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APPROXIMATION BY GENERALIZED BERNSTEIN POLYNOMIALS

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For the generalized Bernstein polynomials introduced by Cao (see J. Math. Anal. Appl., 209, 140-146 (1997)), we further study their approximation properties. We obtain the direct and inverse theorems and some characterizations on the approximation behaviour. We also give some Voronovskaja type expansions when the parameter s_n satisfies differnt condition. The obtained results generalize the corresponding ones of the classical Bernstein polynomials.

Key Words: Generalized Bernstein Polynomials; Direct Theorem; Inverse Theorem; Approximation Order; Asymptotic Expansion

1. INTRODUCTION

Let $C[a, b]$ be the space of continuous functions on $[a, b]$ in which the norm of any function f is defined by

$$\|f\| := \max_{x \in [a, b]} |f(x)|.$$

Associated with an integer $n \geq 1$, the Bernstein polynomials $\mathcal{B}_n$ in $C[0, 1]$ are defined by

$$(\mathcal{B}_n f)(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) P_{n,k}(x),$$

where

$$x \in [0, 1], f \in C[0, 1]$$

and

$$P_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}.$$

Let N be the set of natural numbers and let s_n be a sequence of natural numbers, and $s_n \geq 1$. In[1], Cao introduced the generalized Bernstein polynomials defined by

$$(C_n f)(x) := \frac{1}{s_n} \sum_{k=0}^{n} \left(\sum_{j=0}^{s_n - 1} f\left(\frac{k+j}{n+s_n-1}\right) \right) P_{n,k}(x).$$

Clearly, when $s_n = 1$, $(C_n f((x) = (\mathcal{B}_n f)(x)$. Furthermore, he proved the following main results.

Theorem 1.1 — *For every function $f \in C\ [0, 1]$, if and only if $\lim_{n \to \infty} (s_n/n) = 0$, then*

$$\lim_{n \to \infty} \| (C_n f) - f \| = 0,$$

and for $n \in Q = \{n : n \in N,$ and $0 < (s_n - 1)/n + 1/\sqrt{n} \le 1\}$ there holds:

$$\| (C_n f) - f \| \le 4\omega \left(f, \frac{s_n - 1}{n} + \frac{1}{\sqrt{n}} \right).$$

Here, $\omega(f, t)$ is the first order modulus of continuity of function f.

In this note, we further investigate the approximation properties of the generalized Bernstein polynomials $C_n f$. In Section 2, we use so-called Ditzian-Totik's modulus of continuity and give the direct and inverse theorems of approximation. Our approach is based on simultaneously establishing upper and lower bopund estimations on approximation speed and then derive the chracterizations of the behaviour of approximation order. In Section 3, we study the Voronoskaja type expansions for the generalized operators. Some asymptotic expansions formulas are given when the parameter s_n satisfies some different conditions. The obtained results include the corresponding ones for the classical Bernstein polynomials.

2. Direct and Inverse Theorems

Assume $f \in C\,[0, 1]$, the second order Ditzian-Totik's modulus of continuity is defined by (see[2]).

$$\omega_{\varphi}^{2}(f, t) := \sup_{0 < h \le t} \| \Delta_{h\varphi}^{2} f \|,$$

where

$$\varphi(x) = \sqrt{x(1-x)},\ x \in [0, 1],$$

and

$$\Delta_h^2 f(x) = \begin{cases} f(x+h) - 2f(x) + f(x-h), & x \pm h \in [0, 1]; \\ 0, & \text{otherwise.} \end{cases}$$

Give two $\mathcal{K}$-functionals which will be used in the sequel (see[2&4]):

$$\mathcal{K}_{\varphi}^{2}(f, t^2) := \inf_{g'' \in [0, 1]} \left\{ \| f - g \| + t^2 \| \varphi^2 g'' \| \right\},$$

and

$$\mathcal{K}(f, t) := \inf_{g' \in C[a, b]} \left\{ \| f - g \| + t \| g' \| \right\},$$

then it is shown in[2] that

$$C^{-1} \mathcal{K}_{\varphi}^{2}(f, t^2) \le \omega_{\varphi}^{2}(f, t) \le C\, \mathcal{K}_{\varphi}^{2}(f, t^2) \qquad \text{... (2.1)}$$

and

$$C^{-1}\,\mathcal{K}(f,t) \le \omega(f,t) \le C\,\mathcal{K}(f,t) \qquad \text{... (2.2)}$$

here and in the following, C denotes the positive constant independent of f, x and n, its value may be different in different occurrence.

Our main results can be stated as follows:

***Theorem* 2.1** — *For $f \in C\,[0,1]$ and $\lim_{n\to\infty}(s_n/n) = 0$, we have*

$$\|(C_n f) - f\| \le C\left(\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right) + \omega\left(f, \frac{s_n - 1}{n}\right)\right).$$

***Theorem* 2.2** — If $f \in C\,[0,1]$ *and* $\lim_{n\to\infty}(s_n/n) = 0$, *then*

$$\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right) \le C n^{-1} \sum_{k=1}^{n} \|(C_k f) - f\|$$

and

$$\omega\left(f, \frac{1}{n}\right) \le C n^{-1} \left(\sum_{k=1}^{n} \|(C_k f) - f\| + \|f\|\right).$$

From Theorem 2.1 and Theorem 2.2 we easily obtain the following corollaries.

Corollary 2.3 — If $f \in C\,[0,1]$ and $0 < \alpha < 1$, then the sufficient and necessary condition for which

$$\|(\mathcal{B}_n f) - f\| = O\left(\frac{1}{n^\alpha}\right)$$

is $$\omega_\varphi^2(f,t) = O(t^{2\alpha})$$

Corollary 2.4 — If $f \in C\,[0,1]$ and $0 < \alpha < 1$, $s_n > 1$ and $\lim_{n\to\infty}(s_n/n) = 0$, then the statement that $\omega_\varphi^2(f,t) = O(t^{2\alpha})$ and $\omega(f,t) = O(t^\alpha)$ implies

$$\|(C_n f) - f\| = O\left(\left(\frac{s_n}{n}\right)^\alpha\right).$$

Corollary 2.5 — If $s_n > 1$ and $s_n = O(n^{1-\varepsilon})$, $0 < \varepsilon \le 1$, then for any $f \in C\,[0,1]$ and $0 < \alpha < 1$, the statement

$$\|(C_n f) - f\| = O(n^{-\varepsilon\alpha})$$

implies that $\omega_\varphi^2(f,t) = O(t^{2\varepsilon\alpha})$ and $\omega(f,t) = O(t^{\varepsilon\alpha})$.

From Corollary 2.4 and Corollary 2.5, we have

Corollary 2.6 — If $f \in C\,[0,1]$ and $0 < \alpha < 1$ and $< s_n = O(1)$, then the sufficient and necessary condition satisfying

$$\| (C_n f) - f \| = O\left(\frac{1}{n^{\alpha}} \right)$$

is $\omega_{\varphi}^2(f, t) = O(t^{2\alpha})$ and $\omega(f, t) = O(t^{\alpha})$.

To prove Theorem 2.1, we use a known upper estimation on Bernstein polynomials as an intermediate step of deduction. Recalling that (see[4] or [3]).

$$\| (\mathcal{B}_n f) - f \| \le C\, \omega_{\varphi}^2 \left(f, \frac{1}{\sqrt{n}} \right)$$

and using the fact that

$$\frac{1}{s_n} \sum_{j=0}^{s_{n}-1} \left| f\left(\frac{k}{n} \right) - f\left(\frac{k+j}{n + s_n - 1} \right) \right|$$

$$\le \frac{1}{s_n} \sum_{j=0}^{s_{n}-1} \omega \left(f, \frac{| k(s_n - 1) - nj |}{n(n + s_n - 1)} \right)$$

$$\le \omega\left(f, \frac{s_n - 1}{n} \right),$$

we get

$$\| (C_n f) - f \| \le \| \mathcal{B}_n f) - f \| + \| (\mathcal{B}_n f) - (C_n f) \|$$

$$\le C\, \omega_{\varphi}^2 \left(f, \frac{1}{\sqrt{n}} \right) + \frac{1}{s_n} \sum_{k=0}^{n} \sum_{j=0}^{s_n - 1} \left| f\left(\frac{k}{n} \right) - f\left(\frac{k+j}{n + s_n - 1} \right) \right| P_{n,k}(x)$$

$$\le C\left(\omega_{\varphi}^2 \left(f, \frac{1}{\sqrt{n}} \right) + \omega\left(f, \frac{s_n - 1}{n} \right) \right).$$

This completes the proof of Theorem 2.1.

The proof of Theorem 2.2 is based on the following lemmas. At first, similar to the estimations for the Bernstein polynomials given in[2,3&4], it is not difficult to derive the following Lemma 2.7.

Lemma 2.7 — The following inequalities hold:

$$\| (C_n f) \| \le \| f \|, \; f \in C[0, 1]; \quad \| (C_n f)' \| \le 2n \| f \|, \; f \in C[0, 1];$$

$$\| (C_n f)'' \| \le 4n^2 \| f \|, \; f \in C[0, 1]; \quad \| \varphi^2 (C_n f)'' \| \le 2n \| f \|, \; f \in C[0, 1];$$

$$\| (C_n f)' \| \le 2n \| f' \|, \; f' \in C[0, 1]; \quad \| (C_n f)'' \| \le \| f'' \|, \; f'' \in C[0, 1].$$

Then, we prove a Bernstein type inequality.

Lemma 2.8 — For $f'' \in C[0, 1]$, one has

$$\| \varphi^2 (C_n f)'' \| \le \| \varphi^2 f'' \| + \frac{1}{n} \| f'' \|.$$

PROOF : Let $h = (n + s_n - 1)^{-1}$, then by simple calculation we have

$$(C_n f)''(x) = \frac{n(n-1)}{s_n} \sum_{k=0}^{n-2} \left(\sum_{j=0}^{s_n - 1} \Delta_h^2 f\left(\frac{k+j+1}{n+s_n-1} \right) \right) P_{n-2,k}(x).$$

Therefore, since $\phi^2 x) P_{n-2,k}(x) = P_{n,n+1}(x)$, we have

$$\left| \varphi^2 (x) (C_n f)''(x) \right|$$

$$= \left| \frac{1}{s_n} \sum_{k=0}^{n-2} (k+1)(n-k-1) \left(\sum_{j=0}^{s_n-1} \Delta_h^2 f\left(\frac{k+j+1}{n+s_n-1} \right) \right) P_{n,k+1}(x) \right|$$

$$= \left| \frac{1}{h^2 s_n} \sum_{k=1}^{n-1} \sum_{j=0}^{s_n-1} \frac{k}{n+s_n-1} \frac{n-k}{n+s_n-1} \left(\Delta_h^2 f\left(\frac{k+j}{n+s_n-1} \right) \right) P_{n,k}(x) \right|$$

$$\doteq \frac{1}{h^2 s_n} \sum_{k=1}^{n-1} \sum_{j=0}^{s_n-1} \varphi^2 \left(\frac{k+j}{n+s_n-1} \right) \left| \Delta_h^2 f\left(\frac{k+j}{n+s_n-1} \right) P_{n,k}(x) \right|$$

$$= \frac{1}{h^2 s_n} \sum_{k=1}^{n-1} \sum_{j=0}^{s_n-1} \varphi^2 \left(\frac{k+j}{n+s_n-1} \right) P_{n,k}(x)$$

$$\left| \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} f'' \left(\frac{k+j}{n+s_n-1} + s + t \right) ds\, dt \right| .$$

Let

$$y = (k+j)/(n+s_n-1),\ 1 \le k \le n-1,\ 0 \le j \le s_n - 1, \text{ then}$$

$$h = \frac{1}{n+s_n-1} \le y \le 1 - \frac{1}{n+s_n-1} = 1 - h$$

and for $|u| \le h$, there holds $|1 - 2y - u| \le 1$. Hence,

$$\varphi^2(y) = \varphi^2(y+u) - u(1-2y-u) \le \varphi^2(y+u) + |u| \le \varphi^2(y+u) + h,$$

which implies that

$$\varphi^2(y)\left|\Delta_h^2(y)\right| \le \varphi^2(y)\left|\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} f''(y+s+t)\,ds\,dt\right|$$

$$\le \int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\left(\varphi^2(y+s+t)+h\right)|f''(y+s+t)\,ds\,dt|$$

$$\le h^2\left(\left\|\varphi^2 f''\right\| + \frac{1}{n}\|f''\|\right)$$

So,

$$\le \left\|\varphi^2(C_n f)''\right\| \le \left\|\varphi^2 f''\right\| + \frac{1}{n}\|f''\|$$

This enables us to end the proof of Lemma 2.8.

We also need the following two interesting results related to the non-negative numerical sequences. The proof of the first result can be find in[5], and the proof of the other is similar to Lemma 2.1 of[5] where the proof of case $v_1 = 0$ and $C = 1$ was given.

Lemma 2.10 — Let μ_n, v_n and ψ_n are all non-negative numerical sequence and $\mu_1 = v_1 = 0$, if for $0 < r < s$ and $1 \le k \le n, n \in N$, there holds

$$\mu_n \le \left(\frac{k}{n}\right)^r \mu_k + v_k + \psi_k, \qquad v_n \le \left(\frac{k}{n}\right)^s v_k + \psi_k,$$

then

$$\mu_n \le C n^{-r} \sum_{k=0}^{n} k^{r-1}\psi_k.$$

Lemma 2.9 — Let v_n and ψ_n are all non-negative numerical sequence, if for $s > 0$ and $1 \le k \le n, n \in N$, there holds

$$v_n \le \left(\frac{k}{n}\right)^s v_k + C\psi_k,$$

then

$$v_n \le C n^{-s}\left(\sum_{k=0}^{n} k^{s-1}\psi_k + v_1\right).$$

Now, we turn to the proof of Theorem 2.2. Let $\mu_n = n^{-1}\|\varphi^2(C_n f)''\|$, $v_n = n^{-2}\|(C_n f)''\|$, $\psi_n = 4\|(C_n f) - f\|$, then $\mu_1 = v_1 = 0$ and from Lemma 2.7 and Lemma 2.8 we have for $1 \le k \le n$

$$\mu_n \le n^{-1}\|\varphi^2(C_n(C_k f))''\| + n^{-1}\|\varphi^2(C_n(C_k f - f - f))''\|$$

$$\leq n^{-1} \| \varphi^2 (C_k f))'' \| + \frac{1}{n^2} \| (C_k f)'' \| + 2 \| (C_k f) - f \|$$

$$\leq \left(\frac{k}{n} \right) \mu_k + v_k + \psi_k.$$

$$v_n \leq n^{-2} \| (C_n (C_k f))'' \| + n^{-2} \| (C_n (C_k f - f - f))'' \|$$

$$\leq n^{-2} \| (C_k f))'' \| + 4 \| (C_k f) - f \|$$

$$\leq \left(\frac{k}{n} \right)^2 v_k + \psi_k,$$

which implies from Lemma 2.9 that $\mu_n \leq C n^{-1} \sum_{k=1}^{n} \psi_k$, i.e.,

$$\| \varphi^2 (C_n f)'' \| \leq C \sum_{k=1}^{n} \| (C_k f) - f \|. \qquad \text{... (2.3)}$$

Let $v_n = \| (C_n f)' \|$ and let ψ_n be the same as the above, then similar to the above deduction, it is follows from Lemma 2.7 that

$$v_n \leq \left(\frac{k}{n} \right)^2 v_k + \psi_k, \quad 1 \leq k \leq n.$$

So, using Lemma 2.10 gives $v_n \leq C n^{-1} \left(\sum_{k=1}^{n} \psi_k + v_1 \right)$, i.e.,

$$(C_n f)' \| \leq C \left(\sum_{k=0}^{n} \| (C_n f) - f \| + \| f \| \right)$$

$$\leq C \left(\sum_{k=0}^{n} \| (C_n f) - f \| + \| f (C_1 f) \| \right). \qquad \text{... (2.4)}$$

For $n \geq 2$, there exists an $m \in N$, such that $n/2 \leq m \leq n$ and for $1 \leq k \leq n$ there holds $\| (C_m f) - f \| + \| (C_k f) - f \|$. Then

$$\| (C_m f) - f \| \leq \frac{4}{n} \cdot \sum_{k=n/2}^{n} \| (C_k f) - f \|. \qquad \text{... (2.5)}$$

So, combining (2.3), (2.5) and (2.1) we see

$$\omega_\varphi^2 \left(f, \frac{1}{\sqrt{n}} \right) \leq C \, \mathcal{K}_\varphi^2 \left(f, \frac{1}{\sqrt{n}} \right) \leq C \left(\| C_m f) - f \| + n^{-1} \| \varphi^2 (C_m f)'' \| \right)$$

$$\leq C n^{-1} \sum_{k=1}^{n} \| (C_k f) - f \|.$$

Also, collecting (2.4), (2.5) and (2.2) gets

$$\omega\left(f, \frac{1}{n}\right) \leq C \mathcal{K}\left(f, \frac{1}{n}\right) \leq C\left(\| C_m f) - f \| + n^{-1} \| \varphi^2 (C_m f)' \| \right)$$

$$\leq C n^{-1} \sum_{k=1}^{n} (\| (C_k f) - f \| + \| f \|).$$

The proof of Theorem 2.2 is complete.

3. VORONOVSKAJA TYPE EXPANSIONS

Theorem 3.1 — *Let* $s_n = qn^{\sigma}$, *where* $0 < \sigma < 1$ *and* q *is a positive constant. Then, for any* $f \in C^1 [0, 1]$, *there holds*

$$\lim_{n \to \infty} n^{1-\sigma} ((C_n f)(x) - f(x)) = q(2x-1) f'(x).$$

Theorem 3.2 — *Let* $s_n = n\lambda(n) \geq 1$, *where* $\lambda(n) > 0$, $n = 1, 2, \ldots$, *and* $\lim_{n \to \infty} \lambda(n) = 0$, $\lim_{n \to \infty} (s_n / n^{\sigma}) = \infty$ *for* $0 \leq \sigma < 1$. *Then, for any* $f \in C^1 [0, 1]$, *we have*

$$\lim_{n \to \infty} \frac{1}{\lambda(n)} ((C_n f)(x) - f(x)) = q(2x-1) f'(x).$$

Theorem 3.3 — *If* $s_n = q \geq 1$, *where* q *is a positive constant, then for any* $f \in C^2 [0, 1]$, *we have*

$$\lim_{n \to \infty} n ((C_n f)(x) - f(x)) = (1-q)(2x-1) f'(x) + \frac{1}{2} x(1-x) f''(x).$$

Now, we prove these theorems. At first, we prove Theorem 3.1. For $f \in C^1 [0, 1]$, we see

$$f(u) - f(x) = (u-x) f'(x) + \alpha(u) |u-x|, \qquad \ldots (3.1)$$

where $\alpha(u)$ is continuous and bounded in u, and $\lim_{|u-x| \to 0} \alpha(u) = 0$. Thus, there is a constant $M > 0$, such that $|\alpha(u)| \leq M$, and for arbitrary $\varepsilon > 0$, there exists a $\delta > 0$, such that when $|u-x| \leq \delta$, $\alpha(u) < \varepsilon$ holds. Therefore, for any $x, u \in T$, we have

$$|\alpha(u)| < \varepsilon + M \delta^{-1} |u-x|. \qquad \ldots (3.2)$$

Since (see also[1]).

$$(C_n (u-x))(x) = \frac{s_n - 1}{n + s_n - 1} (2x-1),$$

$$(C_n(|u-x|^2))(x) = \frac{x(1-x)(n-(s_n-1)^2)}{(n+s_n-1)^2} + \frac{(s_n-1)(2(s_n-1)+1)}{6(n+s_n-1)^2},$$

we get for $s_n = qn^{\sigma}, 0 < \sigma < 1$

$$\lim_{n\to\infty} n^{1-\sigma}(C_n(u-x))(x) = q(2x-1) \qquad \text{... (3.3)}$$

and

$$\lim_{n\to\infty} n^{1-\sigma}(C_n(|u-x|^2))(x) = 0.$$

Moreover, let $\varepsilon < n^{-1/2}$, then for $n \to \infty$, there holds

$$\varepsilon n^{1-\sigma}(C_n(|u-x|))(x) \le n^{1/2-\sigma}\left(C_n(|u-x|^2))(x)\right)^{1/2} \to 0. \qquad \text{... (3.4)}$$

Hence, combining (3.1), (3.2), (3.3) and (3.4) completes the proof of Theorem 3.1.

Similar to the proof of Theorem 3.1, it is easy to finish the proof of Theorem 3.2. We omit the details.

Finally, we prove Theorem 3.3. By straight calculation, we have for $s_n = q > 1$

$$n(C_n(u-x)^4)(x) \to 0, \quad n \to \infty \qquad \text{... (3.5)}$$

For $f \in C^2[0, 1]$, we see

$$f(u) - f(x) = (u-x)f'(x) + \frac{(u-x)^2}{2}f''(x) + \alpha(u)|u-x|^2, \qquad \text{... (3.6)}$$

where the meaning of $\alpha(u)$ is the same as the above. Similar to the discussion of Theorem 3.1, we know

$$|\alpha(u)| < \varepsilon + M\delta^{-2}|u-x|^2. \qquad \text{... (3.7)}$$

So collecting (3.5), (3.6) and (3.7) gives the conclusion of Theorem 3.3.

From Theorem 3.3, it follows that

Corollary 3.4 — If $0 \le s_1 - 1 = o(1), n \to \infty$, then for any $f \in C^2[0, 1]$, we have

$$\lim_{n\to\infty} n((C_n f)(x) - f(x)) = \frac{1}{2}x(1-x)f''(x).$$

In particular, if $s_n \equiv 1$, then Corollary 3.4 deduces a well known result for the Bernstein operators. Also, from Corollary 3.4, we see that when $0 \le s_n - 1 = o(1), n \to \infty$, the asymptotic expansion formula of Voronovskaja type of second order for the generalized Bernstein operators is the same as the one of the classical Bernstein operators.

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