Indian J. pure appl. Math., **35**(7) : 881-888, July 2004

TIMELIKE AND NULL NORMAL CURVES IN MINKOWSKI SPACE $\mathbb{E}_1^3$

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(*Received* 17 *July* 2003; *accepted* 19 *April* 2004)

In this paper, we characterize timelike and null (lightlike) curves for which the position vector always lies in their normal plane in the Minkowski 3-space.

Key Words: Normal Plane; Frenet Equations; Minkowski 3-Space

1. INTRODUCTION

It is well known that to each unit speed curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ in the Euclidean space $\mathbb{R}^3$ whose successive derivatives $\alpha'(s)$, $\alpha''(s)$ and $\alpha'''(s)$ are linearly independent vectors, one can associate three mutually ortogonal unit vector fields T, N and B called respectively the tangent, the principal normal and the binormal vector fields. Moreover, the planes spanned by vectors $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ are respectively known as the osculating plane, the rectifying plane and the normal plane of the curve α. In relation to that, the curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ for which the position vector α always lies in its rectifying plane, is in[2] for simplicity called rectifying curve. Also, in[2] the equations of rectifying curves are obtained explicitly and it is proved that the ratio of torsion and curvature of such curves is a non-constant linear fucntion in arclength function s. Characterizations of spacelike curves with spacelike, timelike and null principal normal for which the position vector always lies in their normal plane, are given in[4]. In this paper, we characterize timelike and null curves for which the position vector always lie in their normal plane. For simplicity, we called such curves normal curves. By definition, for a normal curve, the position vector α satisfies

$\alpha(s) = \lambda(s) N(s) + \mu(s) B(s)$, where s is the arclength function and $\lambda(s)$ and $\mu(s)$ are differentiable functions.

2. PRELIMINARIES

The Minkowski 3-space $\mathbb{E}_1^3$ is the Euclidean 3-space $\mathbb{R}^3$ provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}_1^3$.

Since g is an indefinite metric, recall that a vector $a \in \mathbb{E}_1^3$ can have one of three causal characters: it can be spacelike if $g(a, a) > 0$ or $a = 0$, timelike if $g(a, a) < 0$ and null (lightlike) if $g(a, a) = 0$ and $a \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{E}_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null (lightlike). Recall that the norm of a vector a is given by $\| a \| = \sqrt{| g(a, a) |}$ and that the spacelike (or timelike) curve $\alpha(s)$ is said to be of unit speed if $g(\alpha'(s), \alpha'(s)) = \pm 1$. Moreover, the velocity of curve $\alpha(s)$ is the function $\upsilon(s) = \| \alpha'(s) \|$. We also recall that the pseudosphere S_1^2 (r) or radius r and with center at the origin, is hyperquadric in the space $\mathbb{E}_1^3$ defined by S_1^2 $(r) = \left\{ x \in \mathbb{E}_1^3 \, : \, g(x, x) = r^2 \right\}$.

Denote by $\{T, N, B\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space $\mathbb{E}_1^3$. For an arbitrary curve $\alpha(s)$ in the space $\mathbb{E}_1^3$, the following Frenet equations are given in[3,7].

If α is a timelike curve, then the Frenet equations read

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \qquad \dots (1)$$

where $g(T, T) = -1, g(N, N) = 1, g(B, B) = 1, g(T, N) = 0, g(T, B) = 0, g(N, B) = 0$.

On the other hand, if α is a null curve, the Frenet equations read

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_2 & 0 & -k_1 \\ 0 & -k_2 & \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \qquad \dots (2)$$

where $g(T, T) = 0, g(N, N) = 1, g(B, B) = 0, g(T, N) = 0, g(T, B) = 1, g(N, B) = 0$. In this case, k_1 can take only two values: $k_1 = 0$ when α is a straight null line or $k_1 = 1$ in all other cases. The functions $k_1(s)$ and $k_2(s)$ are called respectively the first curvature (or the curvature) and the second curvature (or the torsion) of α.

3. The Timelike and the Null Normal Curves in $\mathbb{E}_1^3$

In this section we give some characterization theorems for timelike and null normal curves.

Theorem 3.1 — *Let $\alpha(s)$ be a unit speed timelike curve in $\mathbb{E}_1^3$ with curvatures $k_1(s) > 0$, $k_2(s) \neq 0$ for each $s \in I \subset \mathbb{R}$. Then α is a normal curve if and only if the principal normal and binormal components of the position vector α are respectively given by* $g(\alpha, N) = 1/k_1$, $g(\alpha, B) = (1/k_2)(1/k_1)'$.

PROOF : Let us first suppose that $\alpha(s)$ is a normal curve, where s is arclength parameter. Then by definition we have

$$\alpha(s) = \lambda(s) N(s) + \mu(s) B(s).$$

Differentiating this with respect to s and by using the corresponding Frenet equations (1), we find

$$\lambda k_1 = 1, \quad \lambda' - \mu k_2 = 0, \quad \lambda k_2 + \mu' = 0. \qquad \text{... (3)}$$

From the first and second equations in (3), we get

$$\lambda = 1/k_1, \quad \mu = (1/k_2)(1/k_1)'.$$

Thus

$$\alpha = (1/k_1) N + (1/k_2)(1/k_1)' B. \qquad \text{... (4)}$$

It follows that $g(\alpha, N) = 1/k_1$, $g(\alpha, B) = (1/k_2)(1/k_1)'$.

Conversely, suppose that $g(\alpha, N) = 1/k_1$, $g(\alpha, B) = (1/k_2)(1/k_1)'$. Differentiating the equation $g(\alpha, N) = 1/k_1$ with respect to s, we obtain

$$g(T, N) + g(\alpha, k_1 T + k_2 B) = (1/k_1)'. \qquad \text{... (5)}$$

Therefore, by using $g(\alpha, B) = (1/k_2)(1/k_1)'$, eq. (5) becomes

$$k_1 g(\alpha, T) + k_2 (1/k_2)(1/k_1)' = (1/k_1)'.$$

It follows that $g(\alpha, T) = 0$, which means that α is a normal curve.

Theorem 3.2 — *Let $\alpha(s)$ be a unit speed timelike curve in $\mathbb{E}_1^3$, with curvatures $k_1(s) > 0$. $k_2(s) \neq 0$ for each $s \in I \subset \mathbb{R}$. Then α is congruent to a normal curve if and only if*

$$k_2/k_1 = -((1/k_2)(1/k_1)')'. \qquad \text{... (6)}$$

PROOF : First assume that $\alpha(s)$ is congruent to a normal curve. Then all three equations in (3) imply relation (6).

Conversely, suppose that the relation (6) holds. By applying Frenet eq. (1), we easily obtain

$$\frac{d}{ds}\left[\alpha(s)-\frac{1}{k_1}N(s)-\frac{1}{k_2}\left(\frac{1}{k_1}\right)'B(s)\right]=0.$$

Consequently, α is congruent to a normal curve.

Theorem 3.3 — *Let $\alpha(s)$ be a unit speed timelike curve in $\mathbb{E}_1^3$, with curvatures $k_1(s)>0$. $k_2(s)\neq 0$ for each $s\in I\subset\mathbb{R}$. Then α lies on pseudosphere $S_1^2(r)$* if and only if α is a normal curve.

PROOF : Let us first assume that $\alpha(s)$ lies on pseudosphere $S_1^2(r)$. Then

$$g(\alpha-m,\alpha-m)=r^2,\quad r\in\mathbb{R}^+.$$

Differentiating this equation four times and using Frenet formulae, we get

$$\alpha-m=(1/k_1)N+(1/k_2)(1/k_1)'B.$$

Therefore, up to a translation, α is a normal curve.

Conversely, suppose that α is a normal curve. Then by Theorem 3.2, we have $k_2/k_1 = -((1/k_2)(1/k_1)')'$. Let us consider the vector

$$m=\alpha-(1/k_1)N-(1/k_2)(1/k_1)'B.$$

By taking the derivative of m with respect to s, we get $m'=0$. Thus $m=$ constant. Next, the equation $k_2/k_1 = -((1/k_2)(1/k_1)')'$ is the derivative of the equation $(1/k_1)^2+((1/k_2)(1/k_1)')^2 =$ constant >0. Therefore, we obtain

$$g(\alpha-m,\alpha-m)=(1/k_1)^2+((1/k_2)(1/k_1)')^2=\text{constant}=c>0.$$

We may take $c=r^2$. It follows that α lies on $S_1^2(r)$.

Recall that for the null curve α in the space $\mathbb{E}_1^3$, the first curvature $k_1(s)$ can take only two values: $k_1=0$, if α is straight null line, or $k_1=1$, in all other cases. If $k_1=0$, then α is in direction of its tangent vector T and therefore it can not be a normal curve.

In the next four theorems, we give the characterizations of null normal curves parameterized the pseudo arclength function s and with the first curvature $k_1=1$. Recall that the pseudo arclength function s is defined in[1] by

$$s = \int_0^t g(\alpha''(t), \alpha''(t))^{\frac{1}{4}} dt.$$

***Theorem* 3.4** — *Let $\alpha(s)$ be a null curve in $\mathbb{E}_1^3$ with curvatures $k_1(s) = 1, k_2(s) \neq 0$ for each $s \in I \subset \mathbb{R}$. Then α is a normal curve if and only if the tangent and the principal normal components of the position vector α are respectively given by $g(\alpha, T) = (1/k_2)(1/k_2)'$, $g(\alpha, N) = 1/k_2$.*

PROOF : Let us first suppose that $\alpha(s)$ is a normal curve. Then by definition we have

$$\alpha(s) = \lambda(s) N(s) + \mu(s) B(s). \qquad \text{... (7)}$$

Where λ and μ are arbitrary functions in s. By taking the derivative of (7) with respect to s and by applying Frenet equations, we find

$$\lambda' - \mu k_2 = 0, \ \lambda k_2 = 1, \ \mu' - \lambda = 0. \qquad \text{... (8)}$$

It follows that

$$\lambda = 1/k_2, \ \mu = (1/k_2)(1/k_2)'.$$

Thus (7) implies $g(\alpha, T) = \mu = (1/k_2)(1/k_2)'$ and $g(\alpha, N) = \lambda = 1/k_2$.

Conversely, let us assume that $g(\alpha, T) = (1/k_2)(1/k_2)', g(\alpha, N) = 1/k_2$. Differentiating $g(\alpha, N) = 1/k_2$ with respect to s, we obtain

$$g(T, N) + g(\alpha, k_2 T - B) = (1/k_2)',$$

and consequently

$$g(\alpha, B) = 0.$$

Decompose the vector $\alpha(s)$ with respect to $\{T, N, B\}$ by $\alpha(s) = a(s) T(s) + b(s) N(s) + c(s) B(s)$, where $a(s), b(s), c(s)$ are arbitrary functions. Then we easily find

$$g(\alpha, T) = c = (1/k_2)(1/k_2)', \ \ g(\alpha, N) = b = 1/k_2, g(\alpha, B) = a = 0.$$

Therefore, $\alpha = (1/k_2) N + (1/k_2)(1/k_2)' B$, which means that α is a normal curve.

, $\alpha = (1/k_2)\,N + (1/k_2)\,(1/k_2)'\,B$, which means that α is a normal curve.

***Theorem* 3.5** — *Let $\alpha(s)$ be a null curve in the space $\mathbb{E}^3_1$, with curvatures $k_1(s) = 1, k_2(s) \neq 0$ for each $s \in I \subset \mathbb{R}$. Then α is congruent to a normal curve if and only if*

$$((1/k_2)\,(1/k_2)')' = 1/k_2. \qquad \text{... (10)}$$

PROOF : Let us first assume that α is congruent to a normal curve. Then, up to isometries of $\mathbb{E}^3_1$, the curve α has the eq. 7 and there holds the relation[8]. Then relation[8] imply relation[10]

On the other hand, suppose that the relation[10] holds. By applying Frenet formulae[2], we obtain

$$\frac{d}{ds}\left[\ \alpha(s) - \frac{1}{k_2}N(s) - \frac{1}{k_2}\left(\frac{1}{k_2}\right)' B(s)\ \right] = 0.$$

Consequently, $\alpha(s)$ is congruent to a normal curve.

***Theorem* 3.6** — *Let $\alpha(s)$ be a null curve in $\mathbb{E}^3_1$, with curvatures $k_1(s) = 1, k_2(s) \neq 0$ for each $s \in I \subset \mathbb{R}$. Then $\alpha(s)$ is a normal curve if and only if the position vector α satisfies $g(\alpha, \alpha) = (1/k_2)^2$.*

PROOF : Let us first assume that $\alpha(s)$ is a normal curve. Then from the relations 7 and 9 we easily find that $g(\alpha, \alpha) = (1/k_2)^2$.

Conversely, suppose that the eq. $g(\alpha, \alpha) = (1/k_2)^2$ holds. By taking the derivative of the previous equation three times with respect to s, we get

$$g(\alpha, T) = (1/k_2)\,(1/k_2)',\ g(\alpha, N) = ((1/k_2)\,(1/k_2)')',$$

$$g(\alpha, B) = (1/k_2)' - (((1/k_2)\,(1/k_2)')')'. \qquad \text{... (11)}$$

Decompose the vector $\alpha(s)$ with respect to the Frenet base $\{T, N, B\}$ by $\alpha(s) = a(s)\,T(s) + b(s)\,N(s) + c(s)\,B(s)$, where $a(s), b(s), c(s)$ are arbitrary differentiable functions of s. Then we get

$$g(\alpha, T) = c,\ \ g(\alpha, N) = b,\ \ g(\alpha, B) = a, \qquad \text{... (12)}$$

and

$$g(\alpha, \alpha) = (1/k_2)^2 = b^2 + 2ac. \qquad \text{... (13)}$$

Substituting (11) and (12) into (13), we find

$$(1/k_2)^2 = ((1/k_2)(1/k_2)')'^2 + 2(1/k_2)(1/k_2)'[(1/k_2)' - (((1/k_2)\,1/k_2)')')'].$$

It follows that $((1/k_2)(1/k_2)')' = 1/k_2$. Then Theorem 3.5 implies that, up to a isometric of the space $\mathbb{E}_1^3$, $\alpha(s)$ is a normal curve.

Theorem 3.7 — *Let $\alpha(s)$ be a null curve in $\mathbb{E}_1^3$, with curvatures k_1 $(s = 1, k_2(s) \neq 0$ for each $s \in I \subset \mathbb{R}$. Then $\alpha(s)$ is a normal curve if and only if, up to a parametrization, it is given by*

$$\alpha(t) = e^t\, y(t), \qquad \text{... (14)}$$

where y(t) is a unit speed timelike curve lying on pseudosphere S_1^2.

PROOF : Let us first suppose that α is a normal curve. Then Theorem 3.6 implies that there holds $g(\alpha, \alpha) = (1/k_2)^2$. Let $\rho(s) = \|\alpha(s)\|$ be the distance function of α. It follows that $\rho^2 = (1/k_2)^2$. Let us define a curve $y(s)$ by $y(s) = \alpha(s)/\rho(s)$. Then we have

$$\alpha(s) = |1/k_2|\, y(s). \qquad \text{... (15)}$$

By taking the derivative of (15) with respect to s, yields

$$\alpha'(s) = |1/k_2|'\, y + |1/k_2|\, y'. \qquad \text{... (16)}$$

Since $g(y, y) = 1$, it follows that the curve y lies on pseudosphere S_1^2 of radius 1 and center at the origin. Then $g(y, y') = 0$, so the eq. (16) implies that

$$g(y', y') = -(|1/k_2|'/|1/k_2|)^2.$$

Thus y is a timelike curve with the velocity $\upsilon = \|y'(s)\| = |1/k_2|'/|1/k_2|$. Next, let us put

$$t = \int_0^s \|y'(u)\|\, du.$$

Then we obtain

$$t = \int_0^s \frac{|1/k_2(u)|'}{|1/k_2(u)|}\, du = \ln |1/k_2(s)|,$$

and hence $|1/k_2| = e^t$. Substituting this into (15) gives (14).

Conversely, suppose that $\alpha(t)$ is a null curve defined by (14). Then we can reparameterize the curve α by $t = \ln |1/k_2(s)|$, where s is the pseudo arclength function on α. Then the eq. (14) becomes $\alpha(s) = |1/k_2(s)|\, y(s)$ and thus $g(\alpha, \alpha) = (1/k_2)^2$. Finally, by Theorem 3.6, it follows that α is a normal curve.

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