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UNICITY THEOREMS FOR MEROMORPHIC FUNCTIONS†

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Let f be a non-constant meromorphic function. If, $ff' \neq 0$, f and f' have the same fixed-points, then $f \equiv f'$.

Key Words: Meromorphic Function; Entire Function; Uniqueness

1. INTRODUCTION

Let f be a non-constant meromorphic function in the whole complex plane. We use the following standard notation of value distribution theory,

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \ ...$$

(see Hayman[2], Laine[4], Yang[6]). We denote by $S(r,f)$ any function satisfying

$$S(r,f) = o\ \{T(r,f)\},$$

as $r \to +\infty$, possibly outside of a set with finite measure.

Let g be a meromorphic function and let a be a complex number. $f(z) = a$ if and only if $g(z) = a$, we say that f and g share the value a IM. If $f(z) - a$ and $g(z) - a$ assume the same zeros with the same multiplicities, we say that f and g share the value a CM.

Rubel and Yang[5] proved.

Theorem A — *Let a, b be two distinct finite numbers and let f be a non-constant entire function. If f and f′ share a, b CM, then* $f \equiv f'$.

Gundersen[1] proved.

Theorem B — *Let a be a non-zero finite number, and let f be a non-constant meromorphic function. If f and f′ share 0, a CM, then* $f \equiv f'$.

In[1], Gundersen gave an example to show that Theorem B is not valid if f and f' share a IM.

Recently, Zhang[7] proved

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***Theorem* C** — *Let f be a non-constant meromorphic function and let a be a non-zero constant. If f and f' share 0, a CM, f and f' share a IM, then one of the following cases must occur:*

(1) $f \equiv f'$;

(2) $f(z) = \dfrac{2a}{1 - ce^{-2z}}$

where c is a non-zero constant.

In this paper, we prove

***Theorem* 1** — *Let f be a non-constant meromorphic function, and let $a(z)$ be a non-constant polymonial. If f and f' share 0 CM, $f(z) - a(z) = 0$ whenever $f'(z) - a(z) = 0$, then $f \equiv f'$.*

Corollary 1 — Let f be a non-constant meromorphic function. If, $ff' \neq 0$, f and f' have the same fixed-points, then $f \equiv f'$.

***Theorem* 2** — *Let f be a non-constant meromorphic function and let a be a non-zero constant. If f and f' share 0 CM, $f(z) = a$ whenever $f'(z) = a$, then one of the following cases must occur:*

(1) $f \equiv f'$;

(2) $f(z) = \dfrac{2a}{1 - ce^{-2z}}$

where c is a non-zero constant.

From Theorem 2 we obain Theorems B-C. It does not seem that Theorem 2 can be proved using method of[7].

2. Some Lemmas

In order to prove our results, we need the following lemmas.

Lemma 1 — (see[3], Lemma 4). Let f be a transcendental meromorphic function. Then

$$N\left(r, \frac{1}{f'}\right) \leq N\left(r, \frac{1}{f}\right) + \overline{N}(r,f) + S(r,f).$$

Lemma 2. — Let f be a transcendental meromorphic function, $a(z)$ ($\not\equiv 0$) a polynomial and let $h(z) = f'(z)/a(z)$, $h'(z) \not\equiv 0$. Then

$$(1)\ T(r,f) \leq \overline{N}(r,f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{h-1}\right) - N_0\left(r, \frac{1}{h'}\right) + S(r,f); \quad \text{... (2.1)}$$

$$(2)\ N_{1)}(r,f) \leq \overline{N}_{(2}(r,f) + \overline{N}\left(r, \frac{1}{h-1}\right) + N_0\left(r, \frac{1}{h'}\right) + S(r,f); \quad \text{... (2.2)}$$

where $N_{1)}(r,f)$ is the reduced counting function of the simple poles of f; $\overline{N}_{(2}(r,f)$ is the reduced counting function of the multiple poles of f; and $N_0(r, 1/h')$ denotes the counting function for those zero points of h' which are not zero points of $h(z)-1$, and $\overline{N}_0(r, 1/h')$ denotes the corresponding reduced counting function.

PROOF : We first prove (2.1). Obviously,

$$m\left(r,\frac{1}{f}\right)+m\left(r,\frac{1}{h-1}\right)\leq m\left(r,\frac{1}{h}\right)+m\left(r,\frac{1}{h-1}\right)+S(r,f)$$

$$\leq m\left(r,\frac{1}{h}+\frac{1}{h-1}\right)+S(r,f)\leq m\left(r,\frac{1}{h'}\right)+S(r,f)$$

$$=T(r,h')-N\left(r,\frac{1}{h'}\right)+S(r,f).$$

Thus by Nevanlinna's first fundamental theorem, we have

$$T(r,f)\leq\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{h-1}\right)-N\left(r,\frac{1}{h'}\right)+S(r,f)$$

$$\leq\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{h-1}\right)-N_0\left(r,\frac{1}{h'}\right)+S(r,f).$$

(2.1) is proved.

Next we prove (2.2). Set

$$g(z)=\frac{a^2(z)\,[h(z)-1]^3}{[h'(z)]^2}. \qquad \text{... (2.3)}$$

We claim that $g(z)\not\equiv c$, where c is a constant.

Suppose that $g(z)\equiv c$, then c is a non-zero constant. Otherwise, $h(z)\equiv 1$, that is $f'(z)\equiv a(z)$, which contradicts that f is a transcendental function.

Let $z_0\in\mathbb{C}$, such that $h(z_0)\neq 1$, $a(z_0)\neq 0$. Then there exists $\delta>0$ such that in $D_\delta = \{z : |z-z_0|<\delta\}$,

$$\frac{h'}{(h-1)^{\frac{3}{2}}}=\frac{a}{\sqrt{c}}.$$

Thus

$$\frac{1}{\sqrt{h-1}}=b(z),$$

where $b(z)$ is a polynomial satisfying $b'(z)=a(z)/\sqrt{c}$.

Hence we get

$$f'(z) = \frac{a(z)}{b^2(z)} + a(z), \quad z \in D\,\delta.$$

Thus

$$f'(z) = \frac{a(z)}{b^2(z)} + a(z), \quad z \in D\,\mathbb{C},$$

which contradicts that f is a transcendental function.

Let z_0 be a simple pole of f and $a(z_0) \neq 0$. We claim that

$$g(z_0) \neq 0, \infty, \qquad g'(z_0) = 0. \qquad \dots (2.4)$$

In fact, if z_0 is a simple pole of f, then in the neighbourhood of z_0,

$$f(z) = \frac{c}{z - z_0} + O(1), \qquad (c \neq 0). \qquad \dots (2.5)$$

Let $a(z) = b_1 + b_2(z - z_0) + \dots$, $(b_1 \neq 0)$. Then we have

$$\frac{1}{a(z)} = \frac{1}{b_1} - \frac{b_2}{b_1^2}(z - z_0) + O((z - z_0)^2),$$

$$f'(z) = -\frac{c}{(z - z_0)^2} + O(1),$$

$$h(z) = -\frac{c}{b_1(z - z_0)^2} + \frac{cb_2}{b_1^2(z - z_0)} + O(1),$$

$$h'(z) = -\frac{2c}{b_1(z - z_0)^3} + \frac{cb_2}{b_1^2(z - z_0)^2} + O(1).$$

Thus

$$g(z) = -\frac{b_1 c}{4} \frac{\left\{1 - 3\frac{b_2}{b_1}(z - c_0) + O((z - z)^2)\right\}\left\{1 + 2\frac{b_2}{b_1}(z - c_0) + O((z - z)^2)\right\}}{\left\{1 - \frac{b_2}{b_1}(z - c_0) + O((z - z)^2)\right\}}$$

$$= -\frac{b_1 c}{4} + O((z - z_0)^2).$$

(2.4) is proved. By (2.4) and Lemma 1, we have

$$N_{1)}(r, f) \leq N\left(r, \frac{1}{g'}\right) - \left[N\left(r, \frac{1}{g}\right) - \overline{N}\left(r, \frac{1}{g}\right)\right] + S(r, f)$$

$$\leq N\left(r, \frac{1}{g}\right) - \overline{N}(r, g) - N\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g}\right) + S(r, f)$$

$$\leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r, g) + S(r, f)$$

$$\leq \overline{N}_{(2}(r, f) + \overline{N}\left(r, \frac{1}{h-1}\right) + \overline{N}_0\left(r, \frac{1}{h'}\right) + S(r, f).$$

Thus we get (2.2). The proof of the lemma is complete.

Lemma 3 — (see[2], Theorem 2.5). Let f be a transcendental meromorphic function and let a, b be two distinct polynomials. Then

$$T(r, f) \leq \overline{N}(r, f) + N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-b}\right) + S(r, f).$$

3. Proof of Theorem 1

We first consider the case that f is a transcendental meromorphic function. Suppose that $f \not\equiv f'$. Since f and f' share 0 CM, by Lemma 2 we have

$$T(r, f) \leq \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{\frac{f'}{a} - 1}\right) + S(r, f)$$

$$\leq \frac{3}{2} T(r, f') + S(r, f).$$

Obviously,

$$S(r, f) = S(r, f'). \qquad \text{... (3.1)}$$

Since f and f' share 0 CM and $f(z) - a(z) = 0$ whenever $f'(z) - a(z) = 0$, we get Lemma 3 that

$$T(r, f') \leq \overline{N}(r, f') + \overline{N}\left(r, \frac{1}{f'}\right) + \overline{N}\left(r, \frac{1}{f'-a}\right) + S(r, f')$$

$$\leq \overline{N}(r, f) + N\left(r, \frac{1}{\frac{f'}{f} - 1}\right) + S(r, f')$$

$$\leq \overline{N}(r, f) + T\left(r, \frac{f'}{f} - 1\right) + S(r, f')$$

$$\leq 2\overline{N}(r, f) + S(r, f')$$

$$\leq T(r, f') + S(r, f').$$

Hence

$$T(r, f') = 2\overline{N}(r, f) + s(r, f'), \qquad \text{... (3.2)}$$

$$T(r, f') = 2\overline{N}\left(r, \frac{1}{f'-a}\right) + S(r, f'). \qquad \text{... (3.3)}$$

By (3.2), we get

$$N(r,f)+\overline{N}(r,f)=N(r,f')\le T(r,f')=2\overline{N}(r,f)+S(r,f).$$

Thus

$$N(r,f)=\overline{N}(r,f)+S(r,f), \quad \text{... (3.4)}$$

$$N_{(2}(r,f) = S(r,f). \quad \text{... (3.5)}$$

Set

$$\phi(z) = 2\frac{f'(z)}{f(z)} - \frac{f''(z)}{f'(z)}. \quad \text{... (3.6)}$$

Since f and f' share 0 CM, by (3.5)-(3.6), we have

$$N(r,\phi) = S(r,f).$$

Obviously,

$$m(r,\phi) = S(r,f).$$

Thus

$$T(r,\phi) = S(r,f). \quad \text{... (3.7)}$$

Let z_0 be a multiple zero point of $f'(z)-a(z)$ and $a(z_0)\ne 0$. Then by (3.6),

$$\phi(z_0)=2-\frac{a'(z_0)}{a(z_0)}. \quad \text{... (3.8)}$$

Next, we consider two cases:

Case 1 : $\phi(z)+a'(z)/a(z)\not\equiv 2$. Then

$$\overline{N}_{(2}\left(r,\frac{1}{f'-a}\right)\le N\left(r,\frac{1}{\phi+a'/a-2}\right)\le T\left(r,\phi+\frac{a'}{a}\right)+O(1)=S(r,f). \quad \text{... (3.9)}$$

Thus by Nevanlinna's second fundamental theorem, (3.2), (3.3) and (3.9), we get

$$T(r,f') \le \overline{N}(r,f')+\overline{N}\left(r,\frac{1}{f'}\right)+\overline{N}\left(r,\frac{1}{\frac{f'}{a}-1}\right)-N_0\left(r,\frac{1}{\left(\frac{f'}{a}\right)'}\right)+S(r,f')$$

$$\le \overline{N}(r,f')+\overline{N}\left(r,\frac{1}{\frac{f'}{a}-1}\right)-N_0\left(r,\frac{1}{\left(\frac{f'}{a}\right)'}\right)+S(r,f')$$

$$\le \overline{N}(r,f')+-N_0\left(r,\frac{1}{\left(\frac{f'}{a}\right)'}\right)+S(r,f').$$

Hence

$$N_0\left(r,\frac{1}{\left(\frac{f'}{a}\right)'}\right)+S(r,f'). \quad \text{... (3.10)}$$

Thus by (3.9)-(3.10),

$$N\left(r,\frac{1}{\left(\frac{f'}{a}\right)'}\right)+S(r,f'). \quad \text{... (3.11)}$$

Let $b'(z)=a(z)$. Then by Lemma 2 and (3.11), we have

$$\begin{aligned}\overline{N}_{1)}(r,f)&=\overline{N}_{1)}(r,f+b)\\ &\le \overline{N}_{(2}(r,f+b)+\overline{N}\left(r,\frac{1}{f'}\right)-\overline{N}\left(r,\frac{1}{\left(\frac{f'}{a}\right)'}\right)+S(r,f+b)\\ &\le N_{(2}(r,f)\;+\;S(r,f).\end{aligned} \quad \text{... (3.12)}$$

Thus by (3.2), (3.5) and (3.12), $T(r,f')=2\overline{N}(r,f)+S(r,f)=S(r,f)$, a contradiction. Hence $f\equiv f'$.

Case 2 : $\phi(z)+a'(z)/a(z)\equiv 2$, that is

$$2\frac{f'(z)}{f(z)}-\frac{f''(z)}{f'(z)}+\frac{a'(z)}{a(z)}\equiv 2. \quad \text{... (3.13)}$$

Solving (3.13),

$$\frac{f'(z)}{f^2(z)}=c_1\,a(z)\,e^{-2z},$$

where c_1 is a non-zero constant

Thus

$$f(z)\;=\;\frac{1}{A_1+c\displaystyle\int_0^z a(z)\,e^{-2z}\,dz}, \quad \text{... (3.14)}$$

where A_1 and $c\,(\neq 0)$ are two constants.

Hence

$$f(z)=\frac{1}{A+cp(z)\,e^{-2z}}, \quad \text{... (3.15)}$$

where p is a polynomial of deg p = deg a, and $[p(z)\,e^{-2z}]'=a(z)\,e^{-2z}$. Thus,

$$f'(z) = -\frac{ca(z)\,e^{-2z}}{(A + cp(z)\,e^{-2z})^2}. \qquad \text{... (3.16)}$$

By (3.3), $f'(z) - a(z) = 0$ has infinitely many solutions. Let z_0 satisfy $f'(z_0) - a(z_0) = 0$. Then by (3.15)-(3.16) and $f(z) - a(z) = 0$ whenever $f'(z) - a(z) = 0$ we get

$$\frac{1}{p(z_0)}\left(\frac{1}{a(z_0)} - A\right) = -\frac{1}{a^2(z_0)}. \qquad \text{... (3.17)}$$

We claim that

$$\frac{1}{p(z)}\left(\frac{1}{a(z)} - A\right) = -\frac{1}{a^2(z)} \not\equiv 0. \qquad \text{... (3.18)}$$

Indeed, suppose that

$$\frac{1}{p(z)}\left(\frac{1}{a(z)} - A\right) = -\frac{1}{a^2(z)} \equiv 0. \qquad \text{... (3.19)}$$

Then

$$a(z) - Aa^2(z) + p(z) \equiv 0. \qquad \text{... (3.20)}$$

If $A \neq 0$, then by (3.20) and $\deg p = \deg a$, $a(z)$ is a constant, a contradiction; if $A = 0$, then by (3.20),

$$a(z) + p(z) \equiv 0. \qquad \text{... (3.21)}$$

On the other hand, we get from $\left[p(z)\,e^{-2z}\right]' = a(z)\,e^{-2z}$ that

$$p'(z) - 2p(z) \equiv a(z). \qquad \text{... (3.22)}$$

Thus, (3.21) and (3.22) implies $a'(z) - a(z) \equiv 0$. Since $a(z)$ is a polynomial, we obtain a contradiction: $a \equiv 0$.

Hence by (3.17), we have

$$\overline{N}\left(r, \frac{1}{f' - a}\right) \leq N\left(r, \frac{1}{\frac{1}{p}\left(\frac{1}{a} - A\right) + \frac{1}{a^2}}\right) = O(\log r) = S(r, f),$$

which contradicts (3.3). Hence $f \equiv f'$.

Next we assume that f is a rational function. By $f(z)f'(z) \neq 0$, we have

$$f(z) = \frac{1}{(az + b)^n}, \qquad \text{... (3.23)}$$

where n is a positive integer.

Then by $f(z) - a(z) = 0$ whenever $f'(z) - a(z) = 0$ we can deduce that $f \equiv f'$. We omit the details here. The proof of Theorem 1 is complete.

4. Proof of Theorem 2

By Theorem 1, we obtain that either $f \equiv f'$, or

$$f(z) = \frac{1}{A_1 + ca \int_0^z e^{-2z} dz} \qquad \dots (4.1)$$

If $f \not\equiv f'$, then by (3.3), $f'(z) - a = 0$ has infinitely many solutions. Thus by (4.1) and $f(z) = a$ whenever $f'(z) = a$, we get

$$f(z) = \frac{2a}{1 - ce^{-2z}},$$

where c is non-zero constant. The proof of Theorem 2 is complete.

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