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FINITELY COPRESENTED AND COGENERATED DIMENSIONS

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In this paper, we start to study the relations of the finitely copresented dimension of injective equivalent modules and give a shifting theorem. Moreover, we characterize the co-coherent rings with the finitely copresented dimension and give the properties of finitely copresented dimension of Co-coherent ICF-rings. At last, we introduce the concept of finitely cogenerated dimension of modules and obtain some properties.

Key Words: Finitely Copresented Modules; Finitely Copresented Dimension; Finitely Cogenerated Dimension; Co-coherent Rings; ICF-rings

1. INTRODUCTION

In[1&2], Ho Kuen Ng introduced the concept of the finitely presented dimension, for modules and commutative rings. For an R-module, A, defined the finitely presented dimension of A, *f.p.dim* A, as inf $\{n \mid$ there exists an exact sequence $P_{n+1} \to P_n \to \ldots P_0 \to A \to 0$ of R-modules, where each P_i is projective and P_{n+1}, P_n are finitely generated$\}$. If no such exact sequence exists, we say that *f.p.dim* $A = \infty$. Moreover, Ho Kuen Ng defined the finitely presented dimension of the ring R, *f.p.dim* R as sup $\{$*f.p.dim* $A \mid A$ is finitely generated R-modules$\}$. Consequently, we may regard finitely presented dimension as a measure of how far away an R-module is from being finitely presented, and the finitely presented dimension of a ring as a measure of how far away it is from being Noetherian. In[3&4], Li Yuanlin introduced the notion of finitely presented dimension of rings (may not commutative) and obtain some properties. In Wenting Tong[3] described the properties of (semi) hereditary rings, Noetherian rings, Von Neumann rings and coherent rings etc. by the weak dimension, the global dimension and the finitely presented dimension of commutative rings. In[6], Hiremath V. A. introduced the dual concept of finitely presented modules-finitely copresented modules and give the properties of co-Noetherian rings and V-rings etc. In[7], Zhanmin Zhu introduced the concept of finitely copresented dimension of modules and rings, and characterized some properties. In[8], Nanqing Ding introduced the concept of finitely generated dimension of modules and obtain some results.

In this paper, we start to study the relations of the finitely copresented dimension of injective equivalent modules and give a shifting theorem. Moreover, we characterize the co-coherent rings with the finitely copresented dimension and give the properties of finitely copresented dimension of co-coherent ICF-rings. At last, we introduce the concept of finitely cogenerated dimension of modules and obtain some results.

Throughout this paper, all rings are commutative and all modules are unitary. We use *f.c.p.dim*, *f.c.g.dim* and *i.dim* to denote the finitely copresented dimension, finitely cogenerated dimension and injective dimension respectively.

2. Finitely Copresented Dimension

Definition 2.1[4] — A module M is called cofree if M is isomorphic to the direct sum of some injective envelopes of simple modules.

Definition 2.2[1] — A module M is called finitely copresented if there exists an exact sequence $0 \to M \to N \to K \to 0$, where N is finitely cogenerated cofree, K is finitely cogenerated.

Definition 2.3[7] — Let R be a ring and M an R-module, we define the finitely copresented of M (denote by *f.c.p.dim* M) as $\inf\{n \mid$ there exists an exact sequence $0 \to M \to E_0 \to E_1 \to \ldots E_n \to E_{n+1}$, where each E_i is injective and E_n, E_{n+1} are finitely cogenerated. If no such sequence exists for any n, we say that *f.c.p.dim* $M = \infty$. We also define the finitely copresented dimension of R (denoted by *f.c.p.dim* R) as $\sup \{$*f.c.p. dim* $M \mid M$ is a finitely cogenerated R-module$\}$.

Theorem 2.4 — *R is co-noetherian if and only if f.c.p.dim $R = 0$.*

Proof : It follows from the fact that R is co-noetherian if and only if all finitely cogenerated R-modules are finitely copresented.

Theorem 2.5 — No finitely cogenerated module can have finitely copresented dimension 1.

Proof : Otherwise, let M be finitely cogenerated with *f.c.p.dim* $M = 1$. Let $0 \to M \xrightarrow{\varepsilon} E_0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} E_2$ be an exact sequence, where E_1 and E_2 are finitely cogenerated, each E_i is injective, putting $A = im f_0$, then there exists an exact sequence $0 \to M \xrightarrow{\varepsilon} E_0 \xrightarrow{f_0} A \to 0$ with A finitely cogenerated, then M is finitely copresented, i.e. *f.c.p.dim* $M = 0$. This is a contradiction.

Corollary 2.6 — No ring can have finitely copresented dimension 1.

Theorem 2.7 — *Let $0 \to E \to M \to M' \to 0$ be an exact sequence with E injective.*

(1) If E is finitely cogenerated, then f.c.p.dim M' = f.c.p.dim M

(2) If E is nonfinitely cogenerated, then f.c.p.dim $M' \leq$ *f.c.p.dim* M

(3) If E is nonfinitely cogenerated and f.c.p.dim $M' \geq 1$, then f.c.p.dim $M \leq$ *f.c.p.dim* M'.

Hence *f.c.p. dim* M' = *f.c.p.dim* M.

Proof : (1) It's the dual theorem in[1] (see Theorem 1.10[1]).

(2) If *f.c.p.dim* $M = \infty$, obviously, *f.c.p.dim* $M' \leq$ *f.c.p.dim* M. Let *f.c.p.dim* $M = n < \infty$. Since E is nonfinitely cogenerated and injective, then *f.c.p.dim* $E \geq 1$. Hence *f.c.p.dim* M = *f.c.p.dim*

$(M' \oplus E) \geq 1$. Note that there exists a finitely copresented resolution of $M : 0 \to M \to E_0 \to E_1 \to \ldots \to E_n \to E_{n+1}$, where each E_i is injective and E_n, E_{n+1} are cogenerated. It's equivalent to the following $n + 1$ exact sequences:

$$0 \to K_{n-1} \to E_n \to K_n \to 0$$

....

$$0 \to M \xrightarrow{i} E_0 \xrightarrow{\sigma_0} K_0 \to 0$$

Define $\lambda : E \to E \oplus M'$ by $e \to (e, 0)$ and that $0 \to E \xrightarrow{i\lambda} E_0 \xrightarrow{\sigma'} K_0' \to 0$ is excact, where $K_0' = im\,(i\lambda)$ is injective and $K_0' \subset K_0$.

It follows that there exists an exact sequence

$$0 \to M' \xrightarrow{\bar{i}} K_0' \to K_0 \to 0.$$

Hence, we have $n + 1$ exact sequences as follows:

$$0 \to K_{n-1} \to E_n \to K_n \to 0$$

....

$$0 \to M' \to K_0' \to K_0 \to 0$$

It follows that $0 \to M' \to K_0' \to E_1 \to \ldots \to E_n \to K_n \to 0$ is exact, where K_n, E_n are finitely cogenerated and E_i, K_0 are injective. Hence $f.c.p.dim\, M' \leq n$.

(3) If $f.c.p.dim\, M' = \infty$, obviously $f.c.p.dim\, M' \geq f.c.p.dim\, M$. If $f.c.p.dim\, M' = n < \infty$ $(n \geq 1)$, then there exists a finitely copresented resolutoin of $M' : 0 \to M' \to E_0 \to E_1 \to \ldots E_n \to E_{n+1}$ and that $0 \to M' \oplus E \xrightarrow{i \oplus 1} E_0 \oplus E \to E_1 \to \ldots \to E_n \to E_{n+1}$ is exact, where E_i and $E_0 \oplus E$ are injective and E_n and E_{n+1} are finitely cogenerated. Hence $f.c.p.dim\, M \leq n$. By (2), we have $f.c.p.dim\, M' = f.c.p.dim\, M$.

Theorem 2.8 — *Let R be a ring and M' a module and $0 \to M' \to E_0 \to K_0 \to 0$ (*) be an exact sequence, where E_0 is injective, K_0 and E_0 are finitely cogenerated.*

(1) Let K_0 be finitely copresented and E a nonfinitely cogenerated injective R-module. If $M \cong M' \oplus E$, then $f.c.p.dim\, M = 1$.

(1) Let K_0 be finitely copresented and E a nonfinitely cogenerated injective R-module. If $M \cong M' \oplus E$, then f.c.p.dim $M = 1$.

(2) If K_0 is nonfinitely copresented, then there exist M'' with f.c.p.dim $M'' = 0$ and nonfinitely cogenerated injective module E^*, such that f.c.p.dim $(M'' \oplus E^*) > 1$.

(3) R is a co-coherent ring if and only if for any $M \in f.c.p._R \mathcal{M}$ and $E \in Inj_R \mathcal{M}$, such that f.c.p.dim $(M \oplus E) \leq 1$.

PROOF : (1) Since K_0 is finitely copresented, then there exists an exact sequence $0 \to K_0 \to E_1 \to E_2$, where E_1 and E_2 are finitely cogenerated injective. By hypothesis, $0 \to M' \to E_0 \to K_0$ is exact. It follows that there exists an exact sequence

$$0 \to M' \to E_0 \to E_1 \to E_2$$

and that $0 \to M' \oplus E \to E_0 \oplus E \to E_1 \to E_2$ is exact. Since E_0 is finitely cogenerated and E is nonfinitely cogenerated, then $E_0 \oplus E$ is nonfinitely cogenerated. Moreover, E_0 and E are injective, so $E_0 \oplus E$ is injective. Hence $E_0 \oplus E$ is nonfinitely cogenerated injective. In addition, E_1 and E_2 are finitely cogenerated injective. By definition, *f.c.p.dim* $(M'' \oplus E) = 1.$, i.e. *f.c.p.dim* $M = 1$.

(2) If K_0 is nonfinitely copresented and finitely cogenerated, then *f.c.p.dim* $K_0 \geq 2$. If *f.c.p.dim* $M' > 1$, let $M'' = M', E^* = E$, then *f.c.p.dim* $(M'' + E^*) =$ *f.c.p.dim* $(M' \oplus E) > 1$.

If *f.c.p.dim* $M' = 1$, then there exists a finitely copresented resolution of $M : 0 \to M'' \to E_0' \to K_0' \to 0$ (**), where K_0' is injective and K_0' is finitely copresented. By (*), then there exists an exact sequence

$$0 \to M' \to E_0^* \to K_0 \to 0 \quad (***)$$

where $E_0^* = E_0 \oplus E$ ie injective. By Schanuel Lemma in (**) and (***), $K_0' \oplus E_0^* \cong K_0 \oplus E_0'$. Let $E^* = E_0^*, M'' = K_0'$, then *f.c.p.dim* $(M'' + E^*) =$ *f.c.p.dim* $(K_0' \oplus E_0^*) =$ *f.c.p.* $(K_0 \oplus E_0') =$ *f.c.p.dim* $K_0 > 1$.

(3) If R is a co-coherent ring, then K_0 is finitely copresented. Let M be any finitely copresented. If E is nonfinitely cogenerated injective, then *f.c.p.dim* $(M \oplus E) = 1$ by (1). If E is finitely cogenerated injective, then *f.c.p.dim* $(M \oplus P) = 0$. That is, $M \in F.C.P._R \mathcal{M}$ and $E \in Inj_R \mathcal{M}$, *f.c.p.dim* $(M \oplus E) \leq 1$.

Conversely, let $0 \to M \to E \to E/M \to 0$ be an exact sequence, where E is injective. If E/M is not finitely copresented. By (2), then there exist $M' \in F.C.P._R\mathcal{M}$ and $E^* \in Inj_R\mathcal{M}$, such that $f.c.p.dim\,(M' \oplus E^*) > 1$. This is a contradiction. This completes the proof.

Theorem 2.9 *(Dimension Shifting Theorem)* — *Let* $0 \to M \overset{f}{\to} E \overset{g}{\to} K_0 \to 0$ *be an exact sequence, where E is injective.*

(1) $f.c.p.dim\,K_0 \geq f.c.p.dim\,M - 1$

(2) Let $0 \to M \to E_0 \to E_1 \to \ldots \to E_n \to E_{n+1}$ be a finitely copresented resolution of M with $f.c.p.dim\,M \geq 1$ and $K_0' = ker\,(E_0 \to E_1')$, then $f.c.p.dim\,K_0' = n - 1$.

(3) If $f.c.p.dim\,M = n \geq 2$, then $f.c.p.dim\,K_0 = f.c.p.dim\,M - 1$.

(4) Let $0 \to M \to E_0 \to E_1 \to \ldots \to E_n \to E_{n+1}$ be an exact sequence (may not be finitely copresented resolution of M), where E_j' is injective and $K_j = ker\,(E_j \to E_{j+1}')$. If $f.c.p.dim\,M = n \geq 2$, then $f.c.p.dim\,K_j = n - j - 1,\ 0, \leq j \leq n - 2$.

(5) Let R be a co-coherent ring with $f.c.p.dim\,K_0 \geq 2$, then $f.c.p.dim\,M = f.c.p.dim\,K_0 + 1$.

Proof : (1) Let $f.c.p.dim\,K_0 = n$, then there exists an exact sequence $0 \to K_0 \overset{\alpha}{\to} E_0 \overset{\beta}{\to} E_1 \to \ldots \to E_n \to E_{n+1}$, where each E_i is injective, E_n and E_{n+1} are finitely cogenerated. Hence $0 \to M \overset{f}{\to} E \overset{\alpha g}{\longrightarrow} E_0 \geq \overset{\beta}{\to} E_1 \to \ldots \to E_n \to E_{n+1}$ is exact, therefore $f.c.p.dim\,M = n + 1$.

(2) Obviously.

(3) Let $f.c.p.dim\,M \geq 2$. By (2), there exists an exact sequence $0 \to M \to P_0 \to K_0 \to 0$ with $f.c.p.dim\,M = n - 1 \geq 1$. By Schanuel Lemma, $K_0 \oplus E_0 \cong E \oplus K_0$. Hence $f.c.p.dim\,K_0 = f.c.p.dim\,(K_0 \oplus P_0) = f.c.p.dim\,K_0' = n - 1$.

(4) By (2) and (3).

(5) Suppose $f.c.p.dim\,M \leq 1$, consider the following exact sequence $0 \to M \to E \to K_0 \to 0$ and $0 \to M \to E_0 \to K_0 \to 0$. Since R is a co-coherent ring with $f.c.p.dim\,M \leq 1$, then there exists K_0 such that $f.c.p.dim\,K_0 = 0$. By Schanuel Lemma, $K_0 \oplus E_0 \cong E \oplus K_0'$. Hence $2 \leq f.c.p.dim\,K_0 = f.c.p.dim\,(K_0 \oplus E_0) = f.c.p.dim\,(K_0' + E) = \sup\{f.c.p.dim\,K_0',\ f.c.p.dim\,E\} \leq 1$. This is a contradiction. It

follows that $f.c.p.dim\, M > 1$. By (3), $f.c.p.dim\, M = f.c.p.dim\, K_0 + 1$.

3. Finitely Copresented Dimension of Co-Coherent ICF-Rings

Definition 3.1 — Let R be a ring. An R-module A is said to be almost finitely copresented if $A = B \oplus C$, where B is finitely copresented and C is nonfinitely cogenerated cofree.

Definition 3.2 — R is said to be ICF-ring if for each injective module is cofree.

Theorem 3.3 — *Let R be a n ICF-ring and A an R-module with $c.f.p.dim\, A = 1$. Then A is almost finitely copresented.*

Proof : Let $0 \to A \to F \to K \to 0$ be an exact sequence with K finitely copresented and F injective and hence cofree. We fix a finite set of cogenerators of K and a set of cofree cogenerators of F. We express the cogenerators of K as unique linear combinations of the cofree cogenerators of F. Let $f_1, f_2 \ldots f_n$ be involved in these linear combinations. Let $a_1, a_2 \ldots a_n$ be their kernels under the map $A \to F$ and let $B = (a_1, a_2 \ldots a_n)$. Let C be the quotient module of A cogenerated by the kernels of the other cofree cogenerated of F, say $a_j \to f_j$. Clearly $A = B + C$.

If $r_1 a_1 + r_2 a_2 + \ldots r_n a_n = r_{n+1} a_{n+1} + \ldots +_{n+s} a_{n+s}$ where r_i's are in R and $a_{n+1} \ldots a_{n+s}$ are among the cogenerators of C as described above, then $r_1 f_1 + \ldots + r_n f_n - (r_{n+1} f_{n+1} + \ldots + r_{n+s} f_{n+s})$ is in K and hence is a linear combination of $f_1, f_2 \ldots f_n$. By cofreeness of F, $r_{n+1} = \ldots r_{n+s} = 0$, showing that $B \cap C = 0$ and so $A = B \oplus C$.

The exact sequence $0 \to B \to (f_1, f_2 \ldots f_n) \to K \to 0$ shows that B is finitely copresented. As $f.c.p.dim\, A = 1$, A is nonfinitely cogenerated and so F is nonfinitely cogenerated. C is nonfinitely cogenerated cofree because it is isomorphic to quotient module of F cogenerated by the cofree cogenerators other than $f_1, f_2 \ldots f_n$. This completes the proof.

Theorem 3.4 — *Let A be an R-module.*

(1) If R is an ICF-ring with $f.c.p.dim\, A = 1$, then A is almost finitely copresented.

(2) If R is co-coherent and A is almost finitely copresented, then $f.c.p.dim\, A = 1$.

(3) If R is a co-coherent ICF-ring, then $f.c.p.dim\, A = 1$ if and only if A is almost finitely copresented.

Proof : (1) It follows from Theorem 3.3.

(2) It follows from the fact that a finitely cogenerated quotient module of a cofree module over a co-coherent ring is finitely copresented.

(3) It's immediate consequence of (1) and (2).

4. Finitely Cogenerated Dimension

Definition 4.1 — For an R-module A, we define the finitely cogenerated dimension of module A, $f.c.g.dim\, A$, as inf $\{n \mid$ there exists an exact sequence $0 \to A \to E_0 \to E_1 \to \ldots E_n$, where each

E_i is injective and E_n is finitely cogenerated}. If no such exact sequence exists, we say that $f.c.g.dim\, A = \infty$.

Remark 4.2 : It is clear that A is finitely cogenerated module if and onluy if $f.c.g.dim\, A = 0$.

Consequently, we may regard our finitely cogenerated dimension as a measure of how far away an R-module is from being finitely cogenerated.

Proposition 4.3 — Let A be an R-module, then

(*i*) $f.c.g.dim\ A \leq i.dim\, A + 1$

(*ii*) $f.c.g.dim\, A \leq f.c.p.dim\, A$

Obviously.

Proposition 4.4 — Let $0 \to A \xrightarrow{f} E \xrightarrow{g} K \to 0$ be an exact sequence, where E is injective, then $f.c.g.dim\, A \leq f.c.g.dim\ K + 1$.

PROOF : We may assume that $f.c.g.dim\ K$ is finitely. Let $f.c.g.dim\ K = n < \infty$, then there exists an exact sequence $0 \to K \xrightarrow{\alpha} E_0 \xrightarrow{\beta} E_1 \to \dots E_n$, where each E_i is injective and E_n is finitely cogenerated and then there exists such as exact sequence $0 \to A \xrightarrow{f} E \xrightarrow{\alpha g} E_0 \to \xrightarrow{\beta} E_1 \to \dots \to E_n$.

Hence $f.c.p.dim\, A \leq n + 1$.

Note that, it can be strict inequality in Proposition 4.4 that is, $f.c.g.dim\, A < f.c.g.dim\ K + 1$.

Example 4.5 — Let R be non co-notherian, there exists a finitely cogenerated but nonfinitely copresented module A, such that $0 \to A \to E \to K \to 0$ is an exact sequence, where E is finitely cogenerated and K is nonfinitely cogenerated, then $f.c.g.dim\ K \geq 1$, but $f.c.g.dim\, A = 0$ and that $f.c.g.dim\, A < f.c.g.dim\ K + 1$.

Theorem 4.6 — *The following conditions are equivalent*:

(1) R is a co-noetherian ring;

(2) For any R-module A and for any injective resolution $0 \to A \xrightarrow{\varepsilon} E_0 \xrightarrow{d_0} E_1 \to \dots \to E_{n-1} \xrightarrow{d_{n-1}} E_n$, *where* E_n *is finitely cogenerated, we can find finitely cogenerated injective*

modules $E_{n+1}, E_{n+2}, \ldots,$ *such that* $0 \to A \xrightarrow{\varepsilon} E \xrightarrow{d_0} \ldots E_{n-1} \xrightarrow{d_{n-1}} E_n \xrightarrow{d_n} E_{n+1} \to E_{n+2} \to$ ... *is exact.*

PROOF : (1) $\Rightarrow$ (2). Since $imd_{n-1} \leq E_n$ and E_n is finitely cogenerated, then imd_n is quotient module of finitely cogenerated injective module E_n. As R is co-noetherian, then imd_{n-1} is finitely copresented and hence E_n/imd_{n-1} is finitely cogenerated. Let $E_{n+1} = E(E_n/imd_{n-1})$, define π_n : $E_n \to E_n/imd_{n-1}$ be a natural epimorphism and $i_n : E_n/imd_{n-1} \to E_{n+1}$ be an inclusion map. Let $d_n = i_n \pi_n$, then E_{n+1} is finitely cogenerated injective and $0 \to A \xrightarrow{\varepsilon} E \xrightarrow{d_0} E_1 \to \ldots \to E_{n-1} \xrightarrow{d_{n-1}} E_n \xrightarrow{d_n} E_{n+1}$ is exact. By induction, we can find finitely cogenerated injective modules $E_{n+1}, E_{n+2}, \ldots$ such that $0 \to A \xrightarrow{\varepsilon} E \xrightarrow{d_0} E_1 \to \ldots \to E_{n-1} \xrightarrow{d_{n-1}} E_n \xrightarrow{d_n} E_{n+1} \to$ ... is exact.

(2) $\Rightarrow$ (1). Let E be injective and $B = E/A$ finitely cogenerated. Define $\pi : E \to E/A$ is a natural epimorphism and $i : B \to E(B)$ is an inclusion map, then $E(B)$ is finitely cogenerated. Let $\alpha = i\pi$, then $0 \to A \to E \xrightarrow{\alpha} E(B)$ is exact and that $0 \to A \to E \xrightarrow{\alpha} E(B) \xrightarrow{\beta} E_1$, where E_1 is finitely cogenerated injective. Hence $B = E/A$ is finitely copresented. This completes the proof.

Theorem 4.7 — *R is co-noetherian if and only if for any module A, f.c.g.dim A = f.c.p.dim A.*

PROOF : Suppose A is finitely cogenerated, then *f.c.p.dim* $A = 0$. By hypothesis, *f.c.p.dim* $A = 0$ and that A is finitely copresented. That is, each finitely cogenerated module is finitely copresented and that R is co-noetherian.

Conversely, for any module A, *f.c.g.dim* $A \leq$ *f.c.p.dim* A *by Proposition 4.3. By Theorem 4.6, we have f.c.p.dim* $A \leq$ *f.c.g.dim* A. *Hence f.c.g.dim* $A =$ *f.c.g.dim* A.

Lemma 4.8 — Let R be co-coherent and $n \in \mathbb{N}$. For any module, A and for any injective resolution $0 \to A \to E_0 \to E_1 \to \ldots \to E_{n-1} \xrightarrow{d_{n-1}} E_n$, where E_n is finitely cogenerated, we can find finitely cogenerated injective modules E_{n+1}, E_{n+2} ... such that $0 \to A \to E_0 \to E_1 \to \ldots E_{n-1} \to E_n \to E_{n+1}$ ... is exact.

Lemma 4.9 — Let R be co-coherent and A any module. If *f.c.g.dim* $A > 0$, then *f.c.g.dim* A = *f.c.p.dim* A.

PROOF : By Proposition 4.3 and Lemma 4.8.

Theorem 4.10 — *The following conditions are equivalent:*

(1) R is co-coherent;

(2) If E is injective and Q is a quotient module of E, then f.c.g.dim Q = f.c.p.dim Q;

(3) If M is finitely copresented and N is a quotient module of M, then f.c.g.dim N = f.c.p.dim N.

PROOF : (1) $\Rightarrow$ (2). Let Q be a quotient module of an injective module E. If Q is finitely coquenerated, then Q is finitely corepresented since R is co-coherent and that *f.c.g.dim* Q = *f.c.p.dim* Q. If Q is nonfinitely cogenerated, then *f.c.g.dim* $Q > 0$. By Lemma 4.8, we have *f.c.g.dim* Q = *f.c.p.dim* Q..

(1) $\Rightarrow$ (3) Similar to (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). Let E be injective. If Q is finitely cogenerated then *f.c.g.dim* $Q = 0$ and that *f.c.p.dim* $Q = 0$. Hence Q is finitely copresented and that R is co-coherent.

(3) $\Rightarrow$ (1). Obviously.

Theorem 4.11 — *Let R be a co-coherent ring and $0 \to A \to B \to C \to 0$ be an exact sequence, where f.c.g.dim $A = a$, f.c.g.dim $B = b$ and f.c.g.dim $C = c$. If two of these are finite, so is the third. Furthermore, $b \leq sup\{a, c\}$, $c \leq sup\ \{b, a + 1\}$ and $a \leq sup\ (b, c - 1\}$.*

PROOF : (1) If a and c are finite, then, let $n = \sup\ \{a, c\}$. Since R is co-coherent, then thete exist exact sequences $0 \to A \to A_0 \to A_1 \to \ldots \to A_n$ and $0 \to C \to C_0 \to C_1 \to \ldots \to C_n$, where A_n and C_n are finitely cogenerated. By the dual Theorem of Horse Shoe Lemma, there exists an exact sequence $0 \to B \to B_0 \to \ldots \to B_n$, where $B_i = A_i + C_i$. Obviously B_n is finitely cogenerated. Hence $b \leq \sup\ \{a, c\}$.

(2) b and c are finite. Let $0 \to B \xrightarrow{\varepsilon} B_0 \xrightarrow{\alpha_0} \ldots \to B_n \xrightarrow{\alpha_n} \ldots$ and $0 \to C \xrightarrow{\varepsilon'} C_0 \xrightarrow{\beta_0} \ldots \to C_n \xrightarrow{\beta_n} \ldots$, where B_n is finitely cogenerated with $n \geq b$ and C_n is finitely cogenerated with $n \geq c$. By Comparison Theorem, there exists a chain map such the following diagram commute:

$$\begin{array}{ccccccccccccccccc}
0 & \to & B & \xrightarrow{\varepsilon} & B_0 & \xrightarrow{\alpha_0} & B_1 & \xrightarrow{\alpha_1} & B_2 & \to & \ldots & \to & B_{n-1} & \xrightarrow{\alpha_{n-1}} & B_n & \xrightarrow{\alpha_n} & B_{n+1} \to \ldots \\
 & & g\downarrow & & \gamma_0\downarrow & & \gamma_1\downarrow & & \gamma_2\downarrow & & & & \gamma_{n-1}\downarrow & & \gamma_n\downarrow & & \gamma_{n+1}\downarrow \\
0 & \to & C & \xrightarrow{\varepsilon'} & C_0 & \xrightarrow{\beta_0} & C_1 & \xrightarrow{\beta_1} & C_2 & \to & \ldots & \to & C_{n-1} & \xrightarrow{\beta_{n-1}} & C_n & \xrightarrow{\beta_n} & C_{n+1} \to \ldots
\end{array}$$

Define $\sigma : B_0 \to C_0 \oplus B_1$ by $b_0 \rightarrowtail (\gamma_0 (b_0), - \alpha_0 (b_0))$

$\Delta_n : C_n \oplus B_{n+1} \to C_{n+1} \oplus B_{n+2}$ by

$(c_n, b_{n+1}) \rightarrowtail (\beta_n (c_n) + \gamma_{n+1} (b_{n+1}), - \alpha_{n+1} (b_{n+1}))$

Hence

$$0 \to A \xrightarrow{\varepsilon f} B_0 \xrightarrow{\sigma} C_0 \oplus B_1 \xrightarrow{\Delta_c} C_1 \oplus B_2 \xrightarrow{\Delta_1} \ldots$$

$$C_{n-1} \oplus B_n \xrightarrow{\Delta_{n-1}} C_n \oplus B_{n+1} \xrightarrow{\Delta_n} C_{n+1} \oplus B_{n+2} \to \ldots$$

is exact. In fact, we only show that $Im\delta = Ker\, \Delta_0$.

(*i*) $Im\delta \subseteq Ker\, \Delta_0$

In fact, $\Delta_0\, \delta (b_0) = \Delta_0 (\gamma_0 (b_0), - \alpha_0 (b_0))$

$= (\beta_0\, \gamma_0 (b_0) + \gamma_1 (- \alpha_0) (b_0), - \alpha_1 (\alpha_0 (b_0))) = 0,$

that is $Im\delta \subseteq Ker\, \Delta_0$.

(*ii*) $Im\delta \supseteq Ker\, \Delta_0$.

Let $(c_0, b_1) \in Ker\, \Delta_0$, then $\Delta_0 (c_0, b_1) = (\beta_0 (c_0) + \gamma_1 (b_1), - \alpha_1 (b_1)) = 0$, that is $\beta_0 (c_0) + \gamma_1 (b_1) = 0$ and $\alpha_1 (b_1) = 0$. Hence $b_1 \in Ker\, \alpha_1 = Im\alpha_0 \subseteq B_1$. Then there exists $b_0 \in B_0$ such that $\alpha_0 (b_0) = b_1$ and that $\beta_0 (c_0) + \gamma_1\, \alpha_0 (b_0) = 0$. By exactness, we have $\beta_0 (c_0) + \beta_0\, \gamma_0 (b_0) = 0$, that is, $c_0 + \gamma_0 (b_0) \in Ker\, \beta_0 = Im\, \varepsilon'$, so that $c_0 \in C_\alpha - 0$. Hence $(c_0, b_1) \in Im\, \delta$, and that $Ker\, \Delta_0 \subseteq Im\, \delta$. If $n \geq \sup \{b, c - 1\}$, then $C_{n-1} \oplus B_n$ is finitely cogenerated and that $a \leq \{b, c - 1\}$.

(3) If a and b are finite, it if similar to (2).

Corollary 4.12. With the same hypothesis as in Theorem 4.11, if $c \neq a + 1$ then $b = \sup \{a, c\}$.

PROOF : By Theorem 4.11, $b \leq \sup \{a, c\}$. We only show that $c = a + 1$ if $b < \sup \{a, c\}$. In fact, let $b < \sup \{a, c\}$. If $c < a + 1$, then $c \leq a$ and that $b < a$. By Theorem 4.11, $a \leq \sup \{b, c - 1\} < a$. This is a contradiciton. If $c > a + 1$, then $b < c$ and that $c \leq \sup \{b, a + 1\} < c$. This is also a contradiction. Hence $c = a + 1$.

Theorem 4.13 — *With the same hypothesis as in Theorem 4.11 with* $0 \leq i.dim\, B < \infty$,

(1) If $c \geq i.dim\, B + 2$ *or* $a \geq i.dim\, B + 2$, *then* $c = a + 1$

(2) If $c \leq i.dim\, B + 1$, *then* $a \leq i.dim\, B + 1$

(3) If $a \leq i.dim\,B + 1$, then $c \leq i.dim\,B + 2$

PROOF : Let $i.dim\,B = n, 0 \leq n < \infty$.

(1) Since a or $c \geq n + 2$, then sup $\{a, c\} \geq n + 2$. *By Proposition 4.3,* $b = f.c.g.dim\,B \leq i.dim\,B + 1 = n + 1$ and that $b <$ sup $\{a, c\}$. Hence $c = a + 1$ by Corollary 4.12.

(2) If $c \leq n + 1$, then $a \leq$ sup $\{b, c - 1\} \leq$ sup $\{n + 1, n\} = n + 1$ by Corollary 4.12 and that $a \leq i.dim\,B + 1$.

(3) If $a \leq n + 1$ and $b \leq n + 1$, then $c \leq sup\ \{b, a + 1\} \leq sup\ \{n + 1, n + 1 + 1\} = n = 2$ by Corollary 4.12, and that $c \leq i.dim\,B + 2$.

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