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ON FACTORIZATION OF SOLUTIONS TO SECOND ORDER HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

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If a meromorphic solution of second order homogeneous linear differential equation is factorizable, then the right factor of the factorization of the solution has order not more than the coefficient's. Some asymptotic propersities of solutions are studied.

Key Words: Factorization; Meromorphic Solution; Linear Differential Equation; Right Factor

1. INTRODUCTION

A transcendental meromorphic function $f(z)$, in the complex plane C, is factorizable. If $f(z)$ is of the form

$$f(z) = g(h(z)),$$

where $g(z)$ is transcendental, meromorphic and $h(z)$ is transcendental, entire. $g(z)$ is called the left factor of the factorization of $f(z)$, $h(z)$ the right factor of the factorization of $f(z)$. In this paper, we always study the form of $f(z) = g(h(z))$. $f(z)$is prime, if for every factorization of $f(z)$, either $g(z)$ is bilinear or $h(z)$ is linear. If $h(z)$ is always a rational function, or $g(z)$ is always polynomial, $f(z)$ is said to be pseudo prime. Steimetz[6] pointed out that any nontrivial solution of linear differential equations with rational coefficients is pseudo prime. Zheng and He[10] studied some second order homogeneous linear differential equations and prove some solutions are prime and some are factorizable. Below, we continue to research the factorization of solutions of second order equations with meromorphic coefficients. Also some asymptotic propersities of solutions are studied under the consideration of the coefficients with order less than $\frac{1}{2}$.

Consider the following equation

$$f'' + A(z)f = 0 \qquad \text{... (1.1)}$$

where $A(z)$ is meromorphic, transcendental of finite order. It is known that every nontrivial solution $f(z)$ of (1.1) is transcendental. If $A(z)$ is entire, any nontrivial solution of eq. (1.1) is of infinite order and if $f(z) \neq 0$, then $f(z)$ is factorizable, its right factor of the factorization is of the same order as $A(z)'s$; if $A(z)$ is meromorphic, the eq. (1.1) may have a nontrivial meromorphic solution of finite order (see Remarks in §4). It is easy to prove that for any nontrivial meromorphic solution

$f(z)$, every pole of $f(z)$ must be the pole of $A(z)$.

Let $f(z)$ be a meromorphic function and $\sigma(f)$ denote the order of $f(z)$, i.e.

$$\sigma(f) = \limsup_{r \to \infty} \frac{\log^+ T(r,f)}{\log r},$$

$\mu(f)$ denotes the lower order of $f(z)$, $\lambda(f)$ the exponent of convergence of zero-sequence of $f(z)$, respectively, i.e.

$$\mu(f) = \liminf_{r \to \infty} \frac{\log^+ T(r,f)}{\log r},$$

$$\lambda(f) = \limsup_{r \to \infty} \frac{\log^+ N\left(r, \frac{1}{f}\right)}{\log r}.$$

Nevanlinna theory and some standard notations come from[4].

Our main results may be stated as follows:

Theorem 1.1 — *Let $f(z)$ be a nontrivial meromorphic solution of eq. (1.1). If $f(z) = g(h(z))$ is factorizable, then $\sigma(h) \le \sigma(A)$.*

From Theorem 1.1, the corollary below follows:

Coroallary — Let $f(z)$ be a nontrivial meromorphic solution of the eq. (1.1). If $f(z) = g(h(z))$ and $\sigma(h) > \sigma(A)$, then $g(z)$ is rational.

Theorem 1.2 — *Suppose $A(z)$ is a transcendental meromorphic function of order less than $\frac{1}{2}$, $f(z)$ is a nontrivial meromorphic solution of (1.1) and $f(z) = g(h(z))$ is factorizable. If $g(z)$ has a finite deficient value a*, then there exists a line L approaching to ∞ such that

$$\liminf_{|z| \to \infty} \frac{\log \log |f(z)|}{\log |z|} \ge \frac{1}{2}, \quad z \in L.$$

Let $f(z)$ be meromorphic in **C**. If $n(r,f) = O(r^k)$, $k < \frac{1}{2} < \mu(f)$, Hayman asks whether the corresponding Boas' theorem holds or not. (see Hayman: Research problems in function theory, Athlone Press, 1967, or[8]). Our Theorem 1.2 describes this property.

2. Lemmas

The following lemma is from Theorem 4.4.5 in[9], or see[3,6].

Lemma 2.1 — Let $F_j(z)$ and $h_j(z)$ ($j = 0, 1, 2, \ldots, m$) be not identically vanishing meromorphic functions and $g(z)$ be a nonconstant entire function. There exists an unbounded positive sequence $\{r_n\}_{n=1}^{\infty}$ satisfying.

$$\sum_{j=0}^{m} T(r_n, h_j) \leq KT(r_n, g),$$

where K is a positive constant. If $F_j(z)$ and $h_j(z)$ ($j = 0, 1, 2, ..., m$) satisfy

$$F_0(g)\, h_0 + \ldots + F_m(g)\, h_m \equiv 0,$$

then there exist polynomials $P_0, P_1, ..., P_m$ not all identically zero such that

$$P_0(g)\, h_0 + P_1(g)\, h_1 + \ldots + P_m(g)\, h_m \equiv 0.$$

Furthermore, if $h_j's$ are not all identically zero, then there exist not all identically zero polynomials $Q_0, Q_1, ..., Q_m$ such that

$$F_0(z)\, Q_0(z) + F_1(z)\, Q_1(z) + \ldots + F_m(z)\, Q_m(z) \equiv 0.$$

From a result of Valiron[7], we easily prove the following lemma.

Lemma 2.2 — Let $f(z)$ be a transcendental entire solution of (2.1)

$$a_2(z) f'' + a_1(z) f' + a_0(z) f = 0, \qquad \ldots (2.1)$$

where $a_2(z) \neq 0, a_1(z), a_0(z)$ are polynomials. Then $\mu(f) > 0$.

Lemma 2.3 — *Let $H(z)$ and $h(z)$ be transcendental entire functions of finite order. If $\lambda(H) > 0$, then $\lambda(H(h)) = \infty$.*

PROOF : Let $\{r_n\}_{n=1}^{\infty}$ be a sequence zeros of $H(z)$. By Hadamard-Horel's theorem, $H(z)$ can be expressed as

$$H(z) = z^m e^{P(z)} \prod (z),$$

where $P(z)$ is a polynomial with degree $\deg P(z) \leq \sigma(H)$, m is a non-negative integer, $\prod(z)$ is of the form as follows

$$\prod(z) = \sum_{k=1}^{\infty} \left(1 - \frac{z}{s_k}\right) e^{P_k\left(\frac{z}{s_k}\right)},$$

and

$P_k\left(\frac{z}{z_k}\right) = \sum_{t=1}^{q} \frac{1}{t}\left(\frac{z}{z_k}\right)^t$, q is the minimum non-negative integer satisfying

$$\sum_{k=1}^{\infty} \frac{1}{z_k^{q+1}} < \infty.$$

If $q = 0$, then $\sigma(\prod) = 0$, this is a contradiction to $\lambda(\prod) > 0$. So $q \geq 1$ and then $\lambda(H(h)) = \infty$. The proof is complete.

From Theorem 2.5 in[8], or HYAK CCCP, 159 (1964), 968-970, we state the following.

Lemma 2.4 — Let $f(z)$ be an entire function of a finite deficient value. Then $\nu(f) > \frac{1}{2}$.

The following lemma is from Theorem 4.16[8].

Lemma 2.5 — Let $f(z)$ be meromorphic in the plane and satisfy

$$\limsup_{r \to \infty} \frac{\log^+ n(r,f)}{\log r} = k < \rho = \min\left\{ \mu(f), \frac{1}{2} \right\}.$$

Then there exists a line L approaching to infinity such that

$$\liminf_{|x| \to \infty} \frac{\log \log |f(z)|}{\log |z|} \geq \rho$$

on L.

3. The Proof of Theorem 1.1

By Theorem 5.1 in[2], we have

$$\limsup_{r \to \infty} \frac{\log^+ \log^+ T(r,f)}{\log r} \leq \sigma(A). \qquad \text{... (3.1)}$$

Assume that $\sigma(h) > \sigma(A)$. Then there is a small number $\varepsilon > 0$, such that

$$\sigma(h) > \sigma(A) + \varepsilon.$$

There exists an unbounded sequence $\{r_n\}$, $r_n \to \infty$ as $n \to \infty$, such that

$$T(r_n, h) > r_n^{\sigma(A)+\varepsilon}$$

Take $r \in [r_n, 2r_n]$, and then

$$\begin{aligned} T(r,h) &\geq T(r_n, h) \\ &> r_n^{\sigma(A)+\varepsilon} \\ &= 2^{-\sigma(A)-\varepsilon}(2r_n)^{\sigma(A)+\varepsilon} \\ &\geq 2^{-\sigma(A)-\varepsilon} r^{\sigma(A)+\varepsilon} \end{aligned}$$

When r is sufficiently large, we have

$$T(r, A) < r^{\sigma(A)+\frac{\varepsilon}{2}}$$

And then

$$T(r, h) > T(r, A), \ r \in \bigcup_{n=N}^{\infty} [r_n, 2r_n]$$

where N is some positive integer. Since h is entire, by Nevanlinna theory, we obtain

$$T(r, h'') < 2T(r, h), \ r \notin E$$

and

$$T(r, h'^2) < 3T(r, h), \ r \notin E$$

where E is a possible set of values r of finite linear measure. Because $\bigcup_{n=N}^{\infty} [r_n, 2r_n]$ has infinite linear measure, we deduce that there is a sequence

$$\{\bar{r}_k\} \subset r \in \bigcup_{n=N}^{\infty} [r_n, 2r_n] \setminus E,$$

$\bar{r}_k \to \infty$ as $k \to \infty$, such that

$$T(\bar{r}_k, h'^2) + T(\bar{r}_k, h'') + T(\bar{r}_k, A) < 6T(\bar{r}_k, h).$$

Substituting $f(z) = g(h(z))$ into (1.1), we have

$$g''(h)\, h'^2 + g'(h)\, h'' + Ag(h) \equiv 0.$$

By Lemma 2.1, we obtain that there exist not all identically zero polynomials Q_0, Q_1, Q_2 such that

$$g(z)\, Q_0(z) + g'(z)\, Q_1(z) + g''(z)\, Q_2(z) \equiv 0.$$

By Lemma 2.2, we get $\mu(g) > 0$. $g(z)$ has at most finitely many poles. Let

$$g(z) = \frac{H(z)}{Q(z)},$$

where $H(z)$ is entire, and $Q(z)$ is a non-zero polynomial. Obviously $\sigma(H) = \sigma(g)$, $\mu(H) = \mu(g)$ and $T(r, Q(h)) = o\{T(r, H(h))\}$. Since $H(z)$ has positive lower growth order, there is a positive number $\zeta > 0$, such that for all $r > 0$,

$$\log M(r, H) > r^{\zeta}.$$

By a result of Pòlya[5], there is a number c $(0 < c < 1)$, such that for all $r > 0$,

$$\log\log\log M(r, H(h)) \geq \log\log\log M\left(cM\left(\frac{r}{2}, h \right), H \right)$$
$$> \log\log M\left(\frac{r}{2}, h \right) - O(1).$$

By (3.1), we have

$$\sigma(h) = \limsup_{r\to\infty} \frac{\log\log M\left(\frac{r}{2}, h\right)}{\log r} \leq \limsup_{r\to\infty} \frac{\log\log\log M(r, g(h))}{\log r} \leq \sigma(A).$$

This is a contradiction to our assumption. So $\sigma(h) \leq \sigma(A)$. The proof is complete.

PROOF OF THEOREM 1.2

PROOF : If $g(z)$ has at least one pole, then set

$$g(z) = \frac{G(z)}{H(z)},$$

where $g(z), H(z)$ are non-constant entire and $H(z) \not\equiv 0$ is the canonical product of poles of $g(z)$ if $g(z)$ has infinitely many poles, if $g(z)$ has at most finitely many poles, $H(z)$ is a non-zero polynomial. Since every pole of $f(z)$ is a pole of $A(z)$, noting that $\sigma(A) < \infty$, by Lemma 2.3, $\lambda(H) = \sigma(H) = 0$. As a is a deficient value of $g(z)$, 0 is the deficient value of $G(z) - aH(z)$, by Lemma 2.4, $\mu(G) = \mu(G - aH) > \frac{1}{2}$, so does $\mu(g)$.

If $g(z)$ is entire, by Lemma 2.4, $\mu(g) > \frac{1}{2}$.

From $\mu(g) > \frac{1}{2}$, we have $\mu(g(h)) > \frac{1}{2}$. Note that

$$\limsup_{r\to\infty} \frac{\log^+ n(r, f)}{\log r} \leq \sigma(A) < \frac{1}{2}.$$

By Lemma 2.5, there exists a path L tending to ∞, such that

$$\limsup_{|z|\to\infty} \frac{\log\log |f(z)|}{\log |z|} \geq \frac{1}{2}, \quad z \in L.$$

Theorem 1.2 follows.

5. REMARKS

The factorization $f(z) = g(h(z))$, which satisfies eq. (1.1), may have kinds of forms of the factorization, in the sense of the growth order of the left or right factor of the factorization.

(1) $\sigma(f) < \infty$ may occur. For example, $f(z) = e^z + 1$ solves

$$f'' - \frac{e^z}{e^z + 1} f = 0,$$

and $f(z) = (z+1) \circ e^z$ is pseudo prime.

(2) The left (right) factor, of zero order, may occur. Take

$$g(z) = h(z) = \sum_{n=1}^{\infty} e^{-n (\log n)^2} z^n,$$

and

$$A(z) = -\left(\frac{d^2 g(h)}{dh^2} \left(\frac{dh}{dz} \right)^2 + \frac{dg(h)}{dh} \frac{d^2 h}{dz^2} \right) (g(h(z)))^{-1}.$$

then $f(z) = g(h(z))$ solves

$$f'' + A(z) f =$$

But Baker[1] showed that $\sigma(f)$ is finite, non-zero. So is $\sigma(A)$.

(3) If $\sigma(A) = 0$, we can find examples to show that $f(z)$, of zero roder, is factorizable. For example, take

$$g(z) = h(z) = \sum_{n=0}^{\infty} e^{-n^2} z^n,$$

then Baker[1] showed that $\sigma(f) = \sigma(g(h)) = 0$. $f(z)$ is a solution of

$$f'' - \left(\frac{d^2 g(h)}{dh^2} \left(\frac{dh}{dz} \right)^2 + \frac{dg(h)}{dh} \frac{d^2 h}{dz^2} \right) (g(h(z)))^{-1} f = 0.$$

(4) For the case $\lambda(f) < \infty$, Zeng and He[10] studied that $f(z) = \psi(e^z) e^{\Phi(z) + dz}$ solves

$$w'' + A(e^z) w = 0,$$

where $A(x) = \sum_{j=0}^{p} b_j x^j, p_p \neq 0, p \leq 2$, d is not a rational number, $\psi(\xi)$ is a polynomial having at least a non-zero simple zero and $\sigma(\Phi(z)) < \infty$ and obtained that $f(z)$ is prime.

If d is a rational number or $\psi(\xi)$ is a rational function, we may have a factorizable solutions $f(z)$, with $0 < \lambda(f) < \infty$. For example,

$$f(z) = (e^{-z} + 1) e^{\frac{1}{2} e^z + z} = (z+1) e^{\frac{1}{2} z} \circ e^z$$

solves

$$f'' - \frac{1}{4} e^z (e^z + 6) f = 0.$$

Obviously, $\lambda(f) = 1$.

(5) $\sigma(g) = \infty$ may occur. For example, let

$$h(z) = \sum_{n=1}^{\infty} e^{-n(\log n)^2} z^n,$$

and

$$g(z) = e^{h(z)} . f(z) = g(h(z)) \text{ solves}$$

$$f'' - (h''(z) \cdot h'(h(z)) + h'^2(z) \cdot h''(h(z)) + (h'(z) \cdot h''(h(z)))^2) f = 0.$$

By Remark 2, $h''(z) \cdot h'(h(z)) + h'^2(z) \cdot h''(h(z) \cdot h''(h(z)))^2$ has finite order.

But $\sigma(g) = \sigma(e^h) = \infty$.

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