers $m$ and $n$ such that $m \sim n$. So when the set $\{0, 1, 2, ...\}$ is partitioned into equivalence classes, there exists at least one equivalence class say $\Omega_k$ which has more than one element in it. We choose $F \in H^2(B)$ such that $F=F_0(x) = \sum_{i \in \Omega_k} \alpha_i x^i$ where more than one $\alpha_i$'s are non-zero.

Then $F_0(x)$ is transparent and $X_F$ is a minimal reducing subspace of $S$ in $H^2(B)$ such that $X_F \neq X_n$ for any $n$. □

***Theorem*** 4.5 — *In $\{B_n\}$ is of type III, then every reducing subspace of $S$ in $H^2(B)$ must contain a minimal reducing subspac*

PROOF : Since the weight sequence $\{B_n\}$ is of type III, so $\{0, 1, 2, ...\}$ is partitioned into a finite number of equivalence classes. Therefore, the extremal function of every reducing subspace of $S$ in $H^2(B)$ must have a finite canonical decomposition. So by Corollary 4.2, every reducing subspace of $S$ in $H^2(B)$ must contain a minimal reducing subspace. □

Note : As mentioned in the introduction, the paper[3] classifies the reducing subspaces of weighted unilateral shift operators of finite multiplicity. In this paper, if we take $dimK = N$ and denote $x^i y^j$ and $\beta_i^{(n)}$ as $z^{i+Nj}$ and $(\omega_{i+N_n})^{\frac{1}{2}}$ respectively, then we immediately arrive at the results established in[3].

## REFERENCES

1. D. Sarason, *Mathematical Surveys*, **13** (1974), 1-47.
2. A. L. Shields, *Mathematical Surveys*, **13** (1974), 49-128.
3. M. Stessin and K. Zhu, *Proc. Amer. Math. Soc.*, **130**(9) (2002), 2631-39.