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WEAK CONVERGENCE AND TIGHTNESS FOR FUZZY RANDOM VARIABLES†

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In this paper, we establish some characterizations of weak convergence and tightness for a sequence of random elements taking values in the space of normal and upper-semicontinuous fuzzy sets with compact support in the Euclidean space R^p.

Key Words: Fuzzy Random Variables; Random Sets; Weak Convergence Tightness

1. INTRODUCTION

Prokhorov[11] gave the theory of weak convergence of probability measures on complete separable metric spaces, and his results were examined on $C[0, 1]$ and $D[0, 1]$. The notion of tightness plays an important role both in the theory of weak convergence and in its applications. The relationships between weak convergence and tightness can be also found in Bilingsley[2]. These results were extended by Lindvall[9] to the case of $D[0, \infty)$.

The theory of fuzzy random variables and fuzzy stochastic processes have been received much attentions in recent years (e.g. [4,8,10,12 and 14] and so on). But weak convergence for fuzzy random variables has not established yet.

Joo and Kim[5] introduced a new metric d_s on the space $F(R)$ of fuzzy numbers in R so that $F(R)$ is separable and topologically complete. These results were generalized by Joo and Kim[6] to the space $\mathcal{F}(R^p)$ of normal and upper-semicontinuous fuzzy sets with compact support in the Euclidean space R^p. Also, Kim[7] proved that a $\mathcal{F}(R^p)$-valued function is fuzzy random variable in the sense of Puri and Ralescu[12] if and only if it is measurable when considered as a function into the metric space $\mathcal{F}(R^p)$ endowed the metric d_s. Thus we can apply the notions of weak convergence and tightness of probability measures on complete separable metric spaces to the case of fuzzy random variables.

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In this paper, we establish the theory of weak convergence and tightness for fuzzy random variables. In fact, the present paper was motivated by the theory of weak convergence and tightness of probability measures on $D[0, 1]$ which can be found in Prokhorov[11] and Billingsley[2]. It turns out that the techniques used there are also useful in our concerns.

2. PRELIMINARIES

Let $\mathcal{K}(R^p)$ denote the family of non-empty compact subsets of the Euclidean space R^p. Then it is well-known that $\mathcal{K}(R^p)$ is complete and separable with respect to the Hausdorff metric h defined by

$$h(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} |a - b|, \quad \sup_{b \in B} \inf_{a \in A} |a - b| \right\},$$

where $|\cdot|$ denotes the Euclidean norm. A norm of $A \in \mathcal{K}(R^p)$ is defined by

$$\|A\| = a(A, \{0\}) = \sup_{a \in A} |a|.$$

Let $\mathcal{F}(R^p)$ denote the family of all fuzzy sets $\tilde{u} : R^p \to [0, 1]$ with the following properties:

(1) $\tilde{u}$ is normal, i.e., there exists $x \in R^p$ such that $\tilde{u}(x) = 1$;

(2) $\tilde{u}$ is upper semicontinuous;

(3) $supp\, \tilde{u} = cl\ \{x \in R^p : \tilde{u}(x) > 0\}$ is compact, where $cl\ (A)$ denote the closure of A.

For a fuzzy set $\tilde{u}$ in R^p, the α-level set of $\tilde{u}$ is defined by

$$L_\alpha \tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1 \\ supp\, \tilde{u}, & \text{if } \alpha = 0. \end{cases}$$

Then, it follows immediately that

$$\tilde{u} \in \mathcal{F}(R^p) \text{ if and only if } L_\alpha \tilde{u} \in \mathcal{K}(R^p) \text{ for each } \alpha \in [0, 1].$$

Lemma 2.1 For $\tilde{u} \in \mathcal{F}(R^p)$, we define

$$F_{\tilde{u}} : [0, 1] \to (\mathcal{K}(R^p), h),\ F_{\tilde{u}}(\alpha) = L_\alpha \tilde{u}.$$

Then the followings hold:

(1) $F_{\tilde{u}}$ is non-increasing, i.e., $\alpha \leq \beta$ implies $F_{\tilde{u}}(\alpha) \supset F_{\tilde{u}}(\beta)$,

(2) $F_{\tilde{u}}$ is left continuous on $(0, 1]$,

(3) $F_{\tilde{u}}$ has right-limits on $[0, 1)$ and $F_{\tilde{u}}$ is right-continuous at 0.

Conversely, if $F : [0, 1] \to \mathcal{K}(R^p)$ is a function satisfying the above conditions (1)-(3), then there exists a unique $\tilde{u} \in \mathcal{F}(R^p)$ such that $F(\alpha) = L_\alpha \tilde{u}$ fpr all $\alpha \in [0, 1]$.

PROOF : See Lemma 2.2 of Joo and Kim[6].

If we denote the right limit of $F_{\tilde{u}}$ at α by $L_{\alpha}+\tilde{u}$, then

$$L_\alpha + \tilde{u} = cl\ \{x \in R^p : \tilde{u}(x) > \alpha\}.$$

Now if we define,

$$j_{\tilde{u}}(\alpha) = h(L_\alpha \tilde{u}, L_\alpha + \tilde{u})$$

then the function $F_{\tilde{u}}$ is continuous at α if and only if $j_{\tilde{u}}(\alpha) = 0$.

Lemma 2.2 — For each $\tilde{u} \in \mathcal{F}(R^p)$ and $\varepsilon > 0$, there exists a partition $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_r = 1$ of $[0, 1]$ such that

$$h(L_{\alpha_{i-1}} + \tilde{u}, L_{\alpha_i} \tilde{u}) < \varepsilon \text{ for all } i = 1, 2, \ldots, r.$$

PROOF : See Lemma 2.3 of Joo and Kim[6]. □

For $\tilde{u} \in \mathcal{F}(R^p)$ and $0 < \delta < 1$, we define

$$\tau(\tilde{u}, \delta) = \inf_{\{\alpha_i\}} \max_{1 \le i \le r} h(L_{\alpha_{i-1}} + \tilde{u}, L_{\alpha_i} \tilde{u})$$

where the infimum is taken over all partitions $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_r = 1$ of $[0, 1]$ satisfying $\max_{1 \le i \le r}(\alpha_i - \alpha_{i-1}) > \delta$. Then Lemma 2.2 is equivalent to the assertion that

$$\lim_{\delta \to 0} \tau(\tilde{u}, \delta) = 0 \text{ for each } \tilde{u} \in \mathcal{F}(R^p).$$

Note that $\tau(\tilde{u}, \delta)$ was denoted by $w_{\tilde{u}}(\delta)$ in Joo and Kim[6]. Also, if we define, for $\varepsilon > 0$,

$$J_{\tilde{u}}(\varepsilon) = \{\alpha : j_{\tilde{u}}(\alpha) > \varepsilon\}$$

and

$$J_{\tilde{u}} = \{\alpha : j_{\tilde{u}}(\alpha) > 0\},$$

the Lemma 2.2 implies that $J_{\tilde{u}}(\varepsilon)$ is finite and so $J_{\tilde{u}}$ is countable.

The metric on $\mathcal{F}(R^p)$ which generalizes the Hausdorff metric h on $\mathcal{K}(R^p)$ is usually defined as follows:

$$d_\infty(\tilde{u}, \tilde{v}) = \sup_{0 \le \alpha \le 1} h(L_\alpha \tilde{u}, L_\alpha \tilde{v}).$$

Also, the norm of $\tilde{u}$ is defined as

$$\|\tilde{u}\| = d_\infty(\tilde{u}, \tilde{0}) = \sup_{x \in L_0 \tilde{u}} |x|,$$

where $\tilde{0} = I_{\{0\}}$ is the indicator function of $\{0\}$.

Then it is well-known that $\mathcal{F}(R^p)$ is complete but is not separable with respect to the metric d_∞ (see Klement *et al.*[8]). Joo and Kim[6] introduced a metric d_s on $\mathcal{F}(R^p)$ which makes it separable as follows:

Definition 2.3 — Let T denote the class of strictly increasing, continuous mapping of [0, 1] onto itself. For $\tilde{u}, \tilde{v} \in \mathcal{F}(R^p)$, we define

$$d_s(\tilde{u}, \tilde{v}) = \inf \{\varepsilon > 0 : \text{there exists a } t \in T \text{ such that}$$

$$\sup_{0 \le \alpha \le 1} |t(\alpha) - \alpha| \le \varepsilon \text{ and } d_\infty(\tilde{u}, t(\tilde{v})) \le \varepsilon\},$$

where $t(\tilde{v})$ denotes the composition of $\tilde{v}$ and t.

Then it follows immediately that d_s is a metric on $\mathcal{F}(R^p)$ and $d_s(\tilde{u}, \tilde{v}) \le d(\tilde{u}, \tilde{v})$. The metric d_s will be called the Hausdorff-Skorokhod metric. It is known that $\mathcal{F}(R^p)$ is separable and topologically complete with respect to the metric d_s (For details, see Joo and Kim[6]).

Now, for any partition $V : 0 = \beta_0 < \beta_1 < \ldots < \beta_k = 1$ of [0, 1], if we define $\tilde{g}_V : \mathcal{F}(R^p) \to \mathcal{F}(R^p)$ by

$$\tilde{g}_V(\tilde{u})(x) = \sum_{j=k}^{k} \beta_{j-1} I_{A_{j-1} \setminus A_j}(x) + I_{A_k}(x) \quad \ldots (2.1)$$

where $A_j = L_{\beta_j} \tilde{u}, j = 0, 1, \ldots, k$, then it follows that

$$L_\alpha \tilde{g}_V(\tilde{u}) = \begin{cases} L_{\beta_1} \tilde{u} & \text{if } 0 \le \alpha \le \beta_1, \\ L_{\beta_j} \tilde{u} & \text{if } \beta_{j-1} < \alpha \le \beta_j,\ j = 2, 3, \ldots, k. \end{cases} \quad \ldots (2.2)$$

Lemma 2.4 — If $\delta \ge \beta_j - \beta_{j-1}$ for all j, then

$$d_s(\tilde{u}, \tilde{g}_V(\tilde{u})) \le \tau(\tilde{u}, \delta) + \delta.$$

PROOF : Let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_r = 1$ be any partition of [0, 1] such that $\alpha_{i+1} - \alpha_i > \delta$ for all $i = 0, 1, \ldots, r - 1$. Then for each $i \in \{1, \ldots, r - 1\}$, there exists j such that

$$\alpha_i \le \beta_j < \alpha_{i+1}.$$

We take $t \in T$ so that

$$t(\alpha_i) = \min \{\beta_j \mid \alpha_i \le \beta_j\}$$

and linear in (α_i, α_{i+1}). Then

$$\sup_{0 \le \alpha \le 1} \mid t(\alpha) - \alpha \mid \le \max_j (\beta_j - \beta_{j-1}) \le \delta. \qquad \text{... (2.3)}$$

If $\alpha_i < \alpha \le \alpha_{i+1}$, then

$$\alpha_i \le t(\alpha_i) < t(\alpha) \le t(\alpha_{i+1})$$

and, by (2.2),

$$L_{t(\alpha_i)} \tilde{g}_V(\tilde{u}) = L_{\alpha_i} \tilde{u}.$$

Hence,

$$\sup_{\alpha_i < \alpha \le \alpha_{i+1}} h(L_\alpha \tilde{u}, L_{t(\alpha)} \tilde{g}_V(\tilde{u})) \le h(L_{\alpha_{i-1}}^{+} \tilde{u}, L_{\alpha_i} \tilde{u}).$$

Similarly, it can be proved that

$$\sup_{0 \le \alpha \le \alpha_1} h(L_\alpha \tilde{u}, L_{t(\alpha)} \tilde{g}_V(\tilde{u})) \le h(L_0 \tilde{u}, L_{\alpha_1} \tilde{u}).$$

Therefore, we obtain

$$\begin{aligned} d_\infty(t(\tilde{u}), \tilde{g}_V(\tilde{u})) &= \sup_{0 \le \alpha \le 1} h(L_\alpha \tilde{u}, L_{t(\alpha)} \tilde{g}_V(\tilde{u})) \\ &\le \max_i h(L_{\alpha_{i-1}}^{+} \tilde{u}, L_{\alpha_i} \tilde{u}), \end{aligned}$$

which implies together with (2.3)

$$d_s(\tilde{u}, \tilde{g}_V(\tilde{u})) \le \max_i h(L_{\alpha_{i-1}}^{+} \tilde{u}, L_{\alpha_i} \tilde{u}) + \delta.$$

By definition of $\tau_{\tilde{u}}(\delta)$, we conclude that

$$d_s(\tilde{u}, \tilde{g}_V(\tilde{u})) \le \tau_{\tilde{u}}(\delta) + \delta. \qquad \square$$

3. Main Results

Throughout this section, we assume that $\mathcal{K}(R^p)$ and $\mathcal{F}(R^p)$ are considered as the metric spaces endowed with the Hausdorff metric h and Hausdorff-Sokorokhod metric d_s, respectively. Also, it is assumed that $\mathcal{K}^k(R^p)$ is the Cartesian product of k-copies of $\mathcal{K}(R^p)$ endowed with the product topology.

Lemma 3.1 — Let L_α be a function from $\mathcal{F}(R^p)$ to $\mathcal{K}(R^p)$ defined by $\tilde{u} \to L_\alpha \tilde{u}$. Then the following statements hold :

(1) L_0 and L_1 are continuous.

(2) If $0 < \alpha < 1$, then L_α is continuous at $\tilde{u}$ if and only if the function $F_{\tilde{u}}$ defined in Lemma 2.1 is continuous at α.

PROOF : See Lemma 3.1 of Kim[7]. □

Let (Ω, Σ, P) be a probability space. A set-valued function $X : \Omega \to \mathcal{K}(R^p)$ is called a random set if it is measurable, i.e.

$X^{-1}(B) \in \Sigma$ for every Borel subset B of $\mathcal{K}(R^p)$.

Convergence in distribution for random sets can be found in Artstein[1] and Salinetti and Wets[13].

A fuzzy set-valued fnction $\tilde{X} : \Omega \to \mathcal{F}(R^p)$ is called a fuzzy random variable if it is measurable, i.e.

$$\tilde{X}^{-1}(B) \in \Sigma \text{ for every } B \in \mathcal{B}_s,$$

where $\mathcal{B}_s$ denotes the Borel σ-field of $\mathcal{F}(R^p)$ generated by the metric d_s. It is well-known that $\mathcal{B}_s$ coincides with the smallest σ-field for which the maps $L_\alpha : \tilde{u} \to L_\alpha \tilde{u}$ are measurable for all $\alpha \in [0, 1]$. Thus, $\tilde{X} : \Omega \to \mathcal{F}(R^p)$ is a fuzzy random variable if and only if for each $\alpha \in [0, 1]$, $L_\alpha \tilde{X}$ is a random set (For details, see Butnariu[3] and Kim[7]).

For $\alpha_1, \alpha_2, ..., \alpha_k \in [0, 1]$, let us define $L_{\alpha_1, ..., \alpha_k} : \mathcal{F}(R^p) \to \mathcal{K}^k(R^p)$ by

$$L_{\alpha_1, ..., \alpha_k}(\tilde{u}) = (L_{\alpha_1} \tilde{u}, ..., L_{\alpha_k} \tilde{u}),$$

then $L_{\alpha_1, ..., \alpha_k}$ is Borel measurable. As usual, for $A \subset \mathcal{K}^k(R^p)$, we define

$$L^{-1}_{\alpha_1, ..., \alpha_k}(A) = \left\{ \tilde{u} \in \mathcal{F}(R^p) \mid L_{\alpha_1, ..., \alpha_k}(\tilde{u}) \in A \right\}.$$

Now for $I \subset [0, 1]$, let

$$\mathcal{A}_I = \{ L^{-1}_{\alpha_1, ..., \alpha_k}(B) \mid \alpha_1, ..., \alpha_k \in I \text{ and } B \text{ is a Borel subset of } \mathcal{K}^k(R^p) \}.$$

Then $\mathcal{A}_I$ is a field, i.e., $\mathcal{F}(R^p) \in \mathcal{A}_I$ is closed under complementation and finite union.

Let $\mathcal{B}_I = \sigma \{L_\alpha \mid \alpha \in I\}$ be the σ-field generated by the level functions $L_\alpha : f(R^p) \to \mathcal{K}(R^p)$ for $\alpha \in I$, then it is also generated by the class $\mathcal{A}_I$.

Recall that a subclass of $\mathcal{A}$ of $\mathcal{B}_s$ is said to be a *determining class* if two probability measures that agree on $\mathcal{A}$ necessarily agree also on $\mathcal{B}_s$. It is well known that $\mathcal{A}$ is a determining class if it is a field generating the σ-field $\mathcal{B}_s$.

Theorem 3.2 — *If I is dense in [0, 1], then $\mathcal{B}_I = \mathcal{B}_s$ and $\mathcal{A}_I$ is a determining class.*

PROOF : Since I is dense, we can take a sequence $\{\alpha_n\}$ in I so that $\lim_{n \to \infty} \alpha_n = 0$. Then it follows from right-continuity of $L_\alpha \tilde{u}$ at $\alpha = 0$ that for each $\tilde{u} \in \mathcal{F}(R^p)$,

$$L_0 \tilde{u} = \lim_{n \to \infty} L_{\alpha_n} \tilde{u}.$$

Thus L_0 is measurable with respect to $\mathcal{B}_I$. Similarly, it can be proved that L_1 is measurable with respect to $\mathcal{B}_I$. Therefore, $\mathcal{B}_I = \mathcal{B}_{I \cup \{0, 1\}}$. Hence we may assume that I contains 0 and 1.

Now for each positive integer m, we choose a partition $V : 0 = \beta_0 < \beta_1 < \ldots < \beta_k = 1$ of [0.1] in such a way that $\beta_j \in I$ for all j and $\max_j (\beta_j - \beta_{j-1}) < \frac{1}{m}$.

For $\mathbf{A} = (A_0, A_1, \ldots, A_k) \in \mathcal{K}^{k+1}(R^p)$, let $\tilde{f}_V(\mathbf{A})$ be the element of $\mathcal{F}(R^p)$ defined by

$$\tilde{f}_V(\mathbf{A})(x) = \sum_{j=1}^{k} \beta_{j-1} I_{B_{j-1} \setminus B_j}(x) + I_{B_k}(x),$$

where

$$B_0 = \bigcup_{j=0}^{k} A_j, B_1 = \bigcup_{j=1}^{k} A_j, \ldots, B_k = A_k.$$

Then

$$L_\alpha \tilde{f}_V(\mathbf{A}) = \begin{cases} B_1 & \text{if } 0 \le \alpha \le \beta_1, \\ B_j & \text{if } \beta_{j-1} < \alpha \le \beta_j,\ j = 2, \ldots, k. \end{cases}$$

Now we will show that $\tilde{f}_V : \mathcal{K}^{k+1}(R^p) \to \mathcal{F}(R^p)$ is continuous. First, we note that if $h(A_n, A) \to 0$ and $h(A_n', A') \to 0$, then

$$h(A_n \cup A_n', A \cup A') \le \max (h(A_n, A), h(A_n', A')) \to 0.$$

From this fact, it follows that if

$$\mathbf{A}_n = (A_{n0}, A_{n1}, ..., A_{nk}) \to \mathbf{A} = (A_0, A_1, ..., A_k),$$

then

$$\mathbf{B}_n = (B_{n0}, B_{n1}, ..., B_{nk}) \to \mathbf{B} = (B_0, B_1, ..., B_k),$$

where

$$B_{n0} = \bigcup_{j=0}^{k} A_{nj}, B_{n1} = \bigcup_{j=1}^{k} A_{nj}, ..., B_{nk} = A_{nk}.$$

Thus if $\mathbf{A}_n \to \mathbf{A}$, then

$$d_s(\tilde{f}_V(\mathbf{A}_n), \tilde{f}_V(\mathbf{A})) \le d_\infty(\tilde{f}_V(\mathbf{A}_n), \tilde{f}_V(\mathbf{A}))$$

$$= \max_{1 \le j \le k} h(B_{nj}, B_j) \to 0.$$

Hence, $\tilde{f}_V$ is continuous and so is measurable. Since $L_{\beta_0, ..., \beta_k}$ is measurable with respect to $\mathcal{B}_I$, so is the composition $\tilde{f}_V \circ L_{\beta_0, ..., \beta_k}$. But this composition is just the map $\tilde{g}_V$ defined as in (2.1). By Lemma 2.4, we have $\tilde{u} = \lim_{\| V \| \to 0} \tilde{g}_V(\tilde{u})$, where $\| V \| = \max_j (\beta_j - \beta_{j-1})$. This means that the identity function on $\mathcal{F}(R^p)$ is measurable with respect to $\mathcal{B}_I$ which implies $\mathcal{B}_s \subset \mathcal{B}_I$. Therefore, $\mathcal{B}_s = \mathcal{B}_I$. Since $\mathcal{A}_I$ is a field generating the σ -field $\mathcal{B}_s$, it is a determining class. □

For a probability measure Q on $(\mathcal{F}(R^p), \mathcal{B}_s)$, we denote

$$I_Q = \{\alpha \in [0, 1] : L_\alpha \text{ is continuous almost surely [Q]}\}.$$

Since L_0 and L_1 are continuous everywhere, the points 0 and 1 always lie in I_Q. If $0 < \alpha < 1$, then Lemma 3.1, L_α is continuous at $\tilde{u}$ if and only if $j_{\tilde{u}}(\alpha) = 0$. Thus if $J(\alpha) = \{\tilde{u} : j_{\tilde{u}}(\alpha) > 0\}$, then it follows that

$$\alpha \in I_Q \text{ if and only if } Q(J(\alpha)) = 0.$$

Lemma 3.3 — I_Q contains 0 and 1, and $[0, 1]\backslash I_Q$ is at most countable.

PROOF : Let $J_\varepsilon(\alpha)$ $\{\tilde{u} : j_{\tilde{u}}(\alpha) > \varepsilon\}$. First we prove that for any fixed $\varepsilon, \delta > 0$, there can be at most finitely many α for which $Q(J_\varepsilon(\alpha)) \ge \delta$. Suppose that there exist $\varepsilon, \delta > 0$ such that

$$Q(J_\varepsilon(\alpha)) \ge \delta \text{ for infinitely many } \alpha.$$

Then we can choose a sequence $\{\alpha_n\}$ of distinct elements in [0, 1] so that $Q(J_\varepsilon(\alpha_n)) \ge \delta$ for all n. This implies that

$$Q\,(\limsup_{n\to\infty} J_\varepsilon(\alpha_n)) \geq \delta,$$

and so

$$\limsup_{n\to\infty} J_\varepsilon(\alpha_n) \neq \phi.$$

But if $\tilde{u} \in \limsup_{n\to\infty} J_\varepsilon(\alpha_n)$, then $j_{\tilde{u}}(\alpha_n) > \varepsilon$ for infinitely many n. This contradicts to the fact $J_{\tilde{u}}(\varepsilon) = \{\alpha \mid j_{\tilde{u}}(\alpha) > \varepsilon\}$ is finite. Thus, $Q\,(J_\varepsilon(\alpha)) \geq \delta$ for at most finitely many α. Since

$$Q\,(J\,(\alpha)) = \lim_{\varepsilon\to 0} Q\,(J_\varepsilon(\alpha)),$$

we conclude that $Q\,(J\,(\alpha)) > 0$ for at most countably many α. This completes the proof. □

For a fuzzy random variable $\tilde{X} : \Omega \to \mathcal{F}(R^p)$, the probability distribution $P_{\tilde{X}}$ of $\tilde{X}$ is defined as usual;

$$P_{\tilde{X}}(B) = P\,(\tilde{X} \in B) \text{ for } B \in \mathcal{B}_s.$$

For convenience's sake, we write $I_{\tilde{X}}$ for $I_{P_{\tilde{X}}}$. Then

$$\alpha \in I_{\tilde{X}} \text{ if and only if } P\,\{\tilde{X} \in J\,(\alpha)\} = 0. \qquad \text{... (3.1)}$$

Definition 3.4 — (1) Let Q_n, Q be probability measures on $(\mathcal{F}(R^p), \mathcal{B}_s)$. We say that Q_n converges weakly to Q and write $Q_n \Rightarrow Q$ if for any bounded continuous real function f on $\mathcal{F}(R^p)$.

$$\int f dQ_n \to \int f dQ.$$

(2) Let $\tilde{X}_n, \tilde{X}$ be fuzzy random variables. We say that $\tilde{X}_n$ converges in distribution to $\tilde{X}$ and write $\tilde{X}_n \Rightarrow \tilde{X}$ if $P_{\tilde{X}_n} \Rightarrow P_{\tilde{X}}$.

If $\alpha_1, ..., \alpha_k \in I_{\tilde{X}}$, then $L_{\alpha_1, ..., \alpha_k}$ is continuous a.s. $[P_{\tilde{X}}]$, and thus it follows from the well-known mapping theorem (see Theorem 2.7 of Billingsley[2]) that

$$\tilde{X}_n \Rightarrow \tilde{X} \text{ implies } L_{\alpha_1, ..., \alpha_k}(\tilde{X}_n) \Rightarrow L_{\alpha_1, ..., \alpha_k}(\tilde{X}_n). \qquad \text{... (3.2)}$$

But if $\alpha_i \in I_{\tilde{X}}$ for some i, then this is not true.

Example 1 — Let us define

$$\tilde{\upsilon}(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{2} & \text{if } 0 < |x| \le 1, \\ 0 & \text{elsewhere,} \end{cases}$$

and for $n > 2$,

$$\tilde{\upsilon}_n(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{2} - \frac{1}{n} & \text{if } 0 < |x| \le 1, \\ 0 & \text{elsewhere.} \end{cases}$$

We take $\tilde{X}$ to be degenerate at $\tilde{\upsilon}$ and $\tilde{X}_n$ to be degenerate at $\tilde{\upsilon}_n$. Then $\tilde{\upsilon}_n \to \tilde{\upsilon}$ and so $\tilde{X}_n \Rightarrow \tilde{X}$. But since

$$L_\alpha \tilde{\upsilon} = \begin{cases} \{x; |x| \le 1\} & \text{if } 0 \le \alpha \le \frac{1}{2} \\ \{0\} & \text{if } \frac{1}{2} < \alpha \le 1, \end{cases}$$

we have $$I_{\tilde{X}} = [0,\ 1] \setminus \left\{\frac{1}{2}\right\} \text{ by (3.1).}$$

Note that

$$L_{\alpha_1, \ldots, \alpha_k}(\tilde{X}) \text{ is degenerate at } (L_{\alpha_1}\tilde{\upsilon}, \ldots, L_{\alpha_k}\tilde{\upsilon})$$

and

$$L_{\alpha_1, \ldots, \alpha_k}(\tilde{X}_n) \text{ is degenerate at } (L_{\alpha_1}\tilde{\upsilon}_n, \ldots, L_{\alpha_k}\tilde{\upsilon}_n).$$

Thus if $\alpha_i \notin I_{\tilde{X}}$, then $\alpha_i = \frac{1}{2}$ and

$$h(L_{\alpha_i}\tilde{\upsilon}, L_{\alpha_i}\tilde{\upsilon}_n) = 1 \text{ for all } n,$$

which implies $(L_{\alpha_i}\tilde{\upsilon}_n, \ldots, L_{\alpha_k}\tilde{\upsilon}_n) \not\to (L_{\alpha_i}\tilde{\upsilon}, \ldots, L_{\alpha_i}\tilde{\upsilon})$. Therefore, we have

$$L_{\alpha_1, \ldots, \alpha_k}(\tilde{X}_n) \not\Rightarrow L_{\alpha_1, \ldots, \alpha_k}(\tilde{X}).$$ □

Our concern is to find a sufficient condition for which the converse of (3.2) holds. As usual, the concept of tightness plays an important role.

Definition 3.5 — (1) Let $\{Q_n\}$ be a sequence of probability measures on $(\mathcal{F}(R^p), \mathcal{B}_s)$. Then $\{Q_n\}$ is said to be tight if for every $\varepsilon > 0$, there exists a compact subset of $\mathcal{F}(R^p)$ such that

$$Q_n(K^c) < \varepsilon \text{ for all } n.$$

(2) Let $\{\tilde{X}_n\}$ be a sequence of fuzzy random variables. Then $\{\tilde{X}_n\}$ is said to be tight if $\{P_{\tilde{X}_n}\}$ is tight.

Theorem 3.6 — *If $\{\tilde{X}_n\}$ is tight and if for all $\alpha_1, ..., \alpha_k \in I_{\tilde{X}}$ with k arbitrary,*

$$L_{\alpha_1, ..., \alpha_k}(\tilde{X}_n) \Rightarrow L_{\alpha_1, ..., \alpha_k}(\tilde{X}),$$

then $\tilde{X}_n \Rightarrow \tilde{X}$.

PROOF : By tightness of $\{\tilde{X}_n\}$, each subsequence $\{P_{\tilde{X}_{n'}}\}$ contains a further subsequence $\{P_{\tilde{X}_{n''}}\}$ converging weakly to some limit probability measure Q on $\mathcal{F}(R^p)$. By Theorem 2.6 of Billingsley[2], it suffices to show that

$$P_{\tilde{X}} = Q.$$

If $\alpha_1, ..., \alpha_k \in I_{\tilde{X}}$, then by the hypothesis of the theorem,

$$P_{L_{\alpha_1, ..., \alpha_k}}(\tilde{X}_{n''}) \Rightarrow P_{L_{\alpha_1, ..., \alpha_k}}(\tilde{X}).$$

If $\alpha_1, ..., \alpha_k \in I_Q$, then by the above assumption and translation of (3.2) into probability measures

$$(P_{\tilde{X}_{n''}})_{L_{\alpha_1, ..., \alpha_k}} \Rightarrow Q_{L_{\alpha_1, ..., \alpha_k}}$$

Note that

$$(P_{\tilde{X}})_{L_{\alpha_1, ..., \alpha_k}} = P_{L_{\alpha_1, ..., \alpha_k}(\tilde{X})}$$

Thus, if $\alpha_1, ..., \alpha_k \in I_{\tilde{X}} \cap I_Q$, then

$$P_{L_{\alpha_1, ..., \alpha_k}(\tilde{X})} = Q_{L_{\alpha_1, ..., \alpha_k}}$$

This implies that for each Borel subset B of $\mathcal{K}^k(R^p)$,

$$P_{\tilde{X}}\left(L^{-1}_{\alpha_1, ..., \alpha_k}(B) \right) = Q\left(L^{-1}_{\alpha_1, ..., \alpha_k}(B) \right).$$

Since $I_{\tilde{X}} \cap I_Q$ has a countable complement, $\mathcal{A}_{I_{\tilde{X}} \cap I_Q}$ is a determining class by Theorem 3.1, which implies $P_{\tilde{X}} = Q$. This completes the proof. □

The next example shows that the converse of (3.2) is not true without the condition of tightness.

Example 2 — For each $n \geq 2$, let

$$\tilde{u}_n(x) = \begin{cases} 1, & \text{if } x = 0, \\ \frac{1}{2} + \frac{1}{n}(1 - |x|), & \text{if } 0 < |x| \leq 1, \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$\tilde{u}(x) = \begin{cases} 1, & \text{if } x = 0, \\ \frac{1}{2}, & \text{if } 0 < |x| \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Then

$$L_\alpha \tilde{u}_n = \begin{cases} \{x; |x| \leq 1\}, & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\ \left\{x; |x| \leq 1 - \left(\alpha - \frac{1}{2}\right)n\right\}, & \text{if } \frac{1}{2} < \alpha \leq \frac{1}{2} + \frac{1}{n} \\ \{0\} & \text{if } \frac{1}{2} + \frac{1}{n} < \alpha \leq 1, \end{cases}$$

and

$$L_\alpha \tilde{u} = \begin{cases} \{x; |x| \leq 1\}, & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\ \{0\}, & \text{if } \frac{1}{2} < \alpha \leq 1. \end{cases}$$

First we note that $\{\tilde{u}_n\}$ does not converge. For, if $\{\tilde{u}_n\}$ converge to $\tilde{v}$ for some $\tilde{v} \in \mathcal{F}(R^p)$, then there exists a sequence of functions $\{t_n\}$ in T such that

$$\lim_{n \to \infty} t_n(\alpha) = \alpha \text{ uniformly in } [0, 1]$$

and

$$\lim_{n \to \infty} d_\infty(t_n(\tilde{u}_n), \tilde{v}) = 0.$$

For each $\alpha \in [0, 1]$, if we take $\alpha_n = t_n^{-1}(\alpha)$, then

$$\alpha_n \to \alpha \text{ and } L_{\alpha_n} \tilde{u}_n \to L_\alpha \tilde{\upsilon}.$$

Then, we should have $\tilde{\upsilon} = \tilde{u}$. But for each $t \in T, d_\infty(\tilde{u}_n, t(\tilde{u})) \geq 1/2$ and so $d_s(\tilde{u}_n, \tilde{u}) \geq 1/2$ This implies that $\{\tilde{u}_n\}$ does not converge.

Thus if we take $\tilde{X}_n = \tilde{u}_n$ and $\tilde{X} = \tilde{u}$, then

$$\tilde{X}_n \not\Rightarrow \tilde{X}.$$

But if $\alpha \leq 1/2$, then

$$L_\alpha \tilde{X}_n = L_\alpha \tilde{X} = \{x : |x| \leq 1\}.$$

If $\alpha > 1/2$, then for $1/n < \alpha - 1/2$,

$$L_\alpha \tilde{X}_n = L_\alpha \tilde{X} = \{0\}.$$

Thus

$$L_{\alpha_1, \ldots, \alpha_k}(\tilde{X}_n) \Rightarrow L_{\alpha_1, \ldots, \alpha_k}(\tilde{X}) \text{ for all } \alpha_1, \ldots, \alpha_k \in [0, 1].$$

We note that $\tilde{X}_n$ is not tight. For, if $P(\tilde{X}_n \notin K) < \varepsilon$ for all n, then K must contain all the $\tilde{u}_n$. But since $\{\tilde{u}_n\}$ does not converge, K cannot be compact. □

Now we characterize the tightness for a sequence of fuzzy random variables.

***Theorem* 3.7** — *$\{\tilde{X}_n\}$ is tight if and only if*

(1) For each $\eta > 0$, there exists a $\lambda > 0$ such that for all n,

$$P\{\omega : \|\tilde{X}_n(\omega)\| > \lambda\} \leq \eta. \qquad \ldots (3.3)$$

(2) *For each $\varepsilon > 0$ and $\eta > 0$, there exists a $\delta \in (0, 1)$ such that for all n,*

$$P\{\omega : \tau(\tilde{X}_n(\omega), \delta) \geq \varepsilon\} \leq \eta. \qquad \ldots (3.4)$$

PROOF : First we note that $\tau(\tilde{X}_n, \delta)$ are real-valued random variables since $\tau(\tilde{u}, \delta)$ is upper-semicontinuous in $\tilde{u}$ for each δ in the proof of Theorem 4.3 in Joo and Kim[6].

(Necessity) : Suppose that $\{\tilde{X}_n\}$ is tight. For given $\varepsilon > 0$ and $\eta > 0$, there exists a compact subset K of $\mathcal{F}(R^p)$ such that

$$P(\tilde{X}_n \notin K) < \eta \text{ for all } n.$$

By Theorem 4.3 of Joo and Kim[6], we have that

$$K \subset \{\tilde{u} : \|\tilde{u}\| \leq \lambda\} \text{ for large enough } \lambda,$$

and

$$K \subset \{\tilde{u} : \tau(\tilde{u}, \delta) < \varepsilon\} \text{ for small enough } \delta.$$

Therefore, (1) and (2) follow.

(Sufficienty): Suppose that (1) and (2) hold. For given $\eta > 0$, we choose $\lambda > 0$ so that

$$P\left(\left\{\omega : \| \tilde{X}_n(\omega) \| > \lambda\right\}\right) \le \frac{\eta}{2} \text{ for all } n.$$

Then for each k, we choose δ_k so that

$$P\left\{\omega : \tau(\tilde{X}_n(\omega), \delta_k) \ge \frac{1}{k}\right\} \le \frac{\eta}{2^{k+1}} \text{ for all } n.$$

Let $A = \{\tilde{u} : \| \tilde{u} \| \le \lambda\}$ and $A_k = \left[\tilde{u} : \tau(\tilde{u}, \delta_k) < \frac{1}{k}\right\}$. If K is the closure of $A \cap \left(\bigcap_{k=1}^{\infty} A_k\right)$, then K is compact by Theorem 4.1 of Joo and Kim[6].

Since

$$P(\tilde{X}_n \notin K) \le P(\tilde{X}_n \notin A) + \sum_{k=1}^{\infty} P(\tilde{X}_n \notin A_k) < \eta$$

for all n, we conclude that $\{\tilde{X}_n\}$ is tight. □

Since $\mathcal{F}(R^p)$ is separable and topologically complete, a single fuzzy random variable $\tilde{X}$ is tight. By the above theorem, for given $\varepsilon > 0$ and $\eta > 0$, there exist a $\lambda \in (0, 1)$ and $\delta \in (0, 1)$ such that

$$P\{\omega : \| \tilde{X}(\omega) \| > \lambda\} \le \eta,$$

and

$$P\{\omega : \tau(\tilde{X}(\omega), \delta) \ge \varepsilon\} \le \eta.$$

Thus, if (3.3) and (3.4) are satisfied except for infinitely many n, we may ensure that (3.3) and (3.4) hold for all n by increasing λ and decreasing δ if necessary. Therefore, we have the modified form of Theorem 3.7.

Corollary 3.8 — $\{\tilde{X}_n\}$ is tight if and only if

(1) $\lim\limits_{\lambda \to \infty} \limsup\limits_{n \to \infty} P\left\{\omega : \| \tilde{X}(\omega) \| > \lambda\right\} = 0,$

(2) For each $\varepsilon > 0$,

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P\left\{\omega : \tau(\tilde{X}_n(\omega), \delta) \ge \varepsilon\right\} = 0.$$

Lemma 3.9 — For $\tilde{u} \in \mathcal{F}(R^p)$, we define

$$j(\tilde{u}) = \sup_{\alpha} j_{\tilde{u}}(\alpha).$$

Then j is continuous with respect to the two metrices d_∞ and d_s, respectively.

PROOF : First we note that

$$j_{\tilde{u}}(\alpha) = h(L_\alpha \tilde{u}, L_\alpha + \tilde{u})$$

$$\leq h(L_\alpha \tilde{u}, L_\alpha \tilde{v}) + h(L_\alpha \tilde{v}, L_\alpha + \tilde{v}) + h(L_\alpha + \tilde{v}, L_\alpha + \tilde{u})$$

$$\leq 2d_\infty(\tilde{u}, \tilde{v}) + j_{\tilde{u}}(\alpha).$$

Thus, $|j(\tilde{u}) - j(\tilde{v})| \leq 2d_\infty(\tilde{u}, \tilde{v})$ and so j is continuous with respect to the metric d_∞.

Now if $d(\tilde{u}, \tilde{v}) < \delta$, there exists a $t \in T$ such that $d_\infty(\tilde{u}, t(\tilde{v})) < \delta$. Since $j(\tilde{v}) = j(t(\tilde{v}))$, we have

$$|j(\tilde{u}) - j(\tilde{v})| = |j(\tilde{u}) - j(t(\tilde{v}))| \leq 2d_\infty(\tilde{u}, t(\tilde{v})) < 2\delta.$$

Hence, j is continuous with respect to the metric d_s. □

Corollary 3.10 — The following two conditions can be substituted for (1) in Corollary 3.8;

$$(1') \lim_{\lambda \to \infty} \limsup_{n \to \infty} P\left\{\omega : \| L_1 \tilde{X}(\omega) \| > \lambda\right\} = 0.$$

$$(1'') \lim_{\lambda \to \infty} \limsup_{n \to \infty} P\left\{\omega : j(\tilde{X}_n(\omega)) > \lambda\right\} = 0.$$

PROOF : First we note that $j(\tilde{X}_n)$ are real-valued random variables by Lemma 3.9. Since $\| L_1 \tilde{X}_n(\omega) \| \leq \| \tilde{X}_n(\omega) \|$ and $j(\tilde{X}_n(\omega)) \leq \| \tilde{X}_n(\omega) \|$, it follows that (1) implies (1′) and (1″).

Conversely, assume that (1′) and (1″) and (2) of Corollary 3.8 hold. Let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_r = 1$ be a partition of [0, 1] satisfying $\min_i (\alpha_i - \alpha_{i-1}) > \delta$ such that

$$h(L^+_{\alpha_{i-1}} \tilde{u}, L_{\alpha_i} \tilde{u}) < \tau(\tilde{u}, \delta) + 1 \text{ for all } i = 1, \ldots, r.$$

Then

$h(L_{\alpha_i} \tilde{u}, L_{\alpha_{i-1}} \tilde{u}) < \tau_{\tilde{u}}(\delta) + 1 + j(\tilde{u})$, and since $\delta r \leq 1$, we have

$$\| \tilde{u} \| = \| L_0 \tilde{u} \| \leq \| L_1 \tilde{u} \| + r(\tau(\tilde{u}, \delta) + 1 + j(\tilde{u}))$$

$$\leq \| L_1 \tilde{u} \| + \frac{1}{\delta}(\tau(\tilde{u}, \delta) + 1 + j(\tilde{u})).$$

Therefore, (1) of Corollary 3.8 follows from (1′), (1″), (2) and the inequality

$$P\ \{\omega : \| \tilde{X}_n(\omega) \| > 2\lambda\}$$

$$\leq P\ \{\omega : \| L_1 \tilde{X}_n(\omega) \| > \lambda\} + P\left\{\omega : \tau(\tilde{X}_n(\omega), \delta) + 1 + j(\tilde{X}_n(\omega)) > \lambda\,\delta\right\}$$

$$\leq P\ \{\omega : \| L_1 \tilde{X}_n(\omega) \| > \lambda\} + P\left\{\omega : \tau(\tilde{X}_n(\omega), \delta) \geq 1\right\}$$

$$+ P\left\{\omega : j(\tilde{X}_n(\omega)) > \lambda\,\delta - 2\right\}. \qquad \square$$

Corollary 3.11 — Suppose that the following two conditions are satisfied.
(1) For each $\eta > 0$, there exist an $\lambda > 0$ and an integer n_0 such that

$$P\left\{ \| L_1 \tilde{X}_n \| \geq \lambda \right\} \leq \eta \text{ for } n \geq n_0.$$

(2) For each $\varepsilon > 0$ and $\eta > 0$, there exists a $\delta \in (0, 1)$ and an integer n_0 such that

$$P\left\{ \sup_{|\alpha - \beta| \leq \delta} h(L_\alpha \tilde{X}_n, L_\beta \tilde{X}_n) \geq \varepsilon \right\} \leq \eta \text{ for } n \geq n_0.$$

Furthermore, if $L_{\alpha_1, \ldots, \alpha_k}(\tilde{X}_n) \Rightarrow L_{\alpha_1, \ldots, \alpha_k}(\tilde{X})$, whenever the α_i all lie in $I_{\tilde{X}}$ with k arbitrary, then

$$\tilde{X}_n \Rightarrow \tilde{X}.$$

PROOF : It suffices to prove that the two conditions of Corollary 3.8 are satisfied. We let

$$\phi(\tilde{u}, \delta) = \sup_{|\alpha - \beta| \leq \delta} h(L_\alpha \tilde{u}, L_\beta \tilde{u}) = \sup_{0 \leq \alpha \leq 1 - \delta} h(L_\alpha \tilde{u}, L_{\alpha + \delta} \tilde{u}).$$

First we choose an integer m sufficiently large that

$$P\left\{ \omega : \phi(\tilde{X}_n(\omega), 1/m) \geq 1 \right\} \leq \eta \text{ for } n \geq n_0.$$

Then

$$\| \tilde{X}_n(\omega) \| \leq \| L_1 \tilde{X}_n(\omega) \| + \sum_{i=1}^{m} h\left(L_{\frac{i}{m}} \tilde{X}_n(\omega), L_{\frac{i-1}{m}} \tilde{X}_n(\omega)\right)$$

$$\leq \| L_1 \tilde{X}_n(\omega) \| + m\phi(\tilde{X}_n(\omega), 1/m).$$

Thus,

$$P\ \{\omega : \| \tilde{X}_n(\omega) \| \geq \lambda + m\} \leq P\ \{\omega : \| L_1 \tilde{X}_n(\omega) \| \geq \lambda\}$$

$$+ P\ \{\omega : \phi(\tilde{X}_n(\omega), 1/m) \geq 1\}$$

$$\leq 2\eta \text{ for } n \geq n_0.$$

Therefore, the condition (1) of Corollary 3.8 is satisfied. On the other hand, we note that

$$\tau(\tilde{u}, \delta/2) \leq \phi(\tilde{u}, \delta).$$

This follows from the fact that for any partition $0 = \alpha_0 < \alpha_1 < .. < \alpha_r = 1$ satisfying $\delta/2 < \alpha_i - \alpha_{i-1} \leq \delta$,

$$h(L_{\alpha_i}\tilde{u}, L^+_{\alpha_{i-1}}\tilde{u}) \leq \phi(\tilde{u}, \delta) \text{ for all } i.$$

Thus it is trivial that the assumption (2) implies the condition (2) of Corollary 3.8. Therefore, $\tilde{X}_n \Rightarrow \tilde{X}$ follows from Theorem 3.6. □

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