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SUBMANIFOLDS OF COMPLEX SPACE FORMS WITH PARALLEL MEAN CURVATURE VECTOR FIELDS AND EQUAL KÄHLER ANGLES*

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We prove that a closed submanifold M of real dimension $2n$ with parallel mean curvature vector fields and equal Kähler angles, immersed into a complex projective space CP^{2n} of dimension $2n$, must be either a holomorphic or a Lagrangian submanifold, while such a submanifold immersed into a $2n$-dimensional complex Euclidean space C^{2n} must have constant Kähler angles for any positive integer n. As a corollary, we also obtain the same conclusion for a slant submanifold M^{2n} immersed into CP^{2n} without assuming it being closed, which generalizes a result of Chen and Tazawa for minimal submanifolds.

Key Words: Space Form; Parallel Mean Curvature Vector Field; Equal Kähler Angle; Slant

1. INTRODUCTION AND MAIN THEOREM

Let (N, J, g) be a Kähler manifold of complex dimension $2n$, complex structure J, Riemannian metric g, and $x : M \to N$ be an immersed submanifold M of real dimension $2n$. Denote by $\overline{\nabla}$ the Levi-Civita connection of N, we note that we use the sign convention for curvature tensors $\overline{R}$ of $N : \overline{R}(X, Y)Z = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X, Y]} Z$, where X, Y, Z are tangent vector fields of N. Denote by T_pM the tangent space of M at p. For any vector $X \in T_pM$, the angle $\theta(X)$ such that $0 \le \cos\theta(x) \le 1$, is independent of $X \in T_pM$, we call M the submanifold with equal Kähler angles (in this case, wirtinger angles agree with Kähler angles). This concept was first introduced by Chern and Wolfson[6] for real surfaces immersed into Kähler surfaces N, giving, in this case, a single Kähler angles are some functions that at each point p of M measure the deviation of the tangent space T_pM from a complex subspace $T_{x(p)}N$. The slant submanifolds introduced by B-Y. Chen[2] are just submanifolds with constant and equal Kähler angles. Holomorphic and Lagrangian submanifolds are just special slant submanifolds with $\cos\theta = 1$ and $\cos\theta = 0$, respectively.

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A natural question is to ask when the submanifold with equal Kähler angles is holomorphic or Lagrangian. Obviously there are obstructions to the existence of slant submanifolds. For instance, there does not exist totally geodesic proper slant submanifolds ($0 < \cos\theta < 1$) in non-trivial complex space forms by codazzi equations. So the above question has been exhaustedly studied in the past few years. Examples are given in complex space forms by Chen and Tazawa[5], where they proved that minimal surfaces immersed into CP^2 and CH^2 must be either holomorphic or Lagrangian surfaces. By Hopf's fibration, they[3,5] also gave the concrete examples of proper slant submanifolds immersed into complex space forms. In[9], the author also studied slant submanifolds satisfying some equalities. Making use of the weitzenböck formula for the Kähler form of N restricted to M, Wolfson[12] studied the real surfaces immersed into a Kähler surface, without the assumption that Kähler angles being constant. Using the Bochner-type technique, Salavessa and Valli[10,11] studied the same question and generalized Wolfson's theorem[12] to higher dimensions. More precisely,

Proposition 1.1 — Let $x : M^{2n} \to N^{2n}$ be a real $2n$-dimensional minimal submanifold with equal Kähler angles, immersed into a Kähler-Einstein manifold with $Ricci_N = Rg$.

1[12]. If $n = 1$, M is closed, $R > 0$, and has no holomorphic points, then x is Lagrangian.

2.[11] If $n = 2$ and $R \neq 0$, then x is either a holomorphic or a Lagrangian submanifold.

3.[11] If $n \geq 3$, M is closed, and $R > 0$, then x is either a holomorphic or a Lagrangian submanifold.

4.[11] If $n \geq 3$, M is closed, and $R = 0$, then the common Kähler angles must be constant.

Under the assumption of Proposition 1.1, it is unknown if the common Kähler angle is constant when $n = 2$ and $R = 0$. In this paper, we consider the submanifold with parallel mean curvature vector fields case, but assuming the ambient space restricting to the complex space forms. We will see that the above conclusion still holds when the holomorphic sectional curvature of the ambient space is non-negative. Namely, we have

***Theorem* 1.2** — *Let N be a 2n-dimensional complex space form with constant holomorphic sectional curvature 4c, and M a real 2n-dimensional closed submanifold with equal Kähler angles, immersed into N. If the mean curvature vector field of M is paralle, then*

1. When c > 0, M is either a holomorphic or a Lagrangian submanifold.

2. When c = 0, the common Kähler angles of M must be constant.

Remark 1.3 : 1. When restricting to complex space forms, the above theorem generalizes the conclusions in Proposition 1.1, moreover the case $n = 2$ and $c = 0$ now holds in our theorem.

2. We only need the assumption N is a complex space form in the last step of Theorem 1.2's proof (see Section 3 below). So we conjecture that Theorem 1.2 is also true for Kähler-Einstein manifolds.

In Section 2, we list some basic formulae that will be used later. The main theorem's proof is given in Section 3, together with an important corollary.

2. Some Formulas

Let (N, J, g) be a Kähler manifold of complex dimension $2n$, complex structure J, and $x : M \to N$ be an immersed submanifold M of real dimension $2n$. Denote by $\langle\, ,\rangle$ the Riemannian metric g of N compatible with the complex structure J, as well as the induced metric of M from N. We denote by $\nabla, \nabla^{\perp}, A$ and B the induced Levi-Civita connection, the induced normal connection from N, the Weingarten operator and the second fundamental form of the submanifold M, respectively. As usually, TM and $T^{\perp}M$ are the tangent and normal bundles of M in N, respectively.

For any $X, Y, Z \in TM$, the codazzi equation is given by (cf.[1]).

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = (\bar{R}(X, Y) Z)^{\perp}, \quad \text{... (2.1)}$$

where $\nabla_X B$ is defined by

$$(\nabla_X B)(Y, Z) = \overset{\perp}{\nabla}_X (B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z).$$

The Weingartén form A and the second fundamental form B are related by

$$\langle A_v, X, Y \rangle = \langle B(X, Y), v \rangle, \quad v \in T^{\perp}M.$$

For any $X \in TM$, and $v \in T^{\perp}M$, we write

$$JX = PX + NX, \quad Jv = tv + fv.$$

Where PX (resp. tv) and NX (resp. fv) denote the tangent and the normal components of JX (resp. Jv), respectively.

As the complex structure J is g orthogonal, the following are known facts (cf.[13]).

$$P^2 = -I - tN, \quad NP + fN = 0, \quad \text{... (2.2)}$$

$$Pt + tf = 0, \quad f^2 = -I - Nt. \quad \text{... (2.3)}$$

Because J is parallel with respect to ∇ in N, differentiate it along a tangent vector field $X \in TM$ and compare the tangent and normal components, we have (cf.[13]).

$$(\nabla_X P) Y = A_{NY} X + tB(X, Y), \quad \text{... (2.4)}$$

$$(\nabla_X N) Y = -B(X, PY) + fB(X, Y). \quad \text{... (2.5)}$$

Where $\nabla_X P$ and $\nabla_X N$ are the covariant derivatives of P and N, respectively defined by

$$(\nabla_X P) Y = \nabla_X (PY) - P \nabla_X Y, \quad \text{... (2.6)}$$

$$(\nabla_X N) Y = \overset{\perp}{\nabla}_X (NY) - N \nabla_X Y. \quad \text{... (2.7)}$$

Now let us assume that $x : M \to N$ is an immersion with Kähler angle θ, then (cf[2,3]).

$$\langle PX, PY \rangle = \cos^2 \theta \langle X, Y \rangle,, \quad X, Y \in TM,$$

vith $\cos \theta$ a locally Lipschitz function on M, smooth on the open set where it does not vanish. On an open set without holomorphic and lagrangian points, we can choose a locally orthonormal frame $\{e_1, ..., e_{2n}\}$ of TM, such that

$$Pe_i = \cos \theta \, e_{n+i}, \quad Pe_{n+i} = -\cos \theta \, e_i, \quad i = 1, ..., n,$$

and a local orthonormal frame $\{e_{2n+1}, ..., e_{4n}\}$ of $T^{\perp}M$ such that

$$Ne_i = \sin \theta \, e_{2n+i}, \quad Ne_{n+i} = \sin \theta \, e_{3n+i}, \quad i = 1, ..., n.$$

Obviously by the definition of t and f, using (2.2) and (2.3), we have

$$te_{2n+i} = -\sin\theta\, e_i, \quad te_{3n+i} = -\sin\theta\, e_{n+i},$$

$$fe_{2n+i} = -\cos\theta\, e_{3n+i}, \quad fe_{3n+i} = \cos\theta\, e_{2n+i}.$$

By (2.4), for any $X, Y, Z \in TM$

$$\langle (\nabla_X P)\, Y, Z \rangle = \langle A_{NY} X + tB(X, Y), Z \rangle. \qquad \text{... (2.8)}$$

The following index onvention is $\alpha, \beta, \gamma, ..., \in \{1, ..., 2n\}$ and $i, j, k, ..., \in \{1, ..., n\}$. The component of the second fundamental form is denoted by $B^{2n+\gamma}_{\alpha\beta}$. Let $X = e_\alpha$, $Y = e_k$ and $Z = e_{n+l}$ in (2.8), making use of (2.6), by a direct calculation we have

$$B^{2n+k}_{\alpha, n+l} - B^{3n+l}_{\alpha, k} = ctg\,\theta \left(\Gamma^{n+l}_{\alpha, n+k} - \Gamma^{l}_{\alpha, k} \right) + \frac{e_\alpha(\cos\theta)}{\sin\theta}\, \delta_{kl}, \qquad \text{... (2.9)}$$

where $\Gamma^{\gamma}_{\alpha\beta}$ is the connection coefficient of M, defined by

$$\nabla_{e_\alpha} e_\beta = \Gamma^{\gamma}_{\alpha\beta} e_\gamma, \quad \Gamma^{\gamma}_{\alpha\beta} = -\Gamma^{\beta}_{\alpha\gamma}$$

When N is a $2n$-dimensional complex space form with constant holomorphic sectional curvature $4c$, for any $X, Y, Z \in TM$, the curvature tensor is given by

$$\bar{R}(X, Y)Z = c\,\{\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle JY, Z\rangle JX - \langle JX, Z\rangle JY + 2\langle X, JY\rangle JZ\}. \qquad \text{... (2.10)}$$

3. Proof of Theorem 1.2

Let $L = \{p \in M \mid \cos\theta(p) = 0\}$, and L^0 denotes the largest open set contained in L.

Theorem 3.1 — *Let N be a $2n$-dimensional complex space form with constant holomorphic sectional curvature $4c$, and M a $2n$-dimensional submanifold immersed into N with equal Kähler angles. If the mean curvature vector field of M in N is parallel, then on $L^0 \cup (M - L)$*

$$\Delta \cos\theta \le -6\, c \sin^2\theta \cos\theta. \qquad \text{... (3.1)}$$

Proof : First we assume $0 < \cos\theta < 1$, so that we can choose the orthonormal frame fields given in Section 2. Define a function F by

$$F = \sum_{k=1}^{n} < Pe_k, Pe_k > = ncos^2\,\theta. \qquad \text{... (3.2)}$$

The Laplacian of F is given as follows

$$\Delta F = tr(\nabla dF) = \sum_{\alpha=1}^{2n} (e_\alpha^2 F - dF(\nabla_{e_\alpha} e_\alpha)).$$

By (2.4)~(2.8), we can do the following calculations:

$$\begin{aligned}
e_\beta e_\alpha F &= e_\beta \sum_k e_\alpha \langle Pe_k, Pe_k \rangle \\
&= \Delta_{e_\beta} \sum_k 2 \langle \Delta_{e_\alpha}(Pe_k), Pe_k \rangle \\
&= \nabla_{e_\beta} \sum_k 2 \langle (\Delta_{e_\alpha} P)e_k + P\Delta_{e_\alpha} e_k, Pe_k \rangle \\
&= \nabla_{e_\beta} \sum_k 2\left\{ \langle A_{Ne_k} e_\alpha + tB(e_\alpha, e_k), Pe_k \rangle + \langle P\nabla_{e_\alpha} e_k, Pe_k \rangle \right\} \\
&= \nabla_{e_\beta} \sum_k 2\left\{ \sin\theta \langle A_{e_{2n+k}} e_\alpha, Pe_k \rangle - \langle B(e_\alpha, e_k), NPe_k \rangle \right\} \\
&= \nabla_{e_\beta} \sum_k 2\sin\theta\cos\theta \left(B^{2n+k}_{\alpha, n+k} - B^{3n+k}_{\alpha k} \right) \\
&= \sum_k 2e_\beta(\sin\theta\cos\theta) \left(B^{2n+k}_{\alpha, n+k} - B^{3n+k}_{\alpha k} \right) \\
&\quad + \sum_K 2\sin\theta\cos\theta \left\{ \nabla_{e_\beta} B^{2n+k}_{\alpha, n+k} - \nabla_{e\beta} B^{3n+k}_{\alpha k} \right\}. \qquad \text{... (3.3)}
\end{aligned}$$

Similar calculation as above gives

$$dF(\nabla_{e_\beta} e_\alpha) = \sum_k 2\sin\theta\cos\theta$$

$$\left\{ \langle B(\nabla_{e_\beta} e_\alpha, e_{n+k}), e_{2n+k} \rangle - \langle B(\nabla_{e_\beta} e_\alpha, e_k), e_{3n+k} \rangle \right\}. \qquad \text{... (3.4)}$$

Let us denote the last two terms in (3.3) by A_1 and B_1. For A_1 we get

$$\begin{aligned}
A_1 &= \nabla_{e_\beta} B^{2n+k}_{\alpha, n+k} = \nabla_{e_\beta} \langle B(e_\alpha, e_{n+k}), e_{2n+k} \rangle \\
&= \langle \overset{\perp}{\nabla}_{e_\beta} (B(e_\alpha, e_{n+k}), e_{2n+k} \rangle + \langle B(e_\alpha, e_{n+k}), \overset{\perp}{\nabla}_{e_\beta} e_{2n+k} \rangle \\
&= \langle (\nabla_{e_\beta} B)(e_\alpha, e_{n+k}) + B(\nabla_{e_\beta} e_\alpha, e_{n+k}) + B(e_\alpha, \nabla_{e_\beta} e_{n+k}), e_{2n+k} \rangle \\
&\quad + \langle B(e_\alpha, e_{n+k}), \overset{\perp}{\nabla}_{e_\beta} e_{2n+k} \rangle
\end{aligned}$$

$$= \langle (\nabla_{e_{n+k}} B)(e_\alpha, e_\beta) + (\bar{R}(e_\beta, e_{n+k}) e_\alpha)^{\perp}$$

$$+ B(\nabla_{e_\beta} e_\alpha, e_{n+k}) + B(e_\alpha, \nabla_{e_\beta} e_{n+k}), e_{2n+k} \rangle$$

$$+ \langle B(e_\alpha, e_{n+k}), \nabla^{\perp}_{e_\beta} e_{2n+k} \rangle$$

$$= \langle \nabla^{\perp}_{e_{n+k}} (B(e_\alpha, e_\beta)) + (\bar{R}(e_\beta, e_{n+k}) e_\alpha)^{\perp} - B(\nabla_{e_{n+k}} e_\alpha, e_\beta) - B(e_\alpha, \nabla_{e_{n+k}} e_\beta)$$

$$+ B(\nabla_{e_\beta} e_\alpha, e_{n+k}) + B(e_\alpha, \nabla_{e_\beta} e_{n+k}), e_{2n+k} \rangle + \langle B(e_\alpha, e_{n+k}), \nabla^{\perp}_{e_\beta} e_{2n+k} \rangle$$

$$= \langle \nabla^{\perp}_{e_{n+k}} (B(e_\alpha, e_\beta)) + (\bar{R}(e_\beta, e_{n+k}) e_\alpha)^{\perp}, e_{2n+k} \rangle + \langle B(\nabla_{e_\beta} e_\alpha, e_{n+k}), e_{2n+k} \rangle$$

$$+ B(e_\alpha, \nabla_{e_\beta} e_{n+k}) - B(\nabla_{e_{n+k}} e_\alpha, e_\beta) - B(e_\alpha, \nabla_{e_{n+k}} e_\beta), e_{2n+k} \rangle$$

$$+ \langle B(e_\alpha, e_{n+k}), \nabla^{\perp}_{e_\beta} e_{2n+k} \rangle. \qquad \text{... (3.5)}$$

Where we have used the codazzi eq. (2.1) in the fifth equality. By a similar calculation we also have

$$B_1 = \nabla_{e_\beta} B^{3n+k}_{\alpha, k} = \nabla_{e_\beta} \langle B(e_\alpha, e_k), e_{3n+k} \rangle$$

$$= \langle \nabla^{\perp}_{e_k} (B(e_\alpha, e_\beta)) + (\bar{R}(e_\beta, e_k) e_\alpha)^{\perp}, e_{3n+k} \rangle + \langle B(\nabla_{e_\beta} e_\alpha, e_k), e_{3n+k} \rangle$$

$$+ \langle B(e_\alpha, \nabla_{e_\beta} e_k) - B(\nabla_{e_k} e_\alpha, e_\beta) - B(e_\alpha, \nabla_{e_k} e_\beta), e_{3n+k} \rangle$$

$$+ \langle B(e_\alpha, e_k), \nabla^{\perp}_{e_\beta} e_{3n+k} \rangle. \qquad \text{... (3.6)}$$

Combining (3.3), (3.4), (3.5) and (3.6) and using the definition of ΔF we have

$$\Delta F = \sum_{\alpha, k} 2e_\alpha (\sin\theta\cos\theta) \left(B^{2n+k}_{\alpha, n+k} - B^{3n+k}_{\alpha k} \right)$$

$$+ \sum_{\alpha, k} 2\sin\theta\cos\theta \{ \langle \nabla^{\perp}_{e_{n+k}} (B(e_\alpha, e_\alpha)) + (\bar{R}(e_\alpha, e_{n+k})^{\perp}, e_{2n+k} \rangle$$

$$- \langle \nabla^{\perp}_{e_k} (B(e_\alpha, e_\alpha)), + (\bar{R}(e_\alpha, e_k) e_\alpha)^{\perp}, e_{3n+k} \rangle \}$$

$$+ \sum_{\alpha, k} 2\sin\theta\cos\theta$$

$$\{ \langle (B(e_\alpha, \nabla_{e_\alpha} e_{n+k})) - 2B(\nabla_{e_{n+k}} e_\alpha, e_\alpha), e_{2n+k} \rangle$$

$$-\langle B(e_\alpha, \nabla_{e_\alpha} e_k), -2B(\nabla_{e_k} e_\alpha, e_\alpha), e_{3n+k}\rangle\}$$

$$+ \sum_{\alpha, k} 2\sin\theta\cos\theta$$

$$\{\langle B(e_\alpha, e_{n+k}), \overset{\perp}{\nabla}_{e_\alpha} e_{2n+k}\rangle - B(e_\alpha, e_k), \overset{\perp}{\nabla}_{e_\alpha} e_{3n+k}\rangle\}. \qquad \text{... (3.7)}$$

We denote the last two terms by A_2 and B_2, respectively, in the above equation. For A_2,

$$A_2 = \sum_{\alpha, k} 2\sin\theta\cos\theta \langle B(e_\alpha, e_{n+k}), \overset{\perp}{\nabla}_{e_\alpha} e_{2n+k}\rangle$$

$$= \sum_{\alpha, k} 2\sin\theta\cos\theta \langle B(e_\alpha, e_{n+k}), \overset{\perp}{\nabla}_{e_\alpha}\left(\frac{1}{\sin\theta} Ne_k\right)\rangle$$

$$= \sum_{\alpha, k} 2\sin\theta\cos\theta$$

$$\left\{\langle B(e_\alpha, e_{n+k}), \frac{-e_\alpha(\sin\theta)}{\sin^2\theta} Ne_k + \frac{1}{\sin\theta}((\nabla_{e_\alpha} N) e_k + N\nabla_{e_\alpha} e_k)\rangle\right\}$$

$$= \sum_{\alpha, k} 2\sin\theta\cos\theta \left\{\langle B(e_\alpha, e_{n+k}), \frac{-e_\alpha(\sin\theta)}{\sin\theta} e_{2n+k}\rangle\right.$$

$$\left. + \frac{1}{\sin\theta}\langle B(e_\alpha, e_{n+k}), -B(e_\alpha, Pe_k) + fB(e_\alpha, e_k) + N\nabla_{e_\alpha} e_k\rangle\right\}$$

$$= \sum_{\alpha, k}\left\{-2\cos\theta\, e_\alpha(\sin\theta)\, B^{2n+k}_{\alpha, n+k} - 2\cos^2\theta \langle B(e_\alpha, e_{n+k}), B(e_\alpha, e_{n+k})\rangle\right.$$

$$\left. + 2\cos\theta\langle B(e_\alpha, e_{n+k}), fB(e_\alpha, e_k) + N\nabla_{e_\alpha} e_k\rangle\right\}.$$

In the above calculation, we have used (2.5), (2.7) and the property of the chosen frame. Similarly,

$$B_2 = \sum_{\alpha, k} 2\sin\theta\cos\theta \langle B(e_\alpha, e_k), \overset{\perp}{\nabla}_{e_\alpha} e_{3n+k}\rangle$$

$$= \sum_{\alpha, k}\left\{-2\cos\theta\, e_\alpha(\sin\theta)\, B^{3n+k}_{\alpha, k} + 2\cos^2\theta\langle B(e_\alpha, e_k), B(e_\alpha, e_k)\rangle\right.$$

$$\left. + 2\cos\theta\langle B(e_\alpha, e_k), fB(e_\alpha, e_{n+k}) + N\nabla_{e_\alpha} e_{n+k}\rangle\right\}.$$

Therefore

$$A_2 - B_2 = \sum_{\alpha k} \left\{ -2\cos\theta\, e_\alpha (\sin\theta) \left(B^{2n+k}_{\alpha, n+k} - B^{3n+k}_{\alpha, k} \right) - 2\cos^2\theta \| B \|^2 \right.$$

$$+ \sum_{\alpha, k} 2\cos\theta \left\{ \langle B(e_\alpha, e_{n+k}), fB(e_\alpha, e_k) \rangle - \langle B(e_\alpha, e_k), f\beta(e_\alpha, e_{n+k}) \rangle \right\}$$

$$+ \sum_{\alpha, k} 2\cos\theta \left\{ \langle B(e_\alpha, e_{n+k}), N\nabla_{e_\alpha} e_k \rangle - \langle B(e_\alpha, e_k), N\nabla_{e_\alpha} e_{n+k} \rangle \right\}. \qquad \text{... (3.8)}$$

As f is skew-symmetric, inserting (3.8) into (3.7) gives

$$\Delta F = \sum_{\alpha, k} 2\sin\theta\, e_\alpha (\cos\theta) \left(B^{2n+k}_{\alpha, n+k} - B^{3n+k}_{\alpha k} \right) \qquad \text{... (3.9)}$$

$$+ \sum_{\alpha, k} 2\sin\theta\cos\theta \left\{ \langle \nabla^{\perp}_{e_{n+k}} \vec{H} + (\bar{R}(e_\alpha, e_{n+k}) e_\alpha)^{\perp}, e_{2n+k} \rangle \right.$$

$$\left. - \langle \nabla^{\perp}_{e_k} \vec{H} + (\bar{R}(e_\alpha, e_k) e_\alpha)^{\perp}, e_{3n+k} \rangle \right\}$$

$$+ \sum_{\alpha, k} 2\sin\theta\cos\theta \left\{ \langle B(e_\alpha, \nabla_{e_\alpha} e_{n+k}) - 2B(\nabla_{e_{n+k}} e_\alpha, e_\alpha), e_{2n+k} \rangle \right.$$

$$\left. - \langle B(e_\alpha, \nabla_{e_\alpha} e_k) - 2B(\nabla_{e_k} e_\alpha, e_\alpha), e_{3n+k} \rangle \right\} \qquad \text{... (3.10)}$$

$$+ \sum_{\alpha, k} 4\cos\theta \{ \langle B(e_\alpha, e_{n+k}), fB(e_\alpha, e_k) \rangle \} - 2\cos^2\theta \| B \|^2 \qquad \text{... (3.11)}$$

$$+ \sum_{\alpha, k} 2\cos\theta \{ \langle B(e_\alpha, e_{n+k}), N\nabla_{e_\alpha} e_k \rangle - \langle B(e_\alpha, e_k), N\nabla_{e_\alpha} e_{n+k} \rangle \}$$

$$\text{... (3.12)}$$

Where $\vec{H}$ is the mean curvature vector field of M, defined by $\vec{H} = trB$. Since

$$\sum_{\alpha, k} \langle B(\nabla_{e_{n+k}} e_\alpha, e_\alpha), e_{2n+k} \rangle$$

$$= \sum_{\alpha, \beta, k} \Gamma^{\beta}_{n+k, \alpha} B^{2n+k}_{\beta\alpha} = - \sum_{\alpha, \beta, k} \Gamma^{\alpha}_{n+k, \beta} B^{2n+k}_{\beta\alpha}.$$

After rearranging indices in the last term we see that this term vanishes. Similarly

$$\sum_{\alpha, k} \langle B(\nabla_{e_k} e_\alpha, e_\alpha), e_{3n+k} \rangle = 0.$$

Therefore

$$(3.10) = 2\sin\theta\cos\theta \sum_{\alpha, k} \{ \langle B(e_\alpha, \nabla_{e_\alpha} e_{n+k}), e_{2n+k} \rangle - \langle B(e_\alpha, \nabla_{e_\alpha} e_k), e_{3n+k} \rangle \}$$

$$= 2 \sin\theta \cos\theta \sum_{\alpha,\beta,k} \left\{ \Gamma^{\beta}_{\alpha, n+k} B^{2n+k}_{\alpha\beta} - \Gamma^{\beta}_{\alpha k} B^{3n+k}_{\alpha\beta} \right\}.$$

By the property of the chosen frame fields and using the basic inequality, we have

$$(3.11) = 4\cos^2\theta \sum_{\alpha,k,l} \left(B^{2n+l}_{\alpha,n+k} B^{3n+l}_{\alpha,k} - B^{3n+l}_{\alpha n+k} B^{2n+l}_{\alpha,k} \right) - 2\cos^2\theta \| B \|^{2n}$$

$$\leq 2\cos^2\theta \sum_{\alpha,k,l}$$

$$\left\{ \left(B^{2n+l}_{\alpha,n+k} \right)^2 + \left(B^{3n+l}_{\alpha,k} \right)^2 + \left(B^{3n+l}_{\alpha n+k} \right)^2 \left(B^{2n+l}_{\alpha,k} \right)^2 \right\}$$

$$- 2\cos^2\theta \| B \|^2 \leq 0. \qquad \ldots (3.13)$$

By the property of the chosen frame fields again

$$(3.12) = 2\sin\theta\cos\theta \sum_{\alpha,\beta,k} \left(B^{2n+\beta}_{\alpha,n+k} \Gamma^{\beta}_{\alpha k} - B^{2n+\beta}_{\alpha k} \Gamma^{\beta}_{\alpha n+k} \right).$$

Let $k = l$ in (2.9) we see that

$$B^{2n+k}_{\alpha,n+k} - B^{3n+k}_{\alpha k} = \frac{e_\alpha(\cos\theta)}{\sin\theta}.$$

This leads

$$(3.9) = 2 \sum_{\alpha,k} \left\{ e_\alpha(\cos\theta) \right\}^2 = 2n \| \nabla \cos\theta \|^2.$$

Inserting (3.9)~(3.12) into ΔF and using (3.2), we get

$$\Delta(n\cos^2\theta) \leq 2\sin\theta\cos\theta \sum_{\alpha,k} \left\{ \langle \nabla^{\perp}_{e_{n+k}} \vec{H}, e_{2n+k} \rangle - \langle \nabla^{\perp}_{e_k} \vec{H}, e_{3n+k} \rangle \right\}$$

$$2\sin\theta\cos\theta \sum_{\alpha,k} \left\{ \langle (\bar{R}(e_\alpha, e_{n+k}) e_\alpha)^{\perp}, e_{2n+k} \rangle - (\bar{R}(e_\alpha, e_k) e_\alpha)^{\perp}, e_{3n+k} \rangle \right\}$$

$$+ 2n \| \nabla \cos\theta \|^2 + F_1, \qquad \ldots (3.14)$$

where

$$F_1 = 2\sin\theta\cos\theta \sum_{\alpha,k}$$

$$\left\{ \Gamma^{\beta}_{\alpha,n+k} B^{2n+k}_{\alpha,\beta} - \Gamma^{\beta}_{\alpha k} B^{3n+k}_{\alpha,\beta} + B^{2n+\beta}_{\alpha,n+k} \Gamma^{\beta}_{\alpha k} - B^{2n+\beta}_{\alpha k} \Gamma^{\beta}_{\alpha,n+k} \right\}.$$

Now we assume N is a complex space form with constant holomorphic sectional curvature $4c$. By (2.10) we have

$$\sum_{\alpha, k} \langle (\overline{R}(e_\alpha, e_{n+k}) e_\alpha)^\perp, e_{2n+k} \rangle = -3\, nc \sin\theta \cos\theta,$$

$$\sum_{\alpha, k} \langle (\overline{R}(e_\alpha, e_k) e_\alpha)^\perp, e_{3n+k} \rangle = -3\, n \sin\theta \cos\theta.$$

Therefore, when the mean curvature vector field $\vec{H}$ is parallel, (3.14) reads

$$\Delta (n\cos^2\theta) \leq 2n \| \nabla \cos\theta \|^2 - 12\, nc\sin^2\theta \cos^2\theta + F_1. \quad \text{... (3.15)}$$

It is easy to see that formula (3.15) is independent of the chosen local frame fields. Thus for any $p \in M$, we can choose a local normal coordinate for M at p such that $\Gamma^\gamma_{\alpha,\beta}(p) = 0$ and so $F_1 = 0$. An easy calculation shows that

$$\Delta (n\cos^2\theta) = 2n \| \nabla \cos\theta \|^2 + 2\, n\cos\theta\, \Delta \cos\theta.$$

Plugging these into (3.15) we get

$$\Delta \cos\theta \leq -6\, c\sin^2\theta \cos\theta.$$

Generally, $\cos\theta$ is only locally Lipschitz on M, but smooth on the open set of Lagrangian points. Obviously the last formula also holds on holomorphic points and the largest open set of Lagrangian points. Therefore, it surely holds on $L^0 \cup (M - L)$.

PROOF OF THE THEOREM 1.2 — From Theorem 3.1, we see that, (3.1) is valid on all M but the set of Lagrangian points with no interior. Now M is closed and so $\cos\theta$ extends smoothly on all M, i.e. (3.1) holds on the whole submanifold M.

1. When $c > 0$, integrating (3.1) over M and noting that $\cos\theta \geq 0$, we have

$$-\int_M 6\, c\sin^2\theta \cos\theta\, dVolM \geq 0,$$

which implies either $\sin\theta = 0$, or $\cos\theta = 0$, that is M is either a holomorphic or a Lagrangian submanifold.

2. When $c = 0$, (3.1) takes the form

$$\Delta \cos\theta \leq 0.$$

By the maximum principle of Hopf, we see that $\cos\theta$ is constant and therefore M has constant Kähler angles.

When M has constant Kähler angles, as an immediate result of Theorem 3.1, we have the following corollary which generalizes a theorem of Chen and Tazawa[5].

Corollary 3.2 — Let M^{2n} be a slant submanifold with parallel mean curvature vector field, immersed into a complex projective space CP^{2n}. Then M is either a holomorphic or a Lagrangian submanifold.

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