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HANKEL-TOEPLITZ TYPE OPERATORS BETWEEN WEIGHTED BERGMAN SPACES ON THE UNIT DISK*

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In this paper we define some kinds of Hankel-Toeplitz type operators between the weighted Bergman spaces on the unit disk, and develop their boundedness, compactness and Schattenvon Neumann properties.

Key Words : Hankel Type Operator; Toeplitz Type Operator; Besov Space; Schattenvon Neumann Class

1. INTRODUCTION

Let $\mathbb{D}$ be the unit disk of comples plane $\mathbb{C}$ and, for $-1 < \alpha < \infty$, let

$$d\mu_\alpha(z) = \frac{\alpha+1}{\pi}(1-|z|^2)^\alpha \, dm(z),$$

where *dm (z)* is the Lebesgue measure on $\mathbb{D}$. Thus $L^2(\mathbb{D}, d\mu_\alpha)$ denotes the Hilbert space of all measurable functions f on $\mathbb{D}$ for which the norm

$$\|f\|_{L^2(\mathbb{D}, d\mu_\alpha)} = \left\{ \int_{\mathbb{D}} |f(z)|^2 \, d\mu_\alpha(z) \right\}^{\frac{1}{2}}$$

is finite. The weighted Bergman space $A^{\alpha,2}(\mathbb{D})$ is the subspace of all analytic functions in $L^2(\mathbb{D}, d\mu_\alpha)$, and $\overline{A}^{\alpha,2}(\mathbb{D}) = \left\{ f \in L^2(\mathbb{D}, d\mu_\alpha) : \overline{f} \in A^{\alpha,2}(\mathbb{D}) \right\}$ is the conjugate Bergman space. let P_α and $\overline{P}_\alpha$ denote the orthogonal projection operators from $L^2(\mathbb{D}, d\mu_\alpha)$ onto $A^{\alpha,2}(\mathbb{D})$ and

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$\overline{A}^{\alpha,2}(\mathbb{D})$ respectively, and let $b(z)=\sum_{n=0}^{\infty}\hat{b}(n)z^n$ be an analytic function on $\mathbb{D}$. The Toeplitz operators, small and big Hankel operators with symbol b are defined by $T_b = P_\alpha M_{\overline{b}} P_\alpha$, $h_b = \overline{P}_\alpha M_{\overline{b}} P_\alpha$ and $H_b=(I-P_\alpha)M_{\overline{b}}P_\alpha$, respectively. The characterization of the symbols b such that the small and big Hankel operators belong to S_p on the Bergman spaces $A^{\alpha,2}(\mathbb{D})$ was first investigated by Peller[10] and Axler[2]. The general case was considered by Arazy *et al.*[1]. Then many interesting some authors extended the properties of these operators to the unit ball in $\mathbb{C}^n$ (see [3, 7, 14, 15]) and other domains (see [6-9, 12]). We first recall some basic concepts. For $0<p\le\infty, H_1$ and H_2 are two Hilbert spaces, an operator T from H_1 to H_2 belongs to the Schattern-von Neumann class S_p if the sequence of singular numbers $\{S_n(T)\}_{n=0}^{\infty}$ belongs to l^p. S_∞ denotes the class of all bounded operators. Let $0<p\le\infty, -\infty<s<+\infty, m$ is a nonnegative integer with $m>s$. The analytic Besov space B_p^s is defined by

$$B_p^s=\left\{f:(1-|z|^2)^{m-s}D^m f(z)\in L^p((1-|z|^2)^{-1}dm(z))\right\},$$

where $D=\dfrac{d}{dz}$ is the usual differential operator. Let b_∞^s denote the closure of the polynomials in B_∞^s. Then

$$b_\infty^s=\left\{f:(1-|z|^2)^{m-s}|D^m f(z)|\to 0 \text{ as } |z|\to 1\right\}.$$

For the further information of S_p and B_p^s we refer to Peller[10], Janson[5], Zhu[16] and the references therein. In[4, 5, 7, 11, 15], the authors investigated the properties for Hankel and Toeplitz type operators between Bergman spaces with different weights. More precisely, let $\alpha,\beta,\gamma\in(-1,\infty)$ denote three parameters, we define $h_b^{(\alpha,\beta,\gamma)}=\overline{P}_\alpha M_{\overline{b}}P_\beta$, $T_b^{(\alpha,\beta,\gamma)}=P_\alpha M_{\overline{b}}P_\beta$ and $H_b^{(\alpha,\beta,\gamma)}=(I-P_\alpha)M_{\overline{b}}P_\beta$ respectively as the operators from $A^{\beta,2}(\mathbb{D})$ to $L^2(\mathbb{D},d\mu_\gamma)$. Janson[5] considered the case $\alpha=\gamma$ for the operators $h_b^{(\alpha,\beta,\alpha)}$ and $H_b^{(\alpha,\beta,\alpha)}$. The first author of this paper in[4] studied the properties of Schatten-von Neumann for $h_b^{(\alpha,\beta,\gamma)}$ and $T_b^{(\alpha,\beta,\gamma)}$. The purpose of present paper is to discuss the phenomenon of the operators defined in[11] with different weights.

2. Main Theorem and its Proof

Let

$$d^2_{m,\alpha} = (\alpha+1)\int_0^1 x^m(1-x)^\alpha\,dx = \frac{\Gamma(m+1)\,\Gamma(\alpha+1)}{\Gamma(m+\alpha+2)} \asymp (m+1)^{-(\alpha+1)}$$

Then we have

$$A^{\alpha,2}(\mathbb{D}) = span\left\{ e^{\alpha}_m(z) = \frac{z^m}{d_{m,\alpha}} : m \ge 0 \right\}.$$

As in Peng[11] we let $f^{s,\beta}_n = r^{n^s} e^{in\theta}/d_{n^s,\beta}$, and $B^{\beta,2}(\mathbb{D}) = span\left\{ f^{s,\beta}_n : n \ge 0 \right\}$. P^s_β is used to denote the corresponding projection operator. The operator $T^{(s,\alpha,\beta,\gamma)}_b$ with analytic symbol b is defined by

$$T^{(s,\alpha,\beta,\gamma)}_b f = P_\alpha M_{\bar{b}} P^s_\beta f, \text{ for } f \in B^{\beta,2}(\mathbb{D}).$$

If $s = 1$, $\alpha=\beta=\gamma$, $T^{(1,\alpha,\beta,\gamma)}_b = T_b$ is the usual Toeplitz operator on the Bergman space $A^{\alpha,2}(\mathbb{D})$. If $s \ne 1$, $T^{(s,\alpha,\beta,\gamma)}_b$ is called a Hankel-like operator (see[11]), it can also be called a Toeplitz-like operator. Here we call them Hankel-Toeplitz type operators. It is easy to calculate that

$$\left\langle T^{(s,\alpha,\beta,\gamma)}_b (f^{s,\beta}_m), e^{\alpha}_m \right\rangle_{L^2(\mathbb{D}\,d\mu_\gamma)} = \bar{\hat{b}}(n-m)\,\frac{d^2_{(n^s+n)/2,\gamma}}{d_{n^s\beta}\,d_{m,\alpha}}.$$

A direct estimate for the matrix coefficients gives that

$$\frac{d^2_{(n^s+n)/2,\gamma}}{d_{n^s,\beta}\,d_{m,\alpha}} = \frac{(m+1)^{(\alpha+1)/2}\,(n^s+1)^{(\beta+1)/2}}{\left(\frac{1}{2}(n^s+n)+1\right)^{(\gamma+1)}}$$

$$\asymp \begin{cases} (n+1)^{-[(\gamma+1)-(\beta+1)/2]\,s}\,(m+1)^{(\alpha+1)/2}, & \text{for } s \ge 1, \\ (n+1)^{-[(\gamma+1)-s(\beta+1)/2]}\,(m+1)^{(\alpha+1)/2}, & \text{for } 0 < s < 1, \\ (n+1)^{-(\gamma+1)}\,(m+1)^{(\alpha+1)/2}, & \text{for } -\infty < s \le 0 \end{cases}$$

By the result of Example 1 in[11] we can obtain the following theorem.

Theorem 1 — *Let* $\alpha, \beta, \gamma > -1$, *and* $0 < p \le \infty$.

(1) Suppose that $s \ge 1$

(i) If $1/p < s\,[(2\gamma-\beta+1)/2]-(\alpha+1)/2$, *then* $T^{(s,\alpha,\beta,\gamma)}_b \in S_p$ *if and only if*

$b \in B_p^{[(2\gamma-\beta+1)/2]+s(\alpha+1)/2+1/p}$

(ii) If $0<p<\infty$, *and* $1/p \geq s\,[(2\gamma-\beta+1)/2]-(\alpha+1)/2$, *then* $T_b^{(s,\alpha,\beta,\gamma)} \in S_p$ *only if* $b \equiv 0$.

(iii) $T_b^{(s,\alpha,\beta,\gamma)}$ *is compact if, and only if,* $b \in b_\infty^{-s\,[(2\gamma-\beta+1)/2]+(\alpha+1)/2}$

(2) Suppose that $0 < s < 1$.

(i) If $1/p < [(2\gamma-\alpha+1-s(\beta+1)]/2$, *then* $T_b^{(s,\alpha,\beta,\gamma)} \in S_p$ *if and only if* $b \in B_p^{-s\,[(2\gamma-\beta+1)/2]+(\alpha+1)/2+1/p}$

(ii) If $0<p<\infty$, *and* $1<p \geq [(2\gamma-\alpha+1)-s(\beta+1)]/2$, *then* $T_b^{(s,\alpha,\beta,\gamma)} \in S_p$ *only if* $b \equiv 0$.

(iii) $T_b^{(s,\alpha,\beta,\gamma)}$ *is compact if, and only if,* $b \in b_\infty^{-[(2\gamma-\alpha+1)-s(\beta+1)]/2}$

(3) Suppose that $-\infty < s \leq 0$.

(i) If $1/p < (2\gamma-\alpha+1)/2$, *then* $T_b^{(s,\alpha,\beta,\gamma)} \in S_p$ *if and only if* $b \in B_p^{-(2\gamma-\alpha+1)/2+1/p}$

(ii) If $0<p<\infty$ *and* $1/p \geq (2\gamma-\alpha+1)/2$, *then* $T_b^{(s,\alpha,\beta,\gamma)} \in S_p$ *only if* $b \equiv 0$.

(iii) $T_b^{(s,\alpha,\beta,\gamma)}$ *is compact if, and only if,* $b \in b_\infty^{-(2\gamma-\alpha+1)/2}$

When $s=1$, part 1) of this theorem is the result of Theorem 6 in[4]. When $s=1, \gamma=\alpha$, part 1) is also the result of Theorems 3 and 4 in[5].

In order to show the following results, we first establish a useful lemma. Let T denote the unit circle of complex plane $\mathbb{C}$, $H^2(T)$ is the Hardy space. The Toeplitz operator T_b is defined by $T_b f = pM_{\bar{b}}f$, $f \in H^2(T)$, where P is the projection from $L^2(T)$ to $H^2(T)$. Let I^l be the Riesz potential operator of order l $(l \in \mathbb{R})$, which is defined by $\widehat{I^l f}(n) = (n+1)^{-l}\hat{f}(n)$. It is not difficult to compute

$$\left\{\langle I^{-t}\,T_{I^l b}\,I^{-s}(e^{in\theta}), e^{im\theta}\rangle\right\}_{n\geq 0, m\geq 0}$$

$$= \left\{\hat{b}(n-m)(n-m+1)^l(n+1)^s(m+1)^t\right\}_{n\geq 0, m\geq 0}.$$

$I^{-t}T_{I^l b}I^{-s}$ is called the periodic paracummutator. It is natural to consider the following operator

$$\Gamma_b^{st} \sim \left\{\hat{b}(n-m)\,A(n,m)\,(n+1)^s(m+1)^t\right\}_{n\geq 0, m\geq 0},$$

where $A(n,m)$ is called the Fourier kernel of Γ_b^{st}. By[11] for Γ_b^{st} we have

Lemma 1 —

(1) Let $A(n, m) \equiv 1$.

(*i*) If $s \in I\!R, t > \max\left\{\frac{-1}{2}, \frac{-1}{p}\right\}, 0 < p \leq \infty$ and $s + t + \frac{1}{p} < 0$, then $\Gamma_b^{s\,t} \in S_p$ if and only if $b \in B_p^{s+t+\frac{1}{p}}$.

(*ii*) If $0 < p < \infty, s + t + \frac{1}{p} \geq 0$,then $\Gamma_b^{s\,t} \in S_p$ only if $b \equiv 0$.

(*iii*) If $s \in I\!R, t > 0, s + t < 0$, then $\Gamma_b^{s\,t}$ is compact if and only if $b \in b_\infty^{s+t}$.

(*iv*) If $t > 0$, then $\Gamma_b^{-t\,t}$ is bounded if and only if $b \in L^\infty, \Gamma_b^{-t\,t}$ is compact only if $b \equiv 0$.

(2) Let $A(n, m) = \left(\frac{n-m}{n+1}\right)^\nu, \nu > 0, 0 < p \leq \infty$.

(*i*) If $s \in I\!R, t > \max\left\{\frac{-1}{2}, \frac{-1}{p}\right\}$, and $s + t + \frac{1}{p} < \nu$, then $\Gamma_b^{s\,t} \in S_p$ if and only if $b \in B_p^{s+t+\frac{1}{p}}$.

(*ii*) If $s + t + \frac{1}{p} \geq \nu$, then $\Gamma_b^{s\,t} \in S_p$ only if $b \equiv C$.

(*iii*) $\Gamma_b^{s\,t}$ is compact if and only if $b \in b_\infty^{s+t}$.

The proof of this lemma can be found in[4] or in[11].

Let $\mathbb{Z}$ be the set of all integers, and $k = \max\{l \in \mathbb{Z}: l < (\alpha + 1)/2\}$. From Peng[11] we know that

$$L^2(I\!D, d\mu_\alpha) = \bigoplus_{t=0}^{k} A_l^{\alpha, 2}(I\!D) \oplus L^2(\rho(\lambda)\, d\lambda\, d\theta),$$

where $A_l^{\alpha, 2}(I\!D)$ $(l = 0, 1, ..., k)$ are called the discrete parts and $L^2(\rho(\lambda)\, d\lambda\, d\theta)$ is the continuous part. Furthermore, each component $A_l^{\alpha, 2}(I\!D)$ is invariant under the Möbius group $SU(1, 1)$, where

$$SU(1,1) = \left\{\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C},\quad |\alpha|^2 - |\beta|^2 = 1\right\},$$

and $A_0^{\alpha, 2}(I\!D) = A^{\alpha, 2}(I\!D)$ is the weighted Bergman space. Let

$$e_m^{\alpha,1}(z)=\sqrt{\frac{(\alpha-1)(m+\alpha+1)}{(\alpha+1)}}\ \frac{(|z|^2-(m+1)/(m+\alpha+1))\,z^m}{d_{m+1,\alpha-1}(1-|z|^2)}\ (m\geq 0),$$

$$e_{-1}^{\alpha,1}(z)=\sqrt{\frac{\alpha(\alpha-1)}{\alpha+1}}\ \frac{\bar{z}}{1-|z|^2}.$$

Then we know that

$$A_1^{\alpha,2}(\mathbb{D})=span\left\{e_m^{\alpha,1}(z):m\geq -1\right\}.$$

Let P_α^l denote the projection from $L^2(\mathbb{D},d\mu_\alpha)$ onto $A_l^{\alpha,2}(\mathbb{D})$. Define

$$T_b^{(l,l',\alpha,\beta,\gamma)}f=P_\alpha^l M_{\bar{b}} P_\beta^{l'} f \text{ for } f\in A_{l'}^{\beta,2}(\mathbb{D}),$$

where $\alpha,\beta,\gamma>-1$. First we deak with the operators $T_b^{(1,0;\alpha,\beta,\alpha)}$ and $T_b^{(1,1;\alpha,\beta,\beta)}$. For simplicity, we write $T_b^{(1,0;\alpha,\beta)}$ for $T_b^{(1,0;\alpha,\beta,\alpha)}$. Thus we can now prove the following

Theorem 2 — *Let $\alpha>1,\beta>-1$, and $0<p\leq\infty$.*

(i) If $1/p<1+(\alpha-\beta)/2$, then $T_b^{(1,0;\alpha,\beta)}\in S_p$ if and only if $b\in B_p^{(\beta-\alpha)/2+1/p}$

(ii) If $0<p<\infty$ and $1/p\geq 1+(\alpha-\beta)/2$, then $T_b^{(1,0;\alpha,\beta)}\in S_p$ only if $b\equiv C$.

(iii) $T_b^{(1,0;\alpha,\beta)}$ is compact if and only if $b\in b_\infty^{(\beta-\alpha/2}$.

PROOF : We first calculate the matrix coefficients of $T_b^{(1,0;\alpha,\beta)}$ relative to the bases $\{e_n^\beta:n\geq 0\}$ in $A^{\beta,2}(\mathbb{D})$ and $\left\{e_m^{\alpha,1}:m\geq -1\right\}$ in $A_1^{\alpha,2}(\mathbb{D})$. In fact, when $m\geq 0$, we have

$$\langle T_b^{(1,0;\alpha,\beta)}(e_n^\beta),e_m^{\alpha,1}\rangle_{L^2(\mathbb{D},d\mu_\alpha)}$$

$$=\sum_j \frac{\bar{b}(j)}{d_{n,\beta}d_{m+1,\alpha-1}}\sqrt{\frac{(\alpha-1)(m+\alpha+1)}{(\alpha+1)}}$$

$$\int_{\mathbb{D}} z^n\bar{z}^{j+m}\frac{(|z|^2-(m+1)/(m+\alpha+1))}{(1-|z|^2)}\,d\mu_\alpha(z)$$

$$=\bar{b}(n-m)\frac{\sqrt{(\alpha^2-1)(m+\alpha+1)}}{d_{n,\beta}d_{m+1,\alpha-1}}\left(d_{n+1,\alpha-1}^2-(m+1)/(m+\alpha+1)\,d_{n,\alpha-1}^2\right)$$

$$=\bar{b}(n-m)\frac{\sqrt{\alpha^2-1}\,d_{n,\alpha-1}^2(n-m)}{d_{n,\beta}d_{m+1,\alpha-1}(n+\alpha+1)\sqrt{m+\alpha+1}}.$$

Up to the finite rank operators, $T_b^{(1,0;\alpha,\beta)}$ is determined by the following matrix:

$$T_b^{(1,0;\alpha,\beta)} \sim \left\{ \bar{b}(n-m)\frac{n-m}{n+1}(n+1)^{-[\alpha-(\beta+1)/2]}(m+1)^{(\alpha-1/2}\right\}.$$

By Lemma 1 we complete the proof of Theorem 2. □

Let us write $T_b^{(1,1,\alpha,\beta)}$ for $T_b^{(1,1;\alpha,\beta,\beta)}$. We get

***Theorem* 3** — *Let $\alpha>1, \beta>-1$, and $0\le p\le\infty$.*

(i) If $1/p<(\beta-\alpha)/2$, then $T_b^{(1,1;\alpha,\beta)}\in S_p$ if and only if $b\in B_p^{(\alpha-\beta)/2+1/p}$.

(ii) If $0<p<\infty$ and $1/p\ge(\beta-\alpha)/2$, then $T_b^{(1,1;\alpha,\beta)}\in S_p$ only if $b\equiv 0$.

(iii) If $\beta<\alpha$, then $T_b^{(1,1;\alpha,\beta)}$ is compact if and only if $b\in b_\infty^{(\alpha-\beta)/2}$.

(iv) If $\alpha=\beta$, then $T_b^{(1,1;\alpha,\alpha)}\in S_\infty$ if and only if $b\in L^\infty$, $T_b^{(1,1;\alpha,\alpha)}$ is compact only if $b\equiv 0$.

PROOF : For $n, m\ge 0$, we can compute

$$\langle T_b^{(1,1;\alpha,\beta)}(e_n^{\beta,1}, e_m^{\alpha,1}\rangle_{L^2(\mathbb{D},d\mu_\beta)}$$

$$=\sum_j \bar{\hat{b}}(j)\sqrt{\frac{(\alpha-1)(\beta-1)}{(\alpha+1)(\beta+1)}}\;\frac{\sqrt{(n+\beta+1)(m+\alpha+1)}}{d_{n+1,\beta-1}\,d_{m+1,\alpha-1}}\times$$

$$\times\int_{\mathbb{D}}\frac{\left(|z|^2-\dfrac{m+1}{m+\alpha+1}\right)\left(|z|^2-\dfrac{n+1}{n+\beta+1}\right)\bar{z}^{j+m}z^n}{(1-|z|^2)^2}\,d\mu_\beta(z)$$

$$=\bar{\hat{b}}(n-m)\sqrt{\frac{(\alpha-1)(\beta-1)}{(\alpha+1)(\beta-1)}}\;\frac{\alpha(n+1)}{(n+\beta)\sqrt{(n+\beta+1)(m+\alpha+1)}}\;\frac{d_{n,\beta-2}^2}{d_{n+1,\beta-1}\,d_{m+1,\alpha-1}}$$

By the estimates for Γ-functions we find that

$$T_b^{(1,1;\beta,\alpha)}\sim\left\{\hat{b}(n-m)(m+1)^{(\alpha-1)/2}(n+1)^{-(\beta-1)/2}\right\}$$

up to the finite rank operators. By Lemma 1 we can obtain the desired result. □

For $T_b^{(1,0;\alpha,\beta,\gamma)}$, we have the following

***Theorem* 4** — *Let $\alpha>1, \beta,\gamma>-1, \alpha\ne\gamma$, and $0<p\le\infty$.*

(i) If $1/p<(2\gamma-\beta-\alpha)/2$, then $T_b^{(1,0;\alpha,\beta,\gamma)}\in S_p$ if and only if $b\in B_p^{(\alpha+\beta-2\gamma)/2+1/p}$.

(ii) If $0<p<\infty$, *and* $1/p \geq (2\gamma-\beta-\alpha)/2$, *then* $T_b^{(1,0;\alpha,\beta,\gamma)} \in S_p$ *only if* $b \equiv 0$.

(iii) If $\alpha+\beta<2\gamma$ *then* $T_b^{(1,0;\alpha,\beta,\gamma)}$ *is compact if and only if* $b \in b_\infty^{(\alpha+\beta-2\gamma)/2}$

PROOF : By direct straightforward computation one sees easily that

$$\langle T_b^{1,0;\alpha,\beta,\gamma)}(e_n^\beta), e_m^{\alpha,1}\rangle_{L^2(\mathbb{D},d\mu_\gamma)}$$

$$= \overline{\hat{b}}(n-m)\frac{(\gamma+1)}{\gamma}\sqrt{\frac{\alpha-1}{\alpha+1}}\frac{d_{n,\gamma-1}^2}{d_{n,\beta}d_{m+1,\alpha-1}}$$

$$\times\left(\frac{\alpha(n+1)}{(n+\gamma+1)\sqrt{m+\alpha+1}}-\frac{\gamma(m+1)}{(n+\gamma+1)\sqrt{m+\alpha+1}}\right).$$

Write $T_b^{(1,0;\alpha,\beta,\gamma)} = T_1 - T_2$, then

$$T_1 \sim \left\{\overline{\hat{b}}(n-m)\,n+1)^{-(2\gamma-\beta-1)/2}(m+1)^{(\alpha-1)/2}\right\},$$

$$T_2 \sim \left\{\overline{\hat{b}}(n-m)\,n+1)^{-(2\gamma-\beta+1)/2}(m+1)^{(\alpha+1)/2}\right\}.$$

By Lemma 1 we know that if $1/p<(2\gamma-\beta-\alpha)/2$, then $T_i \in S_p$ $(i = 1, 2)$ if and only if $b \in B_p^{(\alpha+\beta-2\gamma)/2+1/p}$ Thus we can see that $T_b^{(1,0;\alpha,\beta,\gamma)} \in S_p$ if and only if $b \in B_p^{(\alpha+\beta-2\gamma)/2+1/p}$ this completes the proof of part (i). Similarly, if $1/p \geq (2\gamma-\beta-\alpha)/2$, then $T_i \in S_p$ $(i = 1, 2)$ only if $b \equiv 0$. Thus $T_b^{(1,0;\alpha,\beta,\gamma)} \in S_p$ only if $b \equiv 0$. Part *(iii)* can be proved in the same way. □

When $\alpha=\gamma$, $T_b^{(1,0;\alpha,\beta,\gamma)}$ is just the operator $T_b^{(1,0;\alpha,\beta)}$ in Theorem 2. Comparing the above result with Theorem 2 we can see the difference of S_p property for the operator $T_b^{(1,0;\alpha,\beta,\gamma)}$ produced by different weights. This is an interesting phenomenon.

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