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ON THE SOLUTIONS OF THE QUADRATIC PENCIL OF A SCHRÖDINGER EQUATION

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In this article we find the solutions of equation

$$-y'' + [q(x) + 2\lambda p(x) - \lambda^2]\, y + \frac{n(n+1)}{x^2}\, y = 0, x \in \boldsymbol{R}_+ = [0, \infty),$$

using the solutions of the quadratic pencil of Schrödinger equation

$$-y'' + [q(x) + 2\lambda p(x) - \lambda^2]\, y = 0, x \in \boldsymbol{R}_+ = [0, \infty),$$

where p and q are real valued function, λ is a spectral parameter and n is a natural number.

Key Words: Schrödinger Equation; Quadratic Pencil; Solutions

1. INTRODUCTION

Let us consider the following boundary value problems

$$-y'' + [q(x) + 2\lambda p(x) - \lambda^2]\, y = 0, \quad x \in \boldsymbol{R}_+ = [0, \infty), \qquad \text{... (1.1)}$$

$$y\,(0) = 0$$

and

$$-y'' + [q(x) + 2\lambda \;\, (x) - \lambda^2]\, y + \frac{n(n+1)}{x^2}\, y = 0, x \in \boldsymbol{R}_+ = [0, \infty), \qquad \text{... (1.2)}$$

where p and q are absolutely continuous real valued functions in each finite subinterval of $\boldsymbol{R}_+$ and satisfying

$$\int_0^\infty |\, p(x)\,|\, dx < \infty, \quad \int_0^\infty x\,[|\, q(x)\,| + |\, p(x)\,|]\, dx < \infty. \qquad \text{... (1.3)}$$

Under the condition (1.3) the eq. (1.1) has the solutions

$$e^+(x, \lambda) = e^{i\,w(z) + i\,\lambda\,x} + \int_x^\infty A^+(x, t)\, e^{i\,\lambda\,t}\, dt, \qquad \text{... (1.4)}$$

and

$$e^{-}(x, \lambda) = e^{iw(x) - t\lambda x} + \int_x^{\infty} A^{-}(x, t)\, e^{-i\lambda t}\, dt, \qquad \text{... (1.5)}$$

for λ in the closed upper and lower half-planes, respectively, where

$$w(x) = \int_x^{\infty} p(t)\, dt$$

and kernels $A^{\pm}(x, t)$ are expressed in terms of p, q and $A^{\pm}(x, t)$ are the solutions of Volterra type integral eqs. (3).

As it is known, the solutions of $e^{+}(x, \lambda)$ and $e^{-}(x, \lambda)$ given (1.4) and (1.5) are important in the investigation of spectral analysis and scattering theories of the boundary value problems (1.1) ([2] - [5]). But the eq. (1.2) has no solutions represented as the solutions (1.4) and (1.5) due to the having a singularity of term $\frac{n(n+1)}{x^2}$ in the neighbourhood of zero.

In this study, our purpose is to find that the eq. (1.2) has similar solutions to (1.4) and (1.5) using the solution of the eq. (1.1).

The similar problem has been studied for Sturm-Lioville and Klein-Gordon equation in [1] and [8].

2. The Solution of (1.2)

Let us consider the following equation

$$-y'' + [q(x) + 2\lambda p(x)]\, y = 0, \; x \in \boldsymbol{R}_{+}. \qquad \text{... (2.1)}$$

Then we get

Theorem 2.1 — *For all λ, eq. (2.1) has the solution $f(x, \lambda)$ which satisfies the initial conditions $f(0, \lambda) = 0, f'(0, \lambda) = 1$ and and $f(x, \lambda)$ has the representation*

$$f(x, \lambda) = x + \int_0^x (x - t)\,[q(t) + 2\lambda p(t)]\, f(t, \lambda)\, dt. \qquad \text{... (2.2)}$$

Moreover the asymptotic equalities

$$f(x, \lambda) = x\,(1 + o(1)), \;\; f'(x, \lambda) = (1 + o(1)) \qquad \text{... (2.3)}$$

are valid.

Proof : If we integrate eq. (2.1) twice and use the initial conditions, we get eq. (2.2), we get asymptotic equalities (2.3) by means of technique[7]

Let $h(x, \lambda)$ be the normalized eigen-function of the boundary value problem (1.1). The solution $h(x, \lambda)$ has the following asymptotic behaviour.

$$h(x, \lambda) = x\,(h'(x, \lambda) + o(1)), \qquad \dots (2.4)$$

for $x \to 0$.

If we consider the function

$$y(x, \lambda) = \frac{f(x, \lambda)\, h'(x, \lambda) - f'(x, \lambda)\, h(x, \lambda)}{\lambda f(x, \lambda)}. \qquad \dots (2.5)$$

we can give following

Theorem 2.2 — *If the function* $f(x, \lambda)$ *does not vanish in the interval* $(0, \infty)$, *then the function* $y(x, \lambda)$ *defined by (2.5) satisfies the equation*

$$-y'' + V(x, \lambda)\, y - \lambda^2 y = 0, \qquad \dots (2.6)$$

where

$$V(x, \lambda = q(x) + 2\lambda p(x) - 2\,[f'(x, \lambda) f^{1}(x, \lambda)]'.$$

PROOF : Let us write the first and second derivatives of $y(x, \lambda)$

$$y'(x, \lambda) = -\lambda h(x, \lambda) - \frac{f'(x, \lambda)}{f(x, \lambda)} \left\{ \frac{f(x, \lambda)\, h'(x, \lambda) - f'(x, \lambda)\, h(x, \lambda)}{\lambda f(x, \lambda)} \right\}$$

and

$$y''(x, \lambda) = -\lambda^2 \frac{f(x, \lambda)\, h'(x, \lambda)}{\lambda f(x, \lambda)} - \left(\frac{f'(x, \lambda)}{f(x, \lambda)} \right) y(x, \lambda) + \left(\frac{f'(x, \lambda)}{f(x, \lambda)} \right)^2$$

$$y(x, \lambda) + \lambda \frac{f'(x, \lambda)\, h(x, \lambda)}{\lambda f(x, \lambda)}$$

$$= -\lambda^2 \left\{ \frac{f(x, \lambda)\, h'(x, \lambda) - f'(x, \lambda)\, h(x, \lambda)}{\lambda f(x, \lambda)} \right\} + \left\{ \left(\frac{f'(x, \lambda)}{f(x, \lambda)} \right)^2 - \left(\frac{f'(x, \lambda)}{f(x, \lambda)} \right) \right\} y(x, \lambda)$$

$$= -\lambda^2 y(x, \lambda) + [q(x) + 2\lambda p(x)]\, y(x, \lambda) + 2 \left[\frac{f'^2(x, \lambda)}{f^2(x, \lambda)} - \frac{f''(x, \lambda)}{f(x, \lambda)} \right] y(x, \lambda;$$

Hence, we find that

$$-y''(x, \lambda) + [q(x) + 2\lambda p(x)]\, y(x, \lambda) - 2\,[f(x, \lambda) - 2\,[f'(x, \lambda) f^{1}(x, \lambda)]' = 0.$$

If we use the asymptotic equalities (2.3) and (2.4) as $x \to 0$, we have

$$\lim_{x \to 0} y(x, \lambda) = \lim_{x \to 0} \frac{f(x, \lambda)\, h'(x, \lambda) - f'(x, \lambda)\, h(x, \lambda)}{\lambda f(x, \lambda)}$$

$$= \lim_{x \to 0} \frac{f(x, \lambda)\, h'(x, \lambda)}{\lambda f(x, \lambda)} - \lim_{x \to 0} \frac{f'(x, \lambda)\, h(x, \lambda)}{\lambda f(x, \lambda)}$$

$$= \lim_{x \to 0} \lambda^{-1} h'(x, \lambda) - \lim_{x \to 0} \lambda^{-1} f'(x, \lambda) \frac{h(x, \lambda)}{f(x, \lambda)}$$

$$= \lambda^{-1} h'(0, \lambda) - \lim_{x \to 0} \lambda^{-1} f'(x, \lambda) \frac{h'(x, \lambda)}{f'(x, \lambda)}$$

$$= 0$$

and

$$2\,[f'(x, \lambda) f^{-1}(x, \lambda)]' = 2\left[(1 + o(1)) \frac{1}{x(1 + o(1))} \right]$$

$$= 2\left(\frac{1}{x} + O\left(\frac{1}{x} \right) \right)$$

$$= -\frac{2}{x^2} + O\left(\frac{1}{x^2} \right), \quad x \to \infty.$$

Hence, by (2.7) it follows that the potential $V(x, \lambda)$ behaves like $-\frac{2}{x^2}$ in the neighbourhood of zero. In this way we use the function (2.5) in the non-singular boundary value problem (1.1) and find the singular boundary value problem (2.6). ■

Now we find the inverse transformation of transformation (2.5). Since

$$y(x, \lambda) = \frac{f(x, \lambda)\, h'(x, \lambda) - f'(x, \lambda)\, h(x, \lambda)}{\lambda f(x, \lambda)}$$

then we get

$$\lambda \frac{y(x, \lambda)}{f(x, \lambda)} = \left[\frac{h(x, \lambda)}{f(x, \lambda)} \right]'$$

and hence, we also find

$$h(x, \lambda) = \lambda f(x, \lambda) \int_0^x \frac{y(t, \lambda)}{f(t, \lambda)}\, dt \qquad \text{... (2.8)}$$

Now in a similar way, we find the differential equation

$$-h''(x, \lambda) + \left[V(x, \lambda) + 2\,[f'(x, \lambda) f^{-1}(x, \lambda)]' \right] h(x, \lambda) - \lambda^2 h(x, \lambda) = 0. \qquad \text{... (2.9)}$$

If we substitute the potential $V(x, \lambda)$ defined in (2.7) in the last statement then we find the non-singular equation

$$-y'' + [q(x) + 2\lambda p(x) - \lambda^2]\, y = 0.$$

So we get the following theorem.

Theorem 2.3 — *If the function* $h(x, \lambda)$ *defined by (2.8) is the solution of (2.9) then the function* $y(x, \lambda)$ *defined by (2.5) is the solution of (2.6).* ■

Remark 2.1 : Now if the boundary value problem defined by (1.1) and the condition $y(0) = 0$ has no negative spectrum then the solution $f(x, \lambda)$ of (1.1) for $\lambda = 0$ which satisfies the initial conditions $f0, \lambda) = 0$. $f'\,0, \lambda) = 1$ does not vanish in the interval $0 < x < \infty$. The normalized eigen-functions of this boundary value problem and their derivatives will have the following asymptotic behaviour at infinity.

$$h(x, \lambda) \cong \sin(\lambda z + \delta x),\; h'(x, \lambda) \cong \lambda \cos(\lambda x + \delta x), \quad x \to \infty.$$

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