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TWISTED FREE ACTIONS OF CYCLIC GROUPS OF PRIME-POWER ORDER

SATYA PRAKASH TRIPATHI

Department of Mathematics, Kirori Mal College, Delhi University, Delhi 110 007

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In this paper, we prove that if $G = \mathbb{Z}_{p^r}$, r an arbitrary positive integer and p a prime, acts freely on a space X, then for each $n \geq 2$, the twisted action of the cyclic group $\tilde{G}_n = \mathbb{Z}_{p^r n}$ on X^n is free if and only if n is also a power of p. This generalizes a result by Cusick and Tannenbaum who have proved the case $p^r = 2$.

Key Words : Diagonal Action; Twisted Action; Free Action

Let X be a topological space and G be a topological group acting continuously on X. Then, every element $g \in G$ can be thought of as a homeomorphism of X. We recall that the action is **free** if each $g\ (\neq e)$ moves every point of X, i.e., for each $x \in X$, $g \cdot x \neq x$. Existence and classification of free actions on various nice spaces has been extensively studied (see Bredon[1], Carlsson[2], Hoffman[3,4] and Cusick[5]. Determining free actions on the products of spheres and various other projective spaces (real or complex) has been an interesting problem. Various tools from homology and cohomology groups to spectral sequences have been successfully applied in such a study. A particular situation, which occurs very frequently in the investigation of free actions on products of spaces, is the following:

Let the cyclic group $\mathbb{Z}_m\ (m \geq 2)$ act freely on a topological space X. This is equivalent to saying that there is a homeomorphism $h : X \to X$ of period m, which has the property that each of $h, h^2, ..., h^{m-1}$ moves every point of X. Consider the n-fold, $(n \geq 2)$, Cartesian product X^n of X. Then the homeomorphism h on X defines a map $t : X^n \to X^n$ given by the formula

$$t(x_1, x_2, ..., x_n) = (x_2, x_3, ..., x_n, h(x_1)). \qquad ... (1)$$

Clearly, $t^{m \cdot n} = I_{X^n}$, which means t is indeed a homeomorphism of the product space X^n of period $m \cdot n$. In other words, t generates an action of the cyclic group $\mathbb{Z}_{m \cdot n}$ on the product space X^n. An action of $\mathbb{Z}_{m \cdot n}$ on the product space X^n defined by the formula (1) is called a twisted action on X^n induced by the given action of $\mathbb{Z}_m$ on the space X.

Now the question is : If the given action of $\mathbb{Z}_m$ on X is free, when can we say that the twisted action of the cyclic group $\mathbb{Z}_{m \cdot n}$ on the product space X^n defined above is also free? Cusick and Tannenbaum[7] have studied this question for the case $m = 2$. They have also given some interesting applications of this case. We proved the case $m = 3$ in (8). In this paper, we answer the above question for the general case $m = p^r$, where r is a positive integer and p is a prime. Our result is the following.

Theorem 1 — *If $G = \mathbb{Z}_{p^r}$, r an arbitrary positive integer and p a prime, acts freely on a space X, then for each $n \geq 2$, the twisted action of the cyclic group $\tilde{G}_n = \mathbb{Z}_{p^2 \cdot n}$ on X^n is free if and only if n is also a power of p.*

Remarks : (*a*) Note that when a group G acts on a space X, then G also acts on the Cartesian product X^n $(n \geq 2)$ under the diagonal action defined by

$$g \cdot (x_1, x_2, ..., x_n) = (g \cdot x_1, g \cdot x_2, ..., g \cdot x_n).$$

Clearly, the action of G on X is free iff the corresponding diagonal action of G on X^n is free.

Let $h : X \to X$ be a homeomorphism of X which generates the action of $G = [h] = \mathbb{Z}/m\mathbb{Z}$, $m \geq 2$, on X. Consider the homeomorphism $t : X^n \to X^n$ defined by $t \cdot (x_1, ..., x_n) = (x_2, ..., h(x_1))$. Let $\tilde{G}_n = [t]$ be the resulting cyclic group $\mathbb{Z}/mn\mathbb{Z}$ generated by t, which acts on X^n. Note that the subgroup of $\tilde{G}_n$ generated by t^n is isomorphic to G and acts on X^n by the formula

$$t^n \cdot (x_1, ..., x_n) = (h(x_1), \ldots, h(x_n)).$$

This action of G on X^n is the same as the diagonal action of $G = [h]$ on X^n. In other words, $\tilde{G}_n$ extends the diagonal action of G on X^n.

(*b*) Let $G = \mathbb{Z}/k\mathbb{Z}$ act on X. Write $Y = X^n$ and $\mathbb{Z} = Y^n$. Consider the subgroup of $\tilde{G}_n = \mathbb{Z}/kn\mathbb{Z} \subseteq \tilde{G}_{m \cdot n}$ acting on X^n. The subgroup $\tilde{G}_n$ also acts on X^n by restriction. This action is identical with the action of the cyclic group of order $n \cdot k$ obtained from regarding $\mathbb{Z}$ as Y^n, where the action of the cyclic group of order k on $Y = X^m$ is the diagonal action of $G = \mathbb{Z}/k\mathbb{Z}$. We will use this fact in our proof.

PROOF OF THE THEOREM 1 : For notational simplicity, we assume $p^r = q$ throughout the proof. To prove the direct part of the theorem, we suppose that n is power of p, i.e., $n = p^k$ for some $k \geq 1$, and show that the action of $\tilde{G}_n$ on X^n is free. This is done by induction on k. Let

$k = 1$. Because the group G acts freely on X, we note that G, regarded as a subgroup of $\tilde{G}_n$, acting on X^n via the diagonal action, acts freely. Hence it suffices to show that no element of $\tilde{G}_n - G$ can fix any point of X^n. If we take any element of $\tilde{G}_n - G$, we observe that it generates the whole group $\tilde{G}_n$, and so it suffices to show that $t \in \tilde{G}_n$ does not have a fixed point. If on the contrary, t has a fixed point, say $t \cdot (z) = z$, where $z = (x_1, ..., x_n)$, then we must have $(x_1, ..., x_n) = (x_2, ..., x_n\, h\,(x_1))$, i.e., $x_1 = x_2 = \ldots = x_n = h\,(x_1)$. Thus, $x_1 = h\,(x_1)$, a contradiction since h does not fix any point of X.

Next, we assume that the result is true for $m = p^k, k \geq 1$. Let $n = p^{k+1} = m \cdot p$. Thus, $\tilde{G}_m$ acts freely on X^m by inductive hypothesis. Let $Y = X^p$ and consider the diagonal action of G on Y. Inductive hypothesis applied to this G-action on Y says that $\tilde{G}_m$ acts on $Y^m = X^n$ freely. The homeomorphism $\tau = Y^m = X^n \to X^n = Y^m$ defined by

$$\tau\,(y_1, ..., y_m) = (y_2, ..., y_m, h\,(y_1))$$

is the generator of the $\tilde{G}_m$ action on Y^m. Writing $y_i = (x_{i_p - p + 1} ..., x_{i_p})$, we see that

$$\begin{aligned} \tau\,(y_1, ..., y_m) &= \tau\,(x_1, ..., x_n) \\ &= (x_{p+1}, ..., x_n, h\,(x_1), ..., h\,(x_p)) \\ &= t^p\,(x_1, ..., x_n), \end{aligned}$$

where $t \in \tilde{G}_n$. Thus, the subgroup $\tilde{G}_m$ of $\tilde{G}_n$ acts freely on the space X^n. To show that $\tilde{G}_n$ acts freely on X^n, note once again that any element of $\tilde{G}_n - \tilde{G}_m$ generates the whole group $\tilde{G}_n$. Hence, it suffices to show that $t \in \tilde{G}_n$ does not have a fixed point in X^n. But that is obvious as before and we conclude that $\tilde{G}_n$ acts freely on X^n. Thus, by the principle of induction, the result is true for all powers p^k, $k \geq 1$, of p.

Now, we prove the converse part of the theorem. Assume that n is not a power of p. With this assumption, we will show that the $\tilde{G}_n$-action on X^n is not free. In fact, we will show that, in this case, the element t^q, where $q = p^r$, fixes some point of X^n. To do this, we consider two cases, viz., (a) when $(n \cdot p) = 1$ and (b) when n is a multiple of p but is not a power of p, i.e., $n = p^s \cdot m$ where $(m, p) = 1$. Here, both m and n are greater than 1.

Case I — If $(n, p) = 1$, we choose integers a and b such that $a \cdot n + b \cdot q = 1$. Let $x \in X$ be an arbitrary element of X. Let $g = h^a \in G$ and $z = (x, g(x), ..., g^{n-1}(x))$. We claim that $t^q(z) = z$. To prove this, let $q = ln + s$ with $l \geq 0$ and $0 \leq s < n$. Then, by definition, we make l rounds of the n-tuples of z and then move s places to the right, multiply each element by h^l and bring them in front. We multiply the first s coordinates by h^{l+1} and move them at the end, that is,

$$t^q(z) = (h^l g^s(x), h^l g^{s+1}(x), ..., h^l g^{n-1}(x), h^{l+1}(x), h^{l+1} g(x), ..., h^{l+1} g^{s-1}(x)).$$

We must show that

$$h^l g^{s+j}(x) = g^j(x), \quad 0 \leq j < n - s,$$

and

$$h^{l+1} g^j(x) = g^{n-s+j}(x), \quad 0 \leq j < s.$$

Note that

$$\begin{aligned} h^l g^{s+j} &= h\, h^{as+aj} \\ &= h^{l+as+aj} \\ &= h^{aln+as+aj} \\ &= h^{aq+aj} \\ &= h^{aj} \\ &= g^j, \end{aligned}$$

where we have used $an \equiv 1 \bmod q$, $ln + s = q$ and $h^q = 1$. Similarly, one shows that

$$\begin{aligned} h^{l+1} g^j &= h^{aln+1+aj} \\ &= h^{aq-as+1+aj} \\ &= h^{-bq+1-as+aj} \\ &= h^{an-as+aj} \\ &= g^{n-s+j}, \text{ for } 0 \leq j < s. \end{aligned}$$

Case II — To complete the proof, we now consider the case where p divides n, but n is not a power of p, i.e., $n = p^s \cdot m$, where p does not divide m. Now, let $Y = X^{p^s}$, so that $X^n = Y^m$. Note that $G = \mathbb{Z}/q$ acts freely on Y via the diagonal action. This leads to an action of $\tilde{G}_m = \mathbb{Z}/mq\mathbb{Z}$ on X^n. As noted earlier, this is the restriction of the action of $\tilde{G}_n = \mathbb{Z}/nq$ on X^n.

Hence, it suffices to show that the group $\tilde{G}_m$ does not act freely on X^n. However, since m is not a multiple of p, $\tilde{G}_m$-action admits a fixed point on $Y^m = X^n$, as proved earlier.

This completes the proof of the theorem.

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