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GLOBAL ASYMPTOTIC STABILITY OF A NONLINEAR SECOND-ORDER DIFFERENCE EQUATION*

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In this paper, we obtain a generally global attractivity result for a non-linear second-order difference equation of the form

$$x_{n+1} = f(x_n, x_{n-1}) \text{ for } n = 0, 1, \ldots,$$

where $f(x, y)$ is decreasing in both arguments. The results are applied to the difference equation :

$$x_{n+1} = \frac{A}{x_n^p} + \frac{1}{x_{n-1}^q} \text{ for } n = 0, 1, \ldots,$$

where $A, p, q \in (0, \infty)$.

Key Words: Difference Equations; Global Asymptotic Stability

1. INTRODUCTION AND PRELIMINARIES

Our aim in this paper is to investigate the global asymptotic stability of the positive equilibrium of a difference equation of the form

$$x_{n+1} = f(x_n, x_{n-1}) \text{ for } n = 0, 1, \ldots, \quad \ldots (1)$$

where $f(z, y)$ is decreasing in both arguments. Then we use our result to obtain a sufficient condition for the global asymptotic stability of the positive equilibrium of the difference equation

$$x_{n+1} = \frac{A}{x_n^p} + \frac{1}{x_{n-1}^q} \text{ for } n = 0, 1, \ldots, \quad \ldots (2)$$

where

$$x_{-1}, x_0, A, p, q \in (0, \infty). \quad \ldots (3)$$

The case $p = 2$ and $q = \frac{1}{2}$ was considered in Arciero, *et al.*[1] where it was shown that the positive equilibrium is locally asymptotically stable when $0 < A < \frac{15}{4}$. It was also shown that a period

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two solution exists when $A > \frac{15}{4}$.

Boundedness, persistence and global asymptotic stability of solutions of eq. (2) have been studied by De Vault *et al.*[3, 4]. In particular, the case when $p = 2$ and $q = \frac{1}{2}$ has been studied in Arciero *et al.*[1]. Furthermore, the case when $p = q = 1$ and some related equations can be found in Philos *et al.*[6].

De Vault and Galminas[2] considered the special case when $pq = 1$ and obtained the following sufficient condition for the global stability of eq. 2.

$$pq = 1,\ p > 1, \text{ and } A \in \left(0, \frac{2\,(1+\sqrt{4p^2-3})^{2p-1}}{(4p^2-4)^p} \right).$$

Recently, Kocić and Stutson[5] considered the general case where (3) holds and obtained the following sufficient condition for the global asymptotic stability of eq. (2).

$$p > 1,\ pq \le 1 \quad \text{and} \quad A \le \left(\frac{1-q}{p-q} \right) \left(\frac{p-q}{p-1} \right)^{\frac{p+1}{q+1}}$$

or

$$q > 1,\ pq \le 1 \quad \text{and} \quad A \ge \left(\frac{q-1}{q-p} \right) \left(\frac{q-p}{1-p} \right)^{\frac{p+1}{q+1}}$$

In this paper, we obtain sufficient conditions for the global attractivity of solutions of eq. (2) in case when either $p > 1$ or $q < 1$.

2. Some Lemmas

Lemma 2.1 — Let $F \in C[(0, \infty), (0, \infty)]$ be a nonincreasing function, $\bar{x}$ denote the (unique) fixed point of F, and let $\lim_{x \to 0+} \frac{F^2(x)}{x} = \ell > 1$ (or ∞). The the following statements are equivalent:

(*a*) $\bar{x}$ is the only fixed point of F^2 in $(0, \infty)$;

(*b*) $F^2(x) > x$ for $0 < x < \bar{x}$;

(*c*) if λ and μ are both positive numbers such that

$$F(\mu) \le \lambda \le \bar{x} \le \mu \le F(\lambda), \qquad \text{... (4)}$$

Then

$$\lambda = \mu = \bar{x}. \qquad \text{... (5)}$$

PROOF : $(a) \Rightarrow (b)$. Otherwise there exists an $x \in (0, \bar{x})$ such that $F^2(x) \le x$. But by the properties of $\lim_{x \to 0+} \frac{F^2(x)}{x} = \ell > 1$ (or ∞) we obtain that there must exist an $0 < x_0 < x$ such that $F^2(x_0) > x_0$ and so $F^2(x)$ has a fixed point in (x_0, x), which is impossible.

$(b) \Rightarrow (c)$. Let λ and μ be both positive numbers such that (4) holds. We now claim that $\lambda = \bar{x}$. Otherwise $\lambda < \bar{x} \le \mu$ and so $\lambda \ge F(\mu) \ge F^2(\lambda) > \lambda$, which is impossible. Thus, $F(\mu) \le \bar{x} \le \mu \le F(\bar{x}) = \bar{x}$ holds. Then $\mu = \bar{x}$ and (5) holds.

$(c) \Rightarrow (a)$. Otherwise there exists an $x_0 \in (0, \bar{x}) \cup (\bar{x}, \infty)$ such that $F^2(x) = x$. If $x_0 \in (0, \bar{x})$, by taking $\lambda = x_0$ and $\mu = F(x_0)$, we can see that (4) holds but not (5). On the other hand, if $x_0 \in (\bar{x}, \infty)$, by taking $\lambda = F(x_0)$ and $\mu = x_0$, we can see that again (4) holds but not (5). Therefore the proof is complete. □

Lemma 2.2 — Let $F, H \in C[(0, \infty), (0, \infty)]$ be nonincreasing in $(0, \infty)$. Let $\bar{x} \in (0, \infty)$ be such that $F(\bar{x}) = H(\bar{x})$, $\lim_{x \to 0+} \frac{H^2(x)}{x} = \ell > 1$ (or ∞), $\lim_{x \to 0+} \frac{F^2(x)}{x} = \ell_2 > 1$ (or ∞) and

$$[H(x) - F(x)](x - \bar{x}) \le 0 \quad \text{for} \quad x > 0. \qquad \dots (6)$$

Assume that $\bar{x}$ is the only fixed point of H^2 in $(0, \infty)$. Then $\bar{x}$ is also the fixed point of F^2 in $(0, \infty)$.

PROOF : Otherwise by Lemma 2.1, there exist positive numbers λ and μ such that (4) holds but not (5). Then in view of (4) and (6) we obtain

$$F(\mu) \le H(\mu) \le \lambda \le \bar{x} \le \mu \le H(\lambda) \le F(\lambda),$$

which by Lemma 2.1 again implies that $\lambda = \mu = \bar{x}$. This is a contradiction and the proof is complete. □

3. MAIN RESULTS

Consider eq. (1) and initial conditions $x_{-1}, x_0 > 0$ and where the function f satisfies the following hypotheses :

(H_1) $f \in C[(0, \infty)^2, (0, \infty)]$;

(H_2) $f(u, v)$ is decreasing in both u and v;

(H_3) the equation $x = f(x, x)$ has a unique positive equilibrium $x = \bar{x}$;

(H_4) $\lim_{x \to 0+} \frac{F^2(x)}{x} = \ell > 1$ (or ∞), where $F(x) = f(x, x)$;

(H_5) $F^2(x) = x$ has a unique fixed point $x = \bar{x}$ in $(0, \infty)$.

Now we obtain the following theorems.

***Theorem* 3.1** — *Assume that the hypotheses $(H_1$–$H_4)$ are satisfied. Then every positive solution of eq. (1) persists.*

PROOF : Let $\{x_n\}$ be a nontrivial solution of eq. (1). We shall first show that $\inf\{x_n\} > 0$. Without loss of generality, we may assume that $x^* = \min\{x_{k-1}, x_k, x_{k+1}, x_{k+2}\}$ for $k \in \{0, 1, ..., n\}$ such that $F^2(x^*) > x^*$. (Otherwise, $\inf\{x_n\} > 0$). Then

$$\begin{aligned} x_{k+3} &= f(x_{k+2}, x_{k+1}) \\ &= f(f(x_{k+1}, x_k),. f(x_k, x_{k-1})) \\ &> x^*. \end{aligned}$$

Thus, $x^* < \min\{x_{k-1}, x_k, x_{k+1}, x_{k+2}\}$. By induction we know that $x_n > x^*$ for $n > k$. Hence, we obtain $\inf\{x_n\} = \lambda > 0$. Since $x_{n+1} = f(x_n, x_{n-1}) < f(\lambda, \lambda) = \mu$ for $n > k_0$, where k_0 is a large number. It follows that $\sup\{x_n\} \leq \mu$. This completes the proof. □

***Theorem* 3.2** — *Assume that the hypotheses $(H_1$–$H_5)$ are satisfied. Then unique positive equilibrium $\bar{x}$ of eq. (1) is a global attractor of all positive solution of eq. (1).*

PROOF : Let $\{x_n\}$ be a positive solution of eq. (1). By Theorem 3.1, we obtain that $\sup\{x_n\}$ and $\inf\{x_n\}$ must exist. We set $\sup\{x_n\} = \mu$ and $\inf\{x_n\} = \lambda$. Then we have $\lambda \leq \bar{x} \leq \mu$ and $\lambda \leq x_n \leq \mu$ for $n \geq n_0$, where n_0 is a large positive number.

By eq. (1), we also get $\sup\{x_{n+1}\} = \sup\{f(x_n, x_{n-1})\}$. Thus, we have $\mu \leq f(\lambda, \lambda) = F(\lambda)$. Similarly, we have $\lambda \geq f(\mu, \mu) = F(\mu)$. Then we have

$$F(\mu) \leq \lambda \leq \bar{x} \leq \mu \leq F(\lambda).$$

By Lemma 2.1, we have that $\lambda = \mu = \bar{x}$. Therefore, $\bar{x}$ is a global attractor of all positive solution of eq. (1). This completes the proof. □

4. APPLICATIONS

Now, we shall apply our results to eq. (2).

***Theorem* 4.1** — *Assume that $p, q \in (0, 1)$. Then the unique positive equilibrium $\bar{x}$ of eq. (2) is globally asymptotically stable.*

PROOF : We shall apply Theorem 3.2. Obviously, $(H_1–H_4)$ are satisfied for

$$0 < p < 1 \quad \text{and} \quad p > q \qquad \text{... (7)}$$

or

$$0 < q < 1 \quad \text{and} \quad p < q. \qquad \text{... (8)}$$

So, it remains to show that (H_5) holds for (7) and (8). We shall prove that (H_5) holds for (7). The proof in case when (8) holds is similar and will be omitted.

Let $F(x) = \dfrac{A}{x^p} + \dfrac{1}{x^q} = \dfrac{A + x^{p-q}}{x^p}$ $\bar{x}$ is the unique equilibrium of eq. (2), $H(x) = \dfrac{A + \bar{x}^{p-q}}{x^p}$. Then $F(x) < H(x)$ for $0 < x < \bar{x}$; $F(x) > H(x)$ for $x > \bar{x}$. It is easy to see that $H^2(x) = x$ has the unique positive root $\bar{x}$ and $F(x)$ and $H(x)$ satisfy the conditions of Lemma 2.2. Hence by Lemma 2.2, we obtain $F^2(x) = x$ has the unique positive root $\bar{x}$. Therefore, we know that $\bar{x}$ is the global attractor of eq. (2).

One can easily prove that the positive equilibrium $\bar{x}$ of eq. (1) is locally asymptotically stable for $p, q \in (0, 1)$. Hence the unique positive equilibrium $\bar{x}$ of eq. (2) is globally asymptotically stable. This completes the proof. □

***Theorem* 4.2** — *Assume that (3) holds and $pq < 1$.* Let $F(x) = \dfrac{A}{x^p} + \dfrac{1}{x^q}$. Then the unique equilibrium $\bar{x}$ of eq. (2) is a global attractor if $F^2(x) = x$ has the unique positive root $x = \bar{x}$.

By Theorem 3.2 and Lemma 4.2, we can easily show the proof and omit it.

By Theorem 4.2 and Kocić and Stutson[5], Theorem 6, we can obtain the following corollary.

Corollary 4.1 — Assume that $A > 0$ and either $p > 1$, $pq < 1$ or $q > 1$, $pq < 1$. If one of the following two conditions is satisfied

$$p > 1,\ pq \le 1 \quad \text{and} \quad A \le \left(\frac{1-q}{p-q}\right)\left(\frac{p-q}{p-1}\right)^{\frac{p+1}{q+1}} \qquad \text{... (9)}$$

or

$$q > 1,\ pq \le 1 \quad \text{and} \quad A \ge \left(\frac{q-1}{q-p}\right)\left(\frac{q-p}{1-p}\right)^{\frac{p+1}{q+1}} \qquad \ldots (10)$$

Then the unique equilibrium $\bar{x}$ of eq. (2) is globally asymptotically stable.

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