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PERIODIC SOLUTIONS FOR A SCALAR INTEGRO-DIFFERENTIAL EQUATION

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Sufficient criteria are established for the existence of periodic solutions to a class of scalar integro-differential equations. The approach is based on the famous Krasnoselskii fixed point theorem. Applications to mathematical models arising in biology are presented.

Key Words: Periodic Solution; Integro-Differential Equations; Krasnoselskii Fixed Point Theorem

1. INTRODUCTION

Recently, the existence of positive periodic solutions of scalar integro-differential equations have attracted much attention from both mathematicians and mathematical biologists[2,4-7,10,12,13,15-17]. Excellent work has been reported[1,8,9,14,18]. However, such problems are far from being systematic and well studied. There is much room for improvement.

The principal aim of this paper is to explore the existence of positive periodic solutions for the following integro-differential equations of the form

$$\dot{y}(t) = -a(t)\,y(t) + \int_{-\infty}^{t} k(t, r)\, g(r, y(r))\, dr, \qquad \text{... (1.1)}$$

and

$$\dot{y}(t) = a(t)\,y(t) - \int_{-\infty}^{t} k(t, r)\, g(r, y(r))\, dr. \qquad \text{... (1.2)}$$

where $a(t) \in C(R, [0, \infty))$ is ω-periodic, $g \in C(R \times [0, \infty), [0, \infty))$ is ω-periodic with respect to the first variable, and $k(t,r) \in C(R \times R, [0, \infty))$ satisfies $k(t + \omega, r + \omega) = k(t, r)$, here ω is a given positive constant denoting the period of eqs. (1.1) and (1.2).

Equations (1.1) and (1.2) are general enough to incorporate as special cases in many mathematical models arising in biology, which have been widely studied in the literature. The readers are referred to Section 4 for detailed justification.

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Since the discussion for (1.2) is exactly the same as that for (1.1), in the following discussion, we will deliberately devote ourself to (1.1) while the details for (1.2) are omitted here.

In Section 2, we shall first present some preliminaries result including the famous Krasnoselskii fixed point theorem and some notations to be widely used throughout this paper. Section 3 is devoted to establishing sufficient criteria for the existence of positive periodic solutions of (1.1). The whole discussion is divided into several cases. In order to illustrate some features of the established sufficient criteria, in Section 4, we carry out some applications of such criteria to some mathematical models arising in biology.

2. Preliminaries

First, let's recall the Krasnoselskii fixed point theorem which will come into play later on.

Definition 2.1 — A nonempty closed subset K of the Banach space X is said to be a cone if $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cup (-K) = \{0\}$.

The following fixed point theorem is due to Krasnoselskii.

Lemma 2.1 (Krasnoselskii fixed point theorem[11]) — Let X be a Banach space and let $K \subset X$ be a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$ and let

$$\Phi : K \cap (\overline{\Omega}_2 \cap \Omega_1) \rightarrow K$$

be a completely continuous operator such that either

$$\| \Phi y \| \geq \| y \| \quad \forall y \in K \cap \partial \Omega_1 \text{ and } \| \Phi y \| \leq \| y \| \quad \forall y \in K \cap \partial \Omega_2$$

or

$$\| \Phi y \| \leq \| y \| \quad \forall y \in K \cap \partial \Omega_1 \text{ and } \| \Phi y \| \geq \| y \| \quad \forall y \in K \cap \partial \Omega_2$$

Then Φ has a fixed point in $K \cap (\overline{\Omega}_2 \cap \Omega_1)$.

Throughout this paper, we will use the following notations

$$\delta = \exp \left\{ - \int_0^{\omega} a(s)\, ds \right\}, \qquad \bar{k} = \frac{1}{\omega} \int_0^{\omega} \int_{-\infty}^{s} k(s, r)\, dr\, ds$$

$$\max g_0 = \lim_{u \to 0} \max_{t \in [0, \omega]} \frac{g(t, u)}{u}, \qquad \min g_0 = \lim_{u \to 0} \max_{t \in [0, \omega]} \frac{g(t, u)}{u},$$

$$\max g_{\infty} = \lim_{u \to +\infty} \max_{t \in [0, \omega]} \frac{g(t, u)}{u}, \qquad \min g_{\infty} = \lim_{u \to +\infty} \min_{t \in [0, \omega]} \frac{g(t, u)}{u}.$$

First, we shall make some preparation for establishing sufficient criteria for the existence of positive periodic solutions of (1.1).

Lemma 2.2 — If $y(t)$ is an ω periodic solution of (1.1), then $y(t)$ is also an ω periodic solution of the following integral equation

$$y(t) = \int_{t}^{t+\omega} G(t, s) \int_{-\infty}^{s} k(s, r)\, g(r, y(r))\, dr\, ds, \quad \ldots (2.1)$$

where

$$G(t, s) = \frac{\exp\left\{ \int_{t}^{s} a(\theta)\, d\theta \right\}}{\exp\left\{ \int_{0}^{\omega} a(\theta)\, d\theta \right\} - 1}, \quad \ldots (2.2)$$

and vice versa.

Lemma 2.2 tells us that to find an ω periodic solution of (1.1) is equivalent to find an ω periodic solution of (2.1), so in the following we will devote ourselves to eq. (2.1).

Lemma 2.3 — The function $G(t, s)$ defined by (2.2) satisfies

$$\frac{\delta}{1-\delta} = G(t, t) \leq G(t, s) \leq G(t, t+\omega) = \frac{1}{1-\delta},$$

$$G(t+\omega, s+\omega) = G(t, s), \qquad \delta \leq \frac{G(t, s)}{G(t, t+\omega)} \leq 1,$$

for any $s \in [t, t+\omega]$.

The conclusion directly follows from the definition of $G(t, s)$.

Take

$$X = \{y : y \in C(R, R), y(t+\omega) = y(t)\},$$

$$K = \{y \in X : y(t) \geq 0 \text{ and } y(t) \geq \delta \| y \| \}, \quad \ldots (2.3)$$

then it is trivial to show that X is a Banach space when endowed with the norm $\| y \| = \sup_{t \in [0, \omega]} | y(t) |$ and K is a cone in X.

Define the operator $\Phi : X \to X$ by

$$(\Phi y)(t) = \int_{t}^{t+\omega} G(t, s) \int_{-\infty}^{s} k(s, r)\, g(r, y(r))\, dr\, ds. \quad \ldots (2.4)$$

One can easily prove that Φ is completely continuous. Now, (2.1) can be rewritten as

$$y(t) = (\Phi y)(t). \quad \ldots (2.5)$$

Lemma 2.4 — $\Phi(K) \subset K$.

PROOF : For any $y \in K$, we have $(\Phi y)(t) \geq 0$. By Lemma 2.3, (2.3) and (2.4), one has

$$\| \Phi y \| \leq \frac{1}{1-\delta} \int_{t}^{t+\omega} \int_{-\infty}^{s} k(s, r)\, g(r, y(r))\, dr\, ds$$

$$= \frac{1}{1-\delta} \int_0^{\omega} \int_{-\infty}^{s} k(s,r)\, g(r, y(r))\, dr\, ds,$$

and

$$(\Phi y)(t) \geq \frac{1}{(t)-\delta} \int_t^{t+\omega} \int_{-\infty}^{s} k(s,r)\, g(r, y(r))\, dr\, ds \geq \delta \| \Phi y \|.$$

Now we can claim that $\Phi y \in K$ for any $y \in K$. The proof is complete. □

3. Existence of Positive Periodic Solutions

3.1 $\min g_0 = \infty$, $\max g_\infty = 0$ or $\max g_0 = 0$, $\min g_\infty = \infty$.

Theorem 3.1 — *Assume that*

(H_1) $\min g_0 = \infty$ *and* $\max g_\infty = 0$.

Then (1.1) has at least one positive ω-periodic solution.

PROOF : Note that $\min g_0 = \infty$, then for any $M_1 > \frac{1-\delta}{\delta^2 \bar{k} \omega}$, there exists a positive constant $\rho_0 > 0$ such that

$$g(t,u) \geq M_1 u, \quad 0 \leq u \leq \rho_0. \qquad \text{... (3.1.1)}$$

Let $\Omega_1 = \{ y \in X, \| y \| < \rho_0 \}$. For any $y \in K \cap \partial \Omega_1$, one has $\| y \| = \rho_0$ and

$$\rho_0 \geq y(t) \geq \delta \| y \| = \delta \rho_0. \qquad \text{... (3.1.2)}$$

Then from (2.4), (3.1.1) and Lemma 2.3, we get

$$(\Phi y)(t) \geq \frac{\delta}{1-\delta} \int_t^{t+\omega} \int_{-\infty}^{s} k(s,r)\, g(r, y(r))\, dr\, ds$$

$$\geq \frac{\delta}{1-\delta} M_1 \int_0^{\omega} \int_{-\infty}^{t} k(s,r)\, y(r)\, dr\, ds$$

$$\geq \frac{\delta}{1-\delta} M_1\, \delta \rho_0 \bar{k}\, \omega > \rho_0 = \| y \|,$$

therefore

$$\| \Phi y \| > \| y \|, \quad y \in K \cap \partial \Omega_1. \qquad \text{... (3.1.3)}$$

On the other hand, since $\max g_\infty = 0$, then for any $0 < \varepsilon < \frac{1-\delta}{2\bar{k}\omega}$, there exists $N_1 > \rho_0$ such that

$$0 \le g(t, u) \le \varepsilon u, \quad u > N_1. \qquad \text{... (3.1.4)}$$

Let

$$\rho_1 > 2N_1 + \frac{2}{1-\delta} \bar{k}\, \omega \max_{\substack{t \in [0, \omega] \\ 0 \le u \le N_1}} \{ g(t, u \}$$

$$\Omega_2 := \{ y \in X : \| y \| < \rho_1 \}.$$

Then for any $y \in K \cap \partial \Omega_2$, we have

$$\| y \| = \rho_1, \quad \delta \rho_1 = \delta \| y \| \le y(t) \le \rho_1.$$

With the help of (2.4), (3.1.4) and Lemma 2.3, we reach

$$(\Phi y)(t) \le \frac{1}{1-\delta} \int_0^{t+\omega} \int_{-\infty}^{s} k(s, r)\, g(r, y(r))\, dr\, ds$$

$$= \frac{1}{1-\delta} \int \int_{y(r) \le N_1} k(s, r)\, g(r, y(r))\, dr\, ds$$

$$+ \frac{1}{1-\delta} \int \int_{y(r) > N_1} k(s, r)\, g(r, y(r))\, dr\, ds$$

$$< \frac{1}{1-\delta} \bar{k}\, \omega \max_{\substack{t \in [0, \omega] \\ 0 \le u \le N_1}} \{ g(t, u \} + \frac{\varepsilon}{1-\delta} \int \int_{y(r) > N_1} k(s, r)\, y(r)\, dr\, ds$$

$$< \frac{\varepsilon}{1-\delta} \rho_1\, \omega \bar{k} + \frac{\varepsilon}{1-\delta} \rho_1\, \omega \bar{k}$$

$$< \frac{\rho_1}{2} + \frac{\| y \|}{2} = \| y \|,$$

which implies that

$$\| \Phi y \| < \| y \|, \text{ for any } y \in K \cap \partial \Omega_2. \qquad \text{... (3.1.5)}$$

Therefore, by (3.1.3), (3.1.5) and Lemma 2.1, we can conclude that the operator Φ has a fixed point $y \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$, i.e. $y(t) = (\Phi y)(t)$. □

Theorem 3.2 — *Assume that*

(H_2) $\max g_0 = \infty$ *and* $\min g_\infty = \infty$.

Then (1.1) has at least one positive ω-periodic solution.

PROOF : Since $\max g_0 = 0$, for any $0 < \varepsilon < \frac{1-\delta}{\bar{k}\,\omega}$, there exists a positive constant $\rho_2 > 0$, such that

$$g(t, u) \le \varepsilon u, \quad 0 < u < \rho_2. \qquad \text{... (3.1.6)}$$

Let $\Omega_1 = \{ y \in X, \| y \| < \rho_2 \}$. For any $y \in K \cap \partial\Omega_1$, one has $\| y \| = \rho_2$ and

$$\rho_2 \ge y(t) \ge \delta \| y \| = \delta \rho_2.$$

Then from (2.4), (3.1.6) and Lemma 2.3, we get

$$\begin{aligned}(\Phi y)(t) &\le \frac{1}{1-\delta} \int_t^{t+\omega} \int_{-\infty}^{s} k(s, r)\, g(r, y(r))\, dr\, ds \\ &\le \frac{\varepsilon}{1-\delta} M_1 \int_0^{\omega} \int_{-\infty}^{s} k(s, r)\, y(r)\, dr\, ds \\ &\le \frac{\varepsilon}{1-\delta} \| y \| \omega \bar{k} < \| y \|,\end{aligned}$$

therefore

$$\| \Phi y \| < \| y \|, \quad y \in K \cap \partial\Omega_1. \qquad \text{... (3.1.7)}$$

On the other hand, since $\min g_\infty = \infty$, then for $M_2 > \frac{1-\delta}{\delta^2 \bar{k}\,\omega}$, there exists a positive constant $\rho_3 > \frac{\rho_2}{\delta}$ such that

$$g(t, u) \ge M_2 u, \quad u \ge \rho_3. \qquad \text{... (3.1.8)}$$

Let $\Omega_2 = \{ y \in X, \| y \| < \rho_3 \}$. For any $y \in K \cap \partial\Omega_2$, we have $\| y \| = \rho_3$ and

$$\rho_3 \ge y(t) \ge \delta \| y \| = \delta \rho_3.$$

With the help of (2.4), (3.18) and Lemma 2.3, we now reach

$$(\Phi y)(t) \ge \frac{\delta}{1-\delta} \int_t^{t+\omega} \int_{-\infty}^{s} k(s, r)\, g(r, y(r))\, dr\, ds$$

$$\geq \frac{\delta}{1-\delta} M_2 \int_0^{\omega} \int_{-\infty}^{s} k(s,r)\, y(r)\, dr\, ds$$

$$\geq \frac{\delta}{1-\delta} M_2\ \delta \| y \| \bar{k}\, \omega > \| y \|,$$

which implies that

$$\| \Phi y \| > \| y \|, \text{ for any } y \in K \cap \partial \Omega_2. \qquad \text{... (3.1.9)}$$

Therefore, by (3.1.7), (3.1.9) and Lemma 2.1, we can conclude that the operator Φ has a fixed point $y \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, i.e. $y(t) = (\Phi y)(t)$.

3.2. $\max g_0 = \max g_\infty = 0$ or $\min g_0 = \min g_\infty = \infty$.

Theorem 3.3 — *Assume that*

(H_3) $\max g_0 = \max g_\infty = 0$,

(H_4) there exists a $r_1 > 0$ such that $g(t,u) > \dfrac{r_1(1-\delta)}{\delta \omega \bar{k}}$, for $u \in [\delta r_1, r_1]$.

Then (1.1) has at least two positive ω-periodic solutions y_1 and y_2 such that

$$0 < \| y_1 \| < r_1 < \| y_2 \|.$$

PROOF : Note that $\max g_0 = 0$, then for any $0 < \varepsilon < \dfrac{1-\delta}{\bar{k}\, \omega}$, there exists a sufficiently small $r_* < r_1$, such that

$$g(t,u) \leq \varepsilon u, \quad 0 < u < r_*. \qquad \text{... (3.2.1)}$$

Let $\Omega_{r_*} =: \{ y \in X : \| y \| < r_* \}$. For any $y \in K \cap \partial \Omega_{r_*}$, one has $\| y \| = r_*$ and

$$r_* \geq y(t) \geq \delta \| y \| = \delta r_*.$$

Then from (2.4), (3.2.1) and Lemma 2.3, we get

$$(\Phi y)(t) \leq \frac{1}{1-\delta} \int_t^{t+\omega} \int_{-\infty}^{s} k(s,r)\, g(r, y(r))\, dr\, ds$$

$$\leq \frac{\varepsilon}{1-\delta} \int_0^{\omega} \int_{-\infty}^{s} k(s,r)\, y(r)\, dr\, ds$$

$$\leq \frac{\varepsilon}{1-\delta} r_*\, \omega \bar{k} < r_* = \| y \|,$$

therefore

$$\|\Phi y\| < \|y\|, \quad y \in K \cap \partial\Omega_{r_*}. \qquad \text{... (3.2.2)}$$

On the other hand, since max $g_\infty = 0$, then for any $0 < \varepsilon < \frac{1-\delta}{2\bar{k}\,\omega}$, there exists $N_2 > r_1$ such that

$$0 \leq g(t, u) \geq \varepsilon\, u, \quad u \leq N_2. \qquad \text{... (3.2.3)}$$

Let

$$r^* > 2N_2 + \frac{2}{1-\delta}\bar{k}\,\omega \max_{\substack{t \in [0,\,\omega] \\ 0 \leq u \leq N_2}} \{g(t, u)\} \qquad \text{... (3.2.4)}$$

$$\Omega_{r^*} := \{y \in X : \|y\| < r^*\}.$$

Then for any $y \in K \cap \partial\Omega_{r^*}$, we have

$$\|y\| = r^*, \quad \delta r^* = \delta\|y\| \leq y(t) \leq r^*.$$

From (2.4), (3.2.3) and Lemma 2.3, it follows that

$$\begin{aligned}
(\Phi y)(t) &\leq \frac{1}{1-\delta}\int_t^{t+\omega}\int_{-\infty}^{s} k(s,r)\, g(r, y(r))\, dr\, ds \\
&= \frac{1}{1-\delta}\int \int_{0 < y(r) \leq N_2} k(s,r)\, g(r, y(r))\, dr\, ds \\
&\quad + \frac{1}{1-\delta}\int \int_{y(r) > N_2} k(s,r)\, g(r, y(r))\, dr\, ds \\
&< \frac{1}{1-\delta}\bar{k}\,\omega \max_{\substack{t \in [0,\,\omega] \\ 0 \leq u \leq N_2}} \{g(t, u\} \\
&\quad + \frac{\varepsilon}{1-\delta}\int \int_{y(r) > N_2} k(s,r)\, y(r)\, dr\, ds \\
&< \frac{r^*}{2} + \frac{\|y\|}{2} = \|y\|,
\end{aligned}$$

which implies that

$$\|\Phi y\| < \|y\| \qquad y \in K \cap \partial\Omega_{r^*}. \qquad \text{... (3.2.5)}$$

Let $\Omega_{r_1} = \{ y \in X : \| y \| < r_1 \}$. For any $y \in K \cap \partial\Omega_{r_1}$, one has $\| y \| = r_1$ and

$$r_1 \geq y(t) \geq \delta \| y \| = \delta r_1. \qquad \text{... (3.2.6)}$$

Then by (2.4) and (H_4), we get

$$\begin{aligned}(\Phi y)(t) &\geq \frac{\delta}{1-\delta} \int_t^{t+\omega} \int_{-\infty}^{s} k(s,r)\, g(r, y(r))\, dr\, ds \\ &> \frac{\delta}{1-\delta} \frac{r_1 (1-\delta)}{\delta \omega \bar{k}} \int_0^{\omega} \int_{-\infty}^{s} k(s,r)\, dr\, ds \\ &> \frac{\delta}{1-\delta} \frac{r_1 (1-\delta)}{\delta \omega \bar{k}} \omega \bar{k} = r_1 = \| y \|,\end{aligned}$$

therefore

$$\| \Phi y \| > \| y \|, \quad y \in K \cap \partial\Omega_{r_1}. \qquad \text{... (3.2.7)}$$

Therefore, by (3.2.2), (3.2.5), (3.2.7) and Lemma 2.1, we can conclude that the operator Φ has a fixed point $y_1 \in K \cap (\overline{\Omega}_{r_1} \setminus \Omega_{r_*})$, and fixed point $y_2 \in K \cap (\overline{\Omega}_{r^*} \setminus \Omega_{r_1})$. Both are positive ω-periodic solution of the eq. (1.1) and

$$0 < \| y_1 \| < r_1 < \| y_2 \|.$$

The proof is complete. □

***Theorem* 3.4** *Assume that*

(H_5) $\min g_0 = \min g_\infty = \infty$,

(H_6) there exists a $r_2 > 0$ such that $g(t,u) > \dfrac{r_2 (1-\delta)}{\omega \bar{k}}, \quad 0 < u \leq r_2$,

then (1.1) has at least two positive ω-periodic solutions y_1 and y_2 such that

$$0 < \| y_1 \| < r_2 < \| y_2 \|.$$

PROOF : Note that $\min g_0 = \infty$, then for any $M_3 > \dfrac{1-\delta}{\delta^2 \bar{k} \omega}$, there exists a positive constant $r' < r_2$, such that

$$g(t,u) \geq M_3 u, \quad 0 < u < r'. \qquad \text{... (3.2.8)}$$

Let $\Omega_{r'} =: \{ y \in X : \| y \| < r' \}$. For any $y \in K \cap \partial\Omega_{r'}$, one has $\| y \| = r'$ and

$$r' \geq y(t) \geq \delta \| y \| = \delta r'. \qquad \text{... (3.2.9)}$$

Then from (2.4), (3.2.8) and Lemma 2.3, we get

$$\begin{aligned}(\Phi y)(t) &\geq \frac{\delta}{1-\delta} \int_t^{t+\omega} \int_{-\infty}^{s} k(s, r)\, g(r, y(r))\, dr\, ds \\ &\geq \frac{\delta}{1-\delta} M_3 \int_0^{\omega} \int_{-\infty}^{s} k(s, r)\, y(r)\, dr\, ds \\ &\geq \frac{\delta}{1-\delta} M_3\, \delta \| y \|\, \omega\, \bar{k} > \| y \|,\end{aligned}$$

therefore

$$\| \Phi y \| > \| y \| \quad y \in K \cap \partial \Omega_{r'}. \qquad \text{... (3.2.10)}$$

On the other hand, since $\min g_{\infty} = \infty$, then for $M_4 > \frac{1-\delta}{\delta^2 \bar{k}\, \omega}$ there exists $r'' > r_2$ such that

$$g(t, u) \geq M_4 u, \quad u > \delta r'' \qquad \text{... (3.2.11)}$$

Let $\Omega_{r''} =: \{ y \in X : \| y \| < r'' \}$. For any $y \in K \cap \partial \Omega_{r''}$, we have $\| y \| = r''$ and

$$r'' \geq y(t) \geq \delta \| y \| = \delta r''.$$

Then from (2.4), (3.2.11) and Lemma 2.3, one has

$$\begin{aligned}(\Phi y)(t) &\geq \frac{\delta}{1-\delta} \int_t^{t+\omega} \int_{-\infty}^{s} k(s, r)\, g(r, y(r))\, dr\, ds \\ &\geq \frac{\delta}{1-\delta} M_4 \int_0^{\omega} \int_{-\infty}^{s} k(s, r)\, y(r)\, dr\, ds \\ &\geq \frac{\delta}{1-\delta} M_4\, \delta r''\, \omega\, \bar{k} > \| y \|,\end{aligned}$$

therefore

$$\| \Phi y \| > \| y \| \quad y \in K \cap \partial \Omega_{r''}. \qquad \text{... (3.2.12)}$$

Let $\Omega_{r_2} =: \{ y \in X : \| y \| < r_2 \}$. For any $y \in K \cap \partial \Omega_{r_2}$, one has $\| y \| = r_2$ and

$$r_2 \geq y(t) \geq \delta \| y \| = \delta r_2. \qquad \text{... (3.2.13)}$$

Then from (2.4) and (H_6), we now reach

$$(\Phi y)(t) \le \frac{1}{1-\delta} \int_t^{t+\omega} \int_{-\infty}^{s} k(s, r)\, g(r, y(r))\, dr\, ds$$

$$< \frac{1}{1-\delta} \frac{r_2(1-\delta)}{\omega \bar{k}} \int_0^{\omega} \int_{-\infty}^{s} k(s, r)\, dr\, ds$$

$$< \frac{1}{1-\delta} \frac{r_2(1-\delta)}{\omega \bar{k}} \omega \bar{k} = r_2 = \| y \|,$$

which implies that

$$\| \Phi y \| < \| y \|, \quad y \in K \cap \partial \Omega_{r_2}. \qquad \text{... (3.2.14)}$$

Therefore, by (3.2.10), (3.2.12), (3.2.14) and Lemma 2.1, we can conclude that the operator Φ has a fixed point $y_1 \in K \cap (\bar{\Omega}_{r_2} \setminus \Omega_{r'})$, and fixed point $y_2 \in K \cap (\bar{\Omega}_{r''} \setminus \Omega_{r_2})$. Both are positive ω-periodic solution of the eq. (1.1) and

$$0 < \| y_1 \| < r_2 < \| y_2 \|.$$

The proof is complete. □

3.3 $\max g_0$, $\min g_\infty$ and $\min g_0$, $\max g_\infty \notin \{0, \infty\}$

***Theorem* 3.5** — *Assume that there exist two positive constants $h_1 \ne h_2$ such that*

$$(H_7) \quad g(t, u) \le \frac{h_1(1-\delta)}{\omega \bar{k}}, \quad 0 \le u \le h_1, \quad t \in [0, \omega],$$

$$(H_8) \quad g(t, u) \ge \frac{h_2(1-\delta)}{\delta \omega \bar{k}}, \quad \delta h_2 \le u \le h_2, \quad t \in [0, \omega],$$

Then (1.1) has at least one positive ω-periodic solution y such that $\| y \|$ between h_1 and h_2.

PROOF : Without loss of generality, we may assume that $h_1 < h_2$.

Let $\Omega_{h_1} = \{ y \in X : \| y \| < h_1 \}$. For any $y \in K \cap \partial \Omega_{h_1}$, one has $\| y \| = h_1$ and

$$h_1 \ge y(t) \ge \delta \| y \| = \delta h_1.$$

Then from (2.4), (H_7) and Lemma 2.3, we get

$$(\Phi y)(t) \le \frac{1}{1-\delta} \int_t^{t+\omega} \int_{-\infty}^{s} k(s, r)\, g(r, y(r))\, dr\, ds$$

$$\le \frac{1}{1-\delta}\,\frac{h_1(1-\delta)}{\omega\bar{k}}\,\omega\bar{k} = h_1 = \|y\|,$$

hence

$$\|\Phi y\| \le \|y\|, \qquad y \in K \cap \partial\Omega_{h_1}. \qquad \text{... (3.3.1)}$$

Let $\Omega_{h_2} = \{y \in X : \|y\| < h_2\}$. For any $y \in K \cap \partial\Omega_{h_2}$, one has $\|y\| = h_2$ and

$$h_2 \ge y(t) \ge \delta\|y\| = \delta h_2. \qquad \text{... (3.3.2)}$$

Then from (2.4), (H_8) and Lemma 2.3, we get

$$(\Phi y)(t) \ \ge\ \frac{\delta}{1-\delta}\int_t^{t+\omega}\int_{-\infty}^{s} k(s,r)\,g(r,y(r))\,dr\,ds$$

$$\ge \frac{\delta}{1-\delta}\,\frac{h_2(1-\delta)}{\delta\,\omega\bar{k}}\,\omega\bar{k} = h_2 = \|y\|,$$

which implies that

$$\|\Phi y\| \ge \|y\|, \qquad y \in K \cap \partial\Omega_{h_2} \qquad \text{... (3.3.3)}$$

Therefore, by (3.3.1), (3.3.3) and Lemma 2.1, we can conclude that the operator Φ has a fixed point $y \in K \cap (\bar{\Omega}_{h_2} \setminus \Omega_{h_1})$, i.e. $y(t) = (\Phi y)(t)$. □

***Theorem* 3.6** — *Assume that*

$$(H_9)\quad \max g_0 = \alpha_1 \in \left(0, \frac{1-\delta}{\omega\bar{k}}\right),$$

$$(H_{10})\quad \min g_\infty = \beta_1 \in \left(\frac{1-\delta}{\delta^2\,\omega\bar{k}}, \infty\right).$$

Then (1.1) has at least one positive ω-periodic solution.

PROOF : Note that $\max g_0 = \alpha_1 \in \left(0, \frac{1-\delta}{\omega\bar{k}}\right)$, then for $\varepsilon = \frac{1-\delta}{\omega\bar{k}} - \alpha_1 > 0$, there exists a sufficiently small $h_1 > 0$, such that

$$\max_{t\in[0,\omega]} \frac{g(t,u)}{u} \le \alpha_1 + \varepsilon = \frac{1-\delta}{\omega\bar{k}}, \quad 0 \le u \le h_1,$$

which implies

$$g(t,u) \le \frac{1-\delta}{\omega\bar{k}}\,u \le \frac{1-\delta}{\omega\bar{k}}\,h_1, \quad 0 \le u \le h_1, \ t \in [0,\omega].$$

Therefore the condition (H_7) of Theorem 3.5 is satisfied.

On the other hand, since $\min g_\infty = \beta_1 \in \left(0, \frac{1-\delta}{\delta^2 \omega \bar{k}}, \infty\right)$, then for $\varepsilon = \beta_1 - \frac{1-\delta}{\delta^2 \omega \bar{k}} > 0$, there exists a sufficiently large $h_2 > 0$ such that

$$\min_{t \in [0,\omega]} \frac{g(t,u)}{u} \geq \beta_1 - \varepsilon = \frac{1-\delta}{\delta^2 \omega \bar{k}}, \quad u \geq \delta h_2,$$

thus, when $u \in [\delta h_2, h_2], t \in [0, \omega]$, one has

$$g(t,u) \geq \frac{1-\delta}{\delta \omega \bar{k}} h_2, \quad u \geq \delta h_2, \qquad t \in [0,\omega],$$

therefore the condition (H_8) of Theorem 3.5 is satisfied. By Theorem 3.5, we complete the proof. □

Theorem 3.7 — *Assume that*

$$(H_{11}) \quad \min g_0 = \alpha_2 \in \left(\frac{1-\delta}{\delta^2 \omega \bar{k}}, \infty\right),$$

$$(H_{12}) \quad \max g_\infty = \beta_2 \in \left(0, \frac{1-\delta}{\omega \bar{k}}\right).$$

Then (1.1) has at least one positive ω-periodic solution.

PROOF : Note that $\min g_0 = \alpha_2 \in \left(\frac{1-\delta}{\delta^2 \omega \bar{k}}, \infty\right)$, then for $\varepsilon = \alpha_2 - \frac{1-\delta}{\delta^2 \omega \bar{k}} > 0$, there exists a sufficiently small $h_2 > 0$, such that

$$\min_{t \in [0,\omega]} \frac{g(t,u)}{u} \geq \alpha_2 - \varepsilon = \frac{1-\delta}{\delta^2 \omega \bar{k}}, \quad 0 \leq u \leq h_2,$$

thus, when $u \in [\delta h_2, h_2], t \in [0, \omega]$, one has

$$g(t,u) \geq \frac{1-\delta}{\delta^2 \omega \bar{k}} \delta h_2 = \frac{h_2(1-\delta)}{\delta \omega \bar{k}},$$

therefore the condition (H_8) of Theorem 3.5 is satisfied.

On the other hand, since $\max g_\infty = \beta_2 \in \left[0, \frac{1-\delta}{\omega \bar{k}}\right)$, then for $\varepsilon = \frac{1-\delta}{\omega \bar{k}} - \beta_2 > 0$, there exists a sufficiently large $\rho_0 > 0$ such that

$$\max_{t \in [0,\omega]} \frac{g(t,u)}{u} \leq \beta_2 + \varepsilon = \frac{1-\delta}{\omega \bar{k}}, \quad u > \rho_0, \qquad \text{... (3.3.4)}$$

In the following, we deal with the two cases:

Case (i) — Suppose that $\max\limits_{t \in [0,\omega]} g(t,u)$ is unbounded, then there exists $u^* \in [0, \infty]$, $u^* = h_1 > \rho_0$ and $t_0 \in [0, \omega]$ such that

$$g(t,u) \le g(t_0, u^*), \quad 0 < u \le u^* = h_1, \qquad \text{... (3.3.5)}$$

then from (3.3.4), (3.3.5), we get

$$g(t,u) \le g(t_0, u^*) \le \frac{1-\delta}{\omega \bar{k}} u^* = \frac{h_1(1-\delta)}{\omega \bar{k}}, \quad 0 < u \le h_1, \quad t \in [0, \omega].$$

Therefore, the condition (H_7) of Theorem 3.5 is satisfied.

Case (ii) — Suppose $\max\limits_{t \in [0,\omega]} g(t,u)$ is bounded, then

$$g(t,u) \le M, \quad (t,u) \in [0,\omega] \times [0, \infty). \qquad \text{... (3.3.6)}$$

In this case, taking sufficiently large $h_1 \ge M \frac{\omega \bar{k}}{1-\delta}$, then from (3.3.6), we have

$$g(t,u) \le M \le \frac{h_1(1-\delta)}{\omega \bar{k}}, \quad 0 < u \le h_1, \quad t \in [0, \omega].$$

Therefore, the condition (H_7) of Theorem 3.5 is satisfied. By Theorem 3.5, we complete the proof. □

Theorem 3.8 — *Assume that (H_7), (H_{10}) and (H_{11}) hold, then (1.1) has at least two positive ω-periodic solution y_1 and y_2 such that*

$$0 < \| y_1 \| < h_1 < \| y_2 \|.$$

PROOF : From (H_{10}) and the proof of Theorem 3.6, we know that there exists a sufficiently large $h_2 > h_1$ such that

$$g(t,u) \ge \frac{h_2(1-\delta)}{\delta \omega \bar{k}}, \quad u \in [\delta h_2, h_2].$$

In view of (H_{11}) and the proof of Theorem 3.7, we see that there exists a sufficiently small $h_2^* \in (0, h_1)$ such that

$$g(t,u) \ge \frac{h_2^*(1-\delta)}{\delta \omega \bar{k}}, \quad u \in [\delta h_2^*, h_2^*].$$

Therefore, from Theorem 3.5, we know that (1.1) has two positive ω-periodic solution y_1 and y_2 such that

$$h_2^* < \| y_1 \| < h_1 < \| y_2 \| < h_2.$$

Thus, the proof is completed. □

Theorem 3.9 — *Assume that* (H_8), (H_9) *and* (H_{12}) *hold, then (1.1) has at least two positive* ω*-periodic solution* y_1 *and* y_2 *such that*

$$0 < \| y_1 \| < h_2 < \| y_2 \|.$$

PROOF : From (H_q) and the proof of Theorem 3.6, we know that there exists a sufficiently small $h_1 \in (0, h_2)$ such that

$$g(t, u) \le \frac{h_1(1-\delta)}{\omega \bar{k}}, \quad 0 < u \le h_1, \quad t \in [0, \omega].$$

In view of (H_{12}) and the proof of Theorem 3.7, we know that there exists a sufficiently large $h_1^* > h_2$ such that

$$g(t, u) \le \frac{h_1^*(1-\delta)}{\omega \bar{k}}, \quad 0 < u \le h_1^*, \quad t \in [0, \omega].$$

Therefore, by Theorem 3.5, we see that (1.1) has two positive ω-periodic solutions y_1 and y_2 such that

$$h_1 < \| y_1 \| < h_2 < \| y_2 \| < h_1^*.$$

The proof is complete. □

3.4 $\min g_0, \max g_0 \in \{0, \infty\}$, $\min g_\infty, \max g_\infty \notin \{0, \infty\}$, or
$\min g_0, \max g_0 \notin \{0, \infty\}$, $\min g_\infty, \max g_\infty \in \{0, \infty\}$

Theorem 3.10 — *Assume that*

$$(H_{13}) \quad \min g_0 = \infty \quad \text{and} \quad \max g_\infty = \beta_2 \in \left[0, \frac{1-\delta}{\omega \bar{k}}\right).$$

Then (1.1) has at least one positive ω*-periodic solution.*

PROOF : Let $\Omega_1 =: \{ y \in X : \| y \| < \rho_0 \}$, from the proof of Theorem 3.1, we get

$$\| \Phi y \| \ge \| y \|, \quad y \in K \cap \partial\Omega_1 \qquad \text{... (3.4.1)}$$

The following let $\Omega_2 =: \{ y \in X : \| y \| < \rho_1 \}$. Since $\max g_\infty = \beta_2 \in \left[0, \frac{1-\delta}{\omega \bar{k}} \right)$, from the proof of Theorem 3.7, we get

$$g(t, u) \le \frac{1-\delta}{\omega \bar{k}} \rho_1, \quad 0 < u \le \rho_1, \qquad t \in [0, \omega],$$

$$\begin{aligned}(\Phi y)(t) &\le \frac{1}{1-\delta} \int_t^{t+\omega} \int_{-\infty}^{s} k(s, r)\, g(r, y(r))\, dr\, ds \\ &\le \frac{1}{1-\delta} \frac{\rho_1 (1-\delta)}{\omega \bar{k}} \omega \bar{k} = \rho_1 = \| y \|,\end{aligned}$$

therefore

$$\| \Phi y \| \le \| y \|, \qquad y \in K \cap \partial \Omega_2 \qquad \text{... (3.4.2)}$$

Therefore, from (3.4.1), (3.4.2) and Lemma 2.1, the proof is complete. □

Theorem 3.11 — *Assume that*

$$(H_{14}) \quad \max g_\infty = 0 \quad \text{and} \quad \min g_0 = \alpha_2 \in \left(\frac{1-\delta}{\delta^2 \omega \bar{k}}, \infty \right).$$

Then (1.1) has at least one positive ω-periodic solution.

PROOF : Let $\Omega_1 =: \{ y \in X : \| y \| < \rho_0 \}$, from the proof of Theorem 3.1, we get

$$\| \Phi y \| \le \| y \|, \; y \in K \cap \partial \Omega_1. \qquad \text{... (3.4.3)}$$

The following let $\Omega_2 =: \{ y \in X : \| y \| < \rho_1 \}$. Since $\min g_0 = \alpha_2 \in \left(\frac{1-\delta}{\delta^2 \omega \bar{k}}, \infty \right)$, from the proof of Theorem 3.7, we get

$$g(t, u) \ge \frac{1-\delta}{\delta \omega \bar{k}} \rho_1, \quad 0 < u \le \rho_1, \qquad t \in [0, \omega],$$

$$\begin{aligned}(\Phi y)(t) &\ge \frac{\delta}{1-\delta} \int_t^{t+\omega} \int_{-\infty}^{s} k(s, r)\, g(r, y(r))\, dr\, ds \\ &\ge \frac{\delta}{1-\delta} \frac{\rho_1 (1-\delta)}{\delta \omega \bar{k}} \delta \omega \bar{k} = \rho_1 = \| y \|,\end{aligned}$$

hence

$$\| \Phi y \| \ge \| y \|, \qquad y \in K \cap \partial \Omega_2 \qquad \text{... (3.4.4)}$$

Therefore, from (3.4.3), (3.4.4) and Lemma 2.1, the proof is complete. □

The following proof is similar to Theorem 3.10 and Theorem 3.11.

Theorem 3.12 — *Assume that*

$$(H_{15})\quad \max g_0 = 0 \quad \text{and} \quad \min g_\infty = \beta_1 \in \left(\frac{1-\delta}{\delta^2 \omega \bar{k}}, \infty \right).$$

Then (1.1) has at least one positive ω-periodic solution.

Theorem 3.13 — *Assume that*

$$(H_{16})\quad \min g_\infty = \infty \quad \text{and} \quad \max g_0 = \alpha_1 \in \left(\frac{1-\delta}{\omega \bar{k}} \right).$$

Then (1.1) has at least one positive ω-periodic solution.

Theorem 3.14 — *Assume that*

$$(H_{17})\quad \max g_0 = \infty \quad \text{and} \quad \min g_\infty = \beta_1 \in \left[0, \frac{1-\delta}{\delta^2 \omega \bar{k}}, \infty \right),$$

(H_{18}) there exists a $r_2 > 0$ such that $g(t, u) < \dfrac{r_2 (1-\delta)}{\omega \bar{k}}$, $0 < u \le r_2, t \in [0, \omega]$.

Then (1.1) has at least two positive ω-periodic solution y_1 and y_2 such that

$$0 < \| y_1 \| < r_2 < \| y_2 \|.$$

PROOF : Let $\Omega_{r_0} =: \{ y \in X : \| y \| < r_0 \}, r_0 < r_2$, since $\min g_0 = \infty$, then from the proof of Theorem 3.1, we get

$$\| \Phi y \| \ge \| y \|, \quad y \in K \cap \partial \Omega_{r_0}. \qquad \text{... (3.4.5)}$$

The following let $\Omega_{r_1} =: \{ y \in X : \| y \| < r_1 \}, r_1 > r_2$,. Since $\min g_\infty = \beta_1 \in \left(\dfrac{1-\delta}{\delta^2 \omega \bar{k}}, \infty \right)$, from the proof of Theorem 3.6, we get

$$g(t, u) \ge \frac{1-\delta}{\delta \omega \bar{k}} r_1, \quad \delta r_1 \le u \le r_1, \qquad t \in [0, \omega],$$

$$(\Phi y)(t) \ \ge \ \frac{\delta}{1-\delta} \int_t^{t+\omega} \int_{-\infty}^{s} k(s, r)\, g(r, y(r))\, dr\, ds$$

$$\ge \frac{\delta}{1-\delta} \frac{r_1 (1-\delta)}{\delta \omega \bar{k}} \omega \bar{k} = r_1 = \| y \|,$$

therefore

$$\| \Phi y \| \ge \| y \|, \qquad y \in K \cap \partial \Omega_{r_1} \qquad \text{.. (3.4.6)}$$

Finally, let $\Omega_{r_2} =: \{ y \in X : \| y \| < r_2 \}$, from (H_{18}) and (2.4), we get

$$(\Phi y)(t) \le \frac{1}{1-\delta} \int_t^{t+\omega} \int_{-\infty}^{s} k(s,r)\, g(r, y(r))\, dr\, ds$$

$$< \frac{1}{1-\delta} \frac{r_2(1-\delta)}{\omega \bar{k}} \omega \bar{k} = r_2 = \| y \|,$$

therefore

$$\| \Phi y \| < \| y \|, \qquad y \in K \cap \partial \Omega_{r_2} \qquad \text{... (3.4.7)}$$

From (3.4.5), (3.4.6) and Lemma 2.1, we can conclude that the operator Φ has a fixed point $y_1 \in K \cap (\bar{\Omega}_{r_2} \setminus \Omega_{r_0})$, and fixed point $y_2 \in K \cap (\bar{\Omega}_{r_1} \setminus \Omega_{r_2})$ Both are positive ω-periodic solution of the eq. (1.1) and

$$0 < \| y_1 \| < r_2 < \| y_2 \|.$$

The proof is complete. □

The following proof is similar to that of Theorem 3.14.

Theorem 3.15 — *Assume that*

(H_{19}) $\min g_\infty = \infty$ and $\min g_0 = \alpha_2 \in \left(\frac{1-\delta}{\delta^2 \omega \bar{k}}, \infty \right)$,

(H_{20}) there exists a $r_3 > 0$ such that $g(t, u) < \frac{r_3(1-\delta)}{\omega \bar{k}}$, $0 < u \le r_3$, $t \in [0, \omega]$.

Then (1.1) has at least two positive ω-periodic solution y_1 and y_2 such that

$$0 < \| y_1 \| < r_3 < \| y_2 \|.$$

Theorem 3.16 — *Assume that*

(H_{21}) $\max g_0 = 0$ and $\max g_\infty = \beta_2 \in \left[0, \frac{1-\delta}{\omega \bar{k}} \right)$,

(H_{22}) there exists a $r_4 > 0$ such that $g(t, u) > \frac{r_4(1-\delta)}{\delta \omega \bar{k}}$, $\delta r_4 < u \le r_4$, $t \in [0, \omega]$.

Then (1.1) has at least two positive ω-periodic solution y_1 and y_2 svch that

$$0 < \| y_1 \| < r_4 < \| y_2 \|.$$

Theorem 3.17 — *Assume that*

$$(H_{23}) \quad \max g_{\infty} = 0 \quad \text{and} \quad \max g_0 = \alpha_1 \in \left[0, \frac{1-\delta}{\omega \bar{k}} \right),$$

$$(H_{24}) \text{ there exists a } r_5 > 0 \text{ such that } g(t, u) > \frac{r_5 (1-\delta)}{\delta \omega \bar{k}}, \quad \delta r_5 < u \le r_5, t \in [0, \omega]. .$$

Then (1.1) has at least two positive ω-periodic solution y_1 and y_2 such that

$$0 < \| y_1 \| < r_5 < \| y_2 \|.$$

4. Applications

In order to illustrate some features of our main results, in this section, we apply the criteria established above to some mathematical models arising in biology, which have been widely explored in the literature.

Example 4.1 — Consider the Hematoroiesis model[3,16]

$$\dot{y}(t) = - a(t)\, y(t) + \int_{-\infty}^{t} k(t, r)\, e^{-\beta(r)\, y(r)}\, dr, \qquad \text{... (4.1.1)}$$

where $a(t) \in C(R, [0, \infty))$, $\beta(r) \in C(R, [0, \infty))$ are ω-periodic; $k(t, r) \in C(R \times R, [0, \infty))$ satisfies $k(t+\omega, r+\omega) = k(t, r)$, where $\omega > 0$ is a given positive constant, denoting the common period of the parameters in (4.1.1).

Applying Theorem 3.1 to (4.1.1), one can easily reach the following conclusion.

Corollary 4.1 — Eq. (4.1.1) has at least one positive ω-periodic solution.

Example 4.2 — Consider the Haematopoiesis (blood cell production)[8,9,11]

$$\dot{y}(t) = - a(t)\, y(t) + \int_{-\infty}^{t} k(t, r) \frac{1}{1 + y(r)^n}\, dr, \quad n > 0 \qquad \text{... (4.1.2)}$$

where $a(t) \in C(R, [0, \infty))$ is ω-periodic; $k(t, r) \in C(R \times R, [0, \infty))$ satisfies $k(t+\omega, r+\omega) = k(t, r)$, $\omega > 0$ is a given positive constant.

Theorem 3.1 implies that

Corollary 4.2 — Eq. (4.1.2) has at least one positive ω-periodic solution.

Example 4.3 — Consider the equation

$$\dot{y}(t) = - a(t)\, y(t) + \int_{-\infty}^{t} k(t, r) \frac{y(r)}{1 + y(r)^n}\, dr, \quad n > 0 \qquad \text{... (4.1.3)}$$

where $a(t) \in C(R, [0, \infty))$ is ω-periodic; $k(t, r) \in C(R \times R, [0, \infty))$ satisfies $k(t+\omega, r+\omega) = k(t, r)$, $\frac{1-\delta}{\delta^2 \omega \bar{k}} < 1$, $\omega > 0$ is a given positive constant.

Theorem 3.11, it follows that

Corollary 4.3 — Eq. (4.1.3) has at least one positive ω-periodic solution.

Example 4.4 — Consider the more general Nicholson's blow flies model[5,7,8,20]

$$y(t) = -a(t)\, y(t) + \int_{-\infty}^{t} k(t, r)\, y(r)\, e^{-\beta(r)\, y(r)}\, dr, \qquad \dots (4.1.4)$$

where $a(t) \in C(R, [0, \infty))$, $\beta(r) \in C(R, [0, \infty))$ are ω-periodic; $k(t, r) \in C(R \times R, [0, \infty))$ satisfies $k(t+\omega, r+\omega) = k(t, r)$, $\frac{1-\delta}{\delta^2 \omega \bar{k}} < 1$, $\omega > 0$ is a given positive constant.

Application of Theorem 3.11, leads to

Corollary 4.4 — Eq. (4.1.4) has at least one positive ω-periodic solution.

5. Conclusive Remarks

In the present paper, we have explored the existence of periodic solutions to a class of scalar integro-differential equations, which incorporate as special cases many models arising in biology. New sufficient criteria are established based on the approach of the famous Krasnoselskii fixed point theorem.

It should be pointed out that similar problems have been widely attacked by many authors. The equations investigated here are more general than those investigated by other authors, for example, the models in Section 4 are special cases of (1.1).

In[8,9,18] Jiang *et al.*, only dealt with (1) : $\min g_0 = \infty$ and $\max g_\infty = 0$, (2) : $\min g_0 = 1$ and $\max g_\infty = 0$, (3) : $\max g_0 = 0$ and $\min g_\infty = \infty$.

As for the common cases investigated by Jiang and us, the results obtained above generalized and improved the main results of[8,9,18]

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